1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates counterclockwise by $45^\circ$, and let $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that scales the $x$-direction by 2. The matrices $A$ and $B$ for $T$ and $U$ are, respectively:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

a) Compute the matrices for $T^{-1}$ and $U^{-1}$.

b) Compute the matrix for the transformation that first rotates counterclockwise by $45^\circ$, then scales the $x$-direction by 2, then rotates clockwise by $45^\circ$.

**Solution.**

a) The matrices for $T^{-1}$ and $U^{-1}$ are $A^{-1}$ and $B^{-1}$, respectively. We compute these using the determinant trick:

$$\det A = \frac{1}{2}(1 + 1) = 1 \quad A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\det B = 2 \quad B^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

b) The transformation in question is $T^{-1} \circ U \circ T$. The matrix for this transformation is

$$A^{-1}BA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$
2. Let \( A \) be an \( n \times n \) matrix which is **not invertible**, and let \( T(x) = Ax \). Which of the following are definitely true? (Circle all that apply.)

   a) \( T(x) = b \) is consistent for all \( b \) in \( \mathbb{R}^n \).

   b) \( Ax = 0 \) has the trivial solution.

   c) There exist \( x \neq y \) in \( \mathbb{R}^n \) such that \( T(x) = T(y) \).

   d) Every vector in \( \mathbb{R}^n \) is a linear combination of the columns of \( A \).

   e) \( A \) has at most \( n - 1 \) pivots.

**Solution.**

a) This means that \( T \) is onto, which implies \( A \) is invertible, so this is **false**.

b) This is always **true**, whether \( A \) is invertible or not.

c) This means that \( T \) is not one-to-one, which is **true** when \( A \) is not invertible.

d) This implies \( A \) is invertible, so it is **false**.

e) This means \( A \) does not have \( n \) pivots, which is **true** when \( A \) is not invertible.