Announcements
Wednesday, October 11

▶ The second midterm is on Friday, October 20.
  ▶ That is one week from this Friday.
  ▶ The exam covers §§1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, and 2.9.

▶ Comments on mid-semester reviews on Piazza.

▶ WeBWorK 2.1, 2.2, 2.3 are due today at 11:59pm.

▶ The quiz on Friday covers §§2.1, 2.2, 2.3.

▶ My office is Skiles 244. Rabin office hours are today, 10–11, 12–1, and 2–3.
Section 2.8

Subspaces of $\mathbb{R}^n$
Today we will discuss **subspaces** of $\mathbb{R}^n$.

A subspace turns out to be the same as a span, except we don’t know *which* vectors it’s the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbb{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.

\[ x + 3y + z = 0 \]
Definition of Subspace

Definition
A subspace of $\mathbb{R}^n$ is a subset $V$ of $\mathbb{R}^n$ satisfying:

1. The zero vector is in $V$. “not empty”
2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$. “closed under addition”
3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu$ is in $V$. “closed under $\times$ scalars”

Fast-forward
Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven’t computed what those vectors are yet.
Definition of Subspace

Definition

A **subspace** of \( \mathbb{R}^n \) is a subset \( V \) of \( \mathbb{R}^n \) satisfying:

1. The zero vector is in \( V \). “not empty”
2. If \( u \) and \( v \) are in \( V \), then \( u + v \) is also in \( V \). “closed under addition”
3. If \( u \) is in \( V \) and \( c \) is in \( \mathbb{R} \), then \( cu \) is in \( V \). “closed under \( \times \) scalars”

What does this mean?

- If \( v \) is in \( V \), then all scalar multiples of \( v \) are in \( V \) by (3). That is, the line through \( v \) is in \( V \).
- If \( u, v \) are in \( V \), then \( xu \) and \( yv \) are in \( V \) for scalars \( x, y \) by (3). So \( xu + yv \) is in \( V \) by (2). So Span\( \{u, v\} \) is contained in \( V \).
- Likewise, if \( v_1, v_2, \ldots, v_n \) are all in \( V \), then Span\( \{v_1, v_2, \ldots, v_n\} \) is contained in \( V \): a subspace contains the span of any set of vectors in it.

If you pick enough vectors in \( V \), eventually their span will fill up \( V \), so:

A subspace is a span of some set of vectors in it.
Examples

Example
A line \( L \) through the origin: this contains the span of any vector in \( L \).

Example
A plane \( P \) through the origin: this contains the span of any vectors in \( P \).

Example
All of \( \mathbb{R}^n \): this contains 0, and is closed under addition and scalar multiplication.

Example
The subset \( \{0\} \): this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)
A **subset** of $\mathbb{R}^n$ is any collection of vectors whatsoever.

All of the following non-examples are still subsets.

A **subspace** is a special kind of subset, which satisfies the three defining properties.

![Subset: yes Subspace: no](image-url)
Non-Examples

Non-Example
A line $L$ (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.

Non-Example
A circle $C$ is not a subspace. Fails: 1, 2, 3. Think: a circle isn't a “linear space.”

Non-Example
The first quadrant in $\mathbb{R}^2$ is not a subspace. Fails: 3 only.

Non-Example
A line union a plane in $\mathbb{R}^3$ is not a subspace. Fails: 2 only.
Spans are Subspaces

**Theorem**
Any \( \text{Span}\{v_1, v_2, \ldots, v_n\} \) is a subspace.

!!!

Every subspace is a span, and every span is a subspace.

**Definition**
If \( V = \text{Span}\{v_1, v_2, \ldots, v_n\} \), we say that \( V \) is the subspace **generated by** or **spanned by** the vectors \( v_1, v_2, \ldots, v_n \).
Question: What is the difference between {} and {0}?
Let \( V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\} \). Let’s check if \( V \) is a subspace or not.

1. Does \( V \) contain the zero vector? 
   \( \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow ab = 0 \)

2. Is \( V \) closed under addition? 
   Let \( \begin{pmatrix} a \\ b \end{pmatrix} \) and \( \begin{pmatrix} a' \\ b' \end{pmatrix} \) be (unknown vectors) in \( V \). 
   \( ab = 0 \) and \( a'b' = 0 \).
   Is \( \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a + a' \\ b + b' \end{pmatrix} \) in \( V \)? 
   This means: \( (a + a')(b + b') = 0 \).
   This is not true for all \( a, a', b, b' \): for instance, \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are in \( V \), but their sum \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is not in \( V \), because \( 1 \cdot 1 \neq 0 \).

We conclude that \( V \) is not a subspace. A picture is above. (It doesn’t look like a span.)
Column Space and Null Space

An $m \times n$ matrix $A$ naturally gives rise to two subspaces.

**Definition**

- The **column space** of $A$ is the subspace of $\mathbb{R}^m$ spanned by the columns of $A$. It is written $\text{Col} \ A$.
- The **null space** of $A$ is the set of all solutions of the homogeneous equation $Ax = 0$:
  \[
  \text{Nul} \ A = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.
  \]
  This is a subspace of $\mathbb{R}^n$.

