The midterm will be returned in recitation on Friday.
  - You can pick it up from me in office hours before then.
  - Keep tabs on your grades on Canvas.

WeBWorK 1.7 is due **Friday** at 11:59pm.

No quiz on Friday!

My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.
Section 1.8/1.9

Linear Transformations
Let $A$ be a matrix, and consider the matrix equation $b = Ax$. If we vary $x$, we can think of this as a function of $x$.

Many functions in real life—the linear transformations—come from matrices in this way.

It makes us happy when a function comes from a matrix, because then questions about the function translate into questions a matrix, which we can usually answer.

For this reason, we want to study matrices as functions.
Change in Perspective. Let $A$ be a matrix with $m$ rows and $n$ columns. Let’s think about the matrix equation $b = Ax$ as a function.

- The independent variable (the input) is $x$, which is a vector in $\mathbb{R}^n$.
- The dependent variable (the output) is $b$, which is a vector in $\mathbb{R}^m$.

As you vary $x$, then $b = Ax$ also varies. The set of all possible output vectors $b$ is the column span of $A$. 

$\mathbb{R}^n \rightarrow b = Ax \rightarrow \mathbb{R}^m$
Matrices as Functions

Projection

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector $x$ is in $\mathbb{R}^3$ and the output vector $b$ is in $\mathbb{R}^3$. Then

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

This is *projection onto the xy-axis*. Picture:
Matrices as Functions

Reflection

\[ A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \). Then

\[ A \begin{pmatrix} x \\ y \end{pmatrix} = \]

This is *reflection over the y-axis*. Picture:
Matrices as Functions

Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

In the equation $Ax = b$, the input vector $x$ is in $\mathbb{R}^2$ and the output vector $b$ is in $\mathbb{R}^2$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} =$$

This is *dilation (scaling) by a factor of 1.5*. Picture:
Matrices as Functions

Identity

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \).

\[ A \begin{pmatrix} x \\ y \end{pmatrix} = \]

This is the identity transformation which does nothing. Picture:
Matrices as Functions
Rotation

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector $x$ is in $\mathbb{R}^2$ and the output vector $b$ is in $\mathbb{R}^2$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

What is this? Let's plug in a few points and see what happens.

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

It looks like \textit{counterclockwise rotation by 90°}.
Matrices as Functions

Rotation

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector $x$ is in $\mathbb{R}^2$ and the output vector $b$ is in $\mathbb{R}^2$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$
In §1.9 of Lay, there is a long list of geometric transformations of $\mathbb{R}^2$ given by matrices. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, . . . ) Please look them over.
We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let’s go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

**Example**
For a vector $x$ in $\mathbb{R}^2$, let $T(x)$ be the counterclockwise rotation of $x$ by an angle $\theta$. Is $T(x) = Ax$ for some matrix $A$?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

But for general $\theta$, it’s not clear.

Our next goal is to answer this kind of question.
Transformations

Vocabulary

Definition

A transformation (or function or map) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a rule \( T \) that assigns to each vector \( x \) in \( \mathbb{R}^n \) a vector \( T(x) \) in \( \mathbb{R}^m \).

- \( \mathbb{R}^n \) is called the domain of \( T \) (the inputs).
- \( \mathbb{R}^m \) is called the codomain of \( T \) (the outputs).
- For \( x \) in \( \mathbb{R}^n \), the vector \( T(x) \) in \( \mathbb{R}^m \) is the image of \( x \) under \( T \).
  Notation: \( x \mapsto T(x) \).
- The set of all images \( \{ T(x) \mid x \text{ in } \mathbb{R}^n \} \) is the range of \( T \).

Notation:

\[ T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \] means \( T \) is a transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

It may help to think of \( T \) as a “machine” that takes \( x \) as an input, and gives you \( T(x) \) as the output.
Many of the functions you know and love have domain and codomain $\mathbb{R}$.

$$\sin: \mathbb{R} \rightarrow \mathbb{R} \quad \sin(x) = \left( \frac{\text{the length of the opposite edge}}{\text{the hypotenuse of a right triangle with angle}} \right) \ x \ \text{in radians}$$

Note how I’ve written down the rule that defines the function $\sin$.

