1. Consider the vectors

\[ v_1 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -4 \\ -2 \\ -8 \end{pmatrix}. \]

Is \( \{v_1, v_2, v_3\} \) linearly independent? If not, give an equation of linear dependence.

**Solution.**

They are linearly dependent: you can see that \( v_3 = -2v_1 \). Hence an equation of linear dependence is

\[ 2v_1 + 0v_2 + v_3 = 0. \]
2. Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 + x_2 + 2x_3 \\ 2x_1 - 2x_2 + x_3 \\ -x_1 + x_2 + 2x_3 \end{pmatrix}. $$

a) Find the standard matrix $A$ for $T$.

b) Is $T$ one-to-one? If not, find two vectors in $\mathbb{R}^3$ with the same image.

c) Is $T$ onto? If not, find a vector in $\mathbb{R}^3$ which is not in the range.

Solution.

a) We plug in the unit coordinate vectors:

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\Rightarrow A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

b) We need to know if $A$ has a pivot in every column. We row reduce:

$$\begin{pmatrix} -1 & 1 & 2 \\ 2 & -2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second column does not have a pivot, so $T$ is not one-to-one. The parametric form of the solution set of $Ax = 0$ is $x_1 = x_2, x_3 = 0$. Any value of $x_2$ gives a solution to $T(x) = 0$, so we have, for instance,

$$T(0) = 0 \quad T \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0.$$

c) The first and last entries of $T(x)$ are the same. Therefore, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in the range, for instance. In particular, $T$ is not onto.