The column space is defined as a span, so we know it is a subspace. It is the range (as opposed to the codomain) of the transformation $T(x) = Ax$.

**Check** that the null space is a subspace:

1. $0$ is in Nul $A$ because $A \cdot 0 = 0$.
2. If $u$ and $v$ are in Nul $A$, then $Au = 0$ and $Av = 0$. Hence $A(u + v) = Au + Av = 0$, so $u + v$ is in Nul $A$.
3. If $u$ is in Nul $A$, then $Au = 0$. For any scalar $c$, $A(cu) = cAu = 0$. So $cu$ is in Nul $A$. 
Column Space and Null Space

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}. \)

Let’s compute the column space:

\[ \text{Col} \ A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \]

This is a line in \( \mathbb{R}^3 \).

Let’s compute the null space:

The reduced row echelon form of \( A \) is
\[ \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

This gives the equation \( x + y = 0 \), or \( x = -y \).

\( y \) parametric vector form \( (x, y) = y (-1, 1) \).

Hence the null space is \( \text{Span} \left\{ (-1, 1) \right\} \), a line in \( \mathbb{R}^2 \).
The column space of a matrix $A$ is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

**Question**

How to find vectors which span the null space?

**Answer:** Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \text{ if, say, } x_3 \text{ and } x_4 \text{ are the free variables. So }$$

$$\text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\} .$$

Refer back to the slides for §1.5 (Solution Sets).

**Note:** It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.
How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of \( \mathbb{R}^n \) or the zero subspace \( \{0\} \)?
- Can it be written as a type of subspace that we’ll learn about later (eigenspaces, …)?

If so, then it’s automatically a subspace.

If all else fails:
- Can you verify directly that it satisfies the three defining properties?
What is the *smallest number* of vectors that are needed to span a subspace?

**Definition**
Let $V$ be a subspace of $\mathbb{R}^n$. A *basis* of $V$ is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $V$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of $V$, and is written $\text{dim } V$.

Why is a basis the smallest number of vectors needed to span?
Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can’t span $V$.

**Important**
A subspace has *many different* bases, but they all have the same number of vectors (see the exercises in §2.9).
Question
What is a basis for $\mathbb{R}^2$?

We need two vectors that span $\mathbb{R}^2$ and are linearly independent. \( \{e_1, e_2\} \) is one basis.

1. They span: \( \begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2 \).
2. They are linearly independent because they are not collinear.

Question
What is another basis for $\mathbb{R}^2$?

Any two nonzero vectors that are not collinear. \( \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \) is also a basis.

1. They span: \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) has a pivot in every row.
2. They are linearly independent: \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) has a pivot in every column.
Bases of $\mathbb{R}^n$

The unit coordinate vectors

\[
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, &
\quad & e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, &
\quad & \ldots, &
\quad & e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, &
\quad & e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}
\end{align*}
\]

are a basis for $\mathbb{R}^n$.

The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: $I_n$ has a pivot in every row.
2. They are linearly independent: $I_n$ has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$ if and only if the matrix

\[
A = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n
\end{pmatrix}
\]

has a pivot in every row and every column, i.e. if $A$ is invertible.

Sanity check: we have shown that $\dim \mathbb{R}^n = n$. 
Example

Let

\[ V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + 3y + z = 0 \right\} \quad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}. \]

Verify that \( \mathcal{B} \) is a basis for \( V \). (So dim \( V = 2 \): it is a plane.)
The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for $\text{Nul } A$.

**Example**

\[
A = \begin{pmatrix}
1 & 2 & 0 & -1 \\
-2 & -3 & 4 & 5 \\
2 & 4 & 0 & -2
\end{pmatrix}
\xrightarrow{\text{rref}}
\begin{pmatrix}
1 & 0 & -8 & -7 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[\text{parametric vector form} \Rightarrow \quad x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}
\]

1. The vectors span $\text{Nul } A$ by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)
**Basis for Col A**

**Fact**

The *pivot columns* of $A$ always form a basis for $\text{Col} \ A$.

**Warning:** I mean the pivot columns of the *original* matrix $A$, not the row-reduced form. (Row reduction changes the column space.)

**Example**

$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & 3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$

Row reduction:

$rref \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

pivot columns = basis ↔ pivot columns in rref

So a basis for $\text{Col} \ A$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$ 

**Why?** See slides on linear independence.
Summary

- A **subspace** is the same as a span of some number of vectors, but we haven’t computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

  \[ \text{Col } A = \text{ the span of the columns of } A \]
  \[ \text{Nul } A = \text{ the solution set of } Ax = 0. \]

- A **basis** of a subspace is a minimal set of spanning vectors; the **dimension** of \( V \) is the number of vectors in any basis.
- The pivot columns form a basis for \( \text{Col } A \), and the parametric vector form produces a basis for \( \text{Nul } A \).

**Warning**
These are not the official definitions!