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2$$

Note that “$x^2$” is sloppy (but common) notation for a function: it doesn’t have a name!

You may be used to thinking of a function in terms of its graph.

The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are $\mathbb{R}$, but it’s hard to do when they’re $\mathbb{R}^2$ and $\mathbb{R}^3$! You need five dimensions to draw that graph.
Matrix Transformations

Definition
Let $A$ be an $m \times n$ matrix. The matrix transformation associated to $A$ is the transformation

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$
defined by $T(x) = Ax$.

In other words, $T$ takes the vector $x$ in $\mathbb{R}^n$ to the vector $Ax$ in $\mathbb{R}^m$.

For example, if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $T(x) = Ax$ then

$$T\begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$

- The domain of $T$ is $\mathbb{R}^n$, which is the number of columns of $A$.
- The codomain of $T$ is $\mathbb{R}^m$, which is the number of rows of $A$.
- The range of $T$ is the set of all images of $T$:

$$T(x) = Ax = \left( \begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_n \end{array} \right) \left( \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right) = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the column span of $A$. It is a span of vectors in the codomain.
Matrix Transformations

Example

Let \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \) and let \( T(x) = Ax \), so \( T : \mathbb{R}^2 \to \mathbb{R}^3 \).

- If \( u = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \) then \( T(u) = \)

- Let \( b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \). Find \( v \) in \( \mathbb{R}^2 \) such that \( T(v) = b \). Is there more than one?
Matrix Transformations
Example, continued

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \to \mathbb{R}^3$.

- Is there any $c$ in $\mathbb{R}^3$ such that there is more than one $v$ in $\mathbb{R}^2$ with $T(v) = c$?

  Translation: is there any $c$ in $\mathbb{R}^3$ such that the solution set of $Ax = c$ has more than one vector $v$ in it?

  The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it. So the solution set to $Ax = c$ has only one vector. So no!

- Find $c$ such that there is no $v$ with $T(v) = c$.

  Translation: Find $c$ such that $Ax = c$ is inconsistent.

  Translation: Find $c$ not in the column span of $A$ (i.e., the range of $T$).

  We could draw a picture, or notice that if $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then our matrix equation translates into

  $x + y = 1 \quad y = 2 \quad x + y = 3$,

  which is obviously inconsistent.
**Note:** All of these questions are questions about the transformation $T$; it still makes sense to ask them in the absence of the matrix $A$.

The fact that $T$ comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T: \mathbb{R}^2 \to \mathbb{R}^3 \quad T \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

**Question:** Is there any $c$ in $\mathbb{R}^3$ such that there is more than one $v$ in $\mathbb{R}^2$ with $T(v) = c$?

Note the question still makes sense, although $T$ has no hope of being a matrix transformation.
The picture of a matrix transformation is the same as the pictures we’ve been drawing all along. Only the language is different. Let

\[ A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

and let \( T(x) = Ax \),

so \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). Then

\[ T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \]

which is still is reflection over the \( y \)-axis. Picture:
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. ($T$ is called a shear.)
So, which transformations actually come from matrices?

Recall: If $A$ is a matrix, $u, v$ are vectors, and $c$ is a scalar, then

$$A(u + v) = Au + Av \quad A(cv) = cAv.$$ 

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Any matrix transformation has to satisfy this property. This property is so special that it has its own name.

**Definition**

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if it satisfies the above equations for all vectors $u, v$ in $\mathbb{R}^n$ and all scalars $c$.

In other words, $T$ “respects” addition and scalar multiplication.

**Check:** if $T$ is linear, then

$$T(0) = 0 \quad T(cu + dv) = cT(u) + dT(v)$$

for all vectors $u, v$ and scalars $c, d$. More generally,

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$

In engineering this is called **superposition**.
We can think of $b = Ax$ as a transformation with input $x$ and output $b$. This gives us a way to draw pictures of the geometry of a matrix.

There are lots of questions that one can ask about transformations.

We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.

**Linear transformations** are the transformations that come from matrices.