Announcements
Wednesday, September 27

- The midterm will be returned in recitation on Friday.
  - You can pick it up from me in office hours before then.
  - Keep tabs on your grades on Canvas.

- WeBWorK 1.7 is due Friday at 11:59pm.

- No quiz on Friday!

- My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.
Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x) = 1.5x$. Is $T$ linear? Check:

\[
T(u + v) = \\
T(cv) =
\]

So $T$ satisfies the two equations, hence $T$ is linear.

**Note:** $T$ is a matrix transformation!

\[
T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,
\]

as we checked before.
Define \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
\[
T(x) = \text{the vector } x \text{ rotated counterclockwise by an angle of } \theta.
\]
Is \( T \) linear? Check:

The pictures show \( T(u) + T(v) = T(u + v) \) and \( T(cu) = cT(u) \).

Since \( T \) satisfies the two equations, \( T \) is linear.
Is every transformation a linear transformation?

**No!** For instance, \( T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix} \) is not linear.

**Why?** We have to check the two defining properties. Let’s try the second:

\[
T \left( c \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \sin(cx) \\ (cx)(cy) \\ \cos(cy) \end{pmatrix} \overset{?}{=} c \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix} = cT \begin{pmatrix} x \\ y \end{pmatrix}
\]

So \( T \) fails the second property. **Conclusion:** \( T \) is *not* a matrix transformation!

(We could also have noted \( T(0) \neq 0 \).)
Which of the following transformations are linear?

A. \( T(x_1 x_2) = |x_1| x_2 \)

B. \( T(x_1 x_2) = (2x_1 + x_2 - 2x_2) \)

C. \( T(x_1 x_2) = (x_1 x_2 x_2) \)

D. \( T(x_1 x_2) = (2x_1 + 1x_1 - 2x_2) \)

Poll

A. \( T((1 0) + (-1 0)) = (0 0) \neq (2 0) = T(1 0) + T((-1 0)) \), so not linear.

B. Linear.

C. \( T(2(1 1)) = (4 2) \neq 2 T(1 1) \), so not linear.

D. \( T(0 0) = (1 0) \neq 0, \) so not linear.

Remark: in fact, \( T \) is linear if and only if each entry of the output is a linear function of the entries of the input, with no constant terms. Check this!
We will see that a *linear* transformation $T$ is a matrix transformation: $T(x) = Ax$.

But what matrix does $T$ come from? What is $A$?

Here’s how to compute it.
Unit Coordinate Vectors

Definition

The unit coordinate vectors in $\mathbb{R}^n$ are

\[
\begin{align*}
e_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \\
e_2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \\
&\quad \ldots, \\
e_{n-1} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \\
e_n &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

Note: if $A$ is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by $e_i$ gives you the $i$th column.
Linear Transformations are Matrix Transformations

Recall: A matrix $A$ defines a linear transformation $T$ by $T(x) = Ax$.

Theorem
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}.$$  

This is an $m \times n$ matrix, and $T$ is the matrix transformation for $A$: $T(x) = Ax$. The matrix $A$ is called the **standard matrix** for $T$.

Take-Away

**Linear transformations are the same as matrix transformations.**

Dictionary

<table>
<thead>
<tr>
<th>Linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$</th>
<th>$m \times n$ matrix $A = \begin{pmatrix} T(e_1) &amp; T(e_2) &amp; \cdots &amp; T(e_n) \end{pmatrix}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(x) = Ax$</td>
<td>$m \times n$ matrix $A$</td>
</tr>
</tbody>
</table>
Why is a linear transformation a matrix transformation?

$T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$

Suppose for simplicity that

$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T\begin{bmatrix} x \cdot e_1 \\ y \cdot e_2 \\ z \cdot e_3 \end{bmatrix} = xT(e_1) + yT(e_2) + zT(e_3) = \begin{bmatrix} | | | \\ T(e_1) \\ T(e_2) \\ T(e_3) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. 
Before, we defined a **dilation** transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\begin{align*}
T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}, \\
T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix}
\end{align*}$$

$$\Rightarrow A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}.$$
Linear Transformations are Matrix Transformations

Example

**Question**

What is the matrix for the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by

\[
T(x) = x \text{ rotated counterclockwise by an angle } \theta
\]
Linear Transformations are Matrix Transformations

Example

**Question**

What is the matrix for the linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that reflects through the \( xy \)-plane and then projects onto the \( yz \)-plane?

\[
T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
Question

What is the matrix for the linear transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that reflects through the \( xy \)-plane and then projects onto the \( yz \)-plane?

\[
T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]
Question
What is the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?

$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. 
Linear Transformations are Matrix Transformations
Example, continued

**Question**
What is the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?

$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
Onto Transformations

Definition
A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** (or **surjective**) if the range of $T$ is equal to $\mathbb{R}^m$ (its codomain). In other words, for every $b$ in $\mathbb{R}^m$, the equation $T(x) = b$ has at least one solution. Or, every possible output has an input. Note that *not* onto means there is some $b$ in $\mathbb{R}^m$ which is not the image of any $x$ in $\mathbb{R}^n$. 

[Interactive diagrams for onto and not onto transformations]
Theorem
Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation with matrix \( A \). Then the following are equivalent:

- \( T \) is onto
- \( T(x) = b \) has a solution for every \( b \) in \( \mathbb{R}^m \)
- \( Ax = b \) is consistent for every \( b \) in \( \mathbb{R}^m \)
- The columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in every row

Question
If \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is onto, what can we say about the relative sizes of \( n \) and \( m \)?

Answer: \( T \) corresponds to an \( m \times n \) matrix \( A \). In order for \( A \) to have a pivot in every row, it must have at least as many columns as rows: \( m \leq n \).

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast \\
0 & 1 & \ast & 0 & \ast \\
0 & 0 & 0 & 1 & \ast
\end{pmatrix}
\]

For instance, \( \mathbb{R}^2 \) is “too small” to map onto \( \mathbb{R}^3 \).
One-to-one Transformations

Definition
A transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one (or into, or injective) if different vectors in $\mathbb{R}^n$ map to different vectors in $\mathbb{R}^m$. In other words, for every $b$ in $\mathbb{R}^m$, the equation $T(x) = b$ has at most one solution $x$. Or, different inputs have different outputs. Note that not one-to-one means at least two different vectors in $\mathbb{R}^n$ have the same image.
Characterization of One-to-One Transformations

Theorem
Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation with matrix $A$. Then the following are equivalent:

▶ $T$ is one-to-one
▶ $T(x) = b$ has one or zero solutions for every $b$ in $\mathbb{R}^m$
▶ $Ax = b$ has a unique solution or is inconsistent for every $b$ in $\mathbb{R}^m$
▶ $Ax = 0$ has a unique solution
▶ The columns of $A$ are linearly independent
▶ $A$ has a pivot in every column.

Question
If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one, what can we say about the relative sizes of $n$ and $m$?

Answer: $T$ corresponds to an $m \times n$ matrix $A$. In order for $A$ to have a pivot in every column, it must have at least as many rows as columns: $n \leq m$.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

For instance, $\mathbb{R}^3$ is “too big” to map into $\mathbb{R}^2$. 
Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- A transformation $T$ is **one-to-one** if $T(x) = b$ has *at most one* solution, for every $b$ in $\mathbb{R}^m$.
- A transformation $T$ is **onto** if $T(x) = b$ has *at least one* solution, for every $b$ in $\mathbb{R}^m$.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.
- There are lots of equivalent conditions for a linear transformation to be one-to-one and/or onto, in terms of its matrix.