Math 1553 Worksheet §1.3
Solutions

If you don’t have a computer, find someone who does.

1. Let \( v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \quad w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}. \)

**Question:** Is \( w \) in \( \text{Span}\{v_1, v_2\} \)?
   a) Formulate this question as a vector equation.
   b) Formulate this question as a system of linear equations.
   c) Formulate this question as an augmented matrix.
   d) Answer the question using the interactive demo.
   e) Answer the question using row reduction.

**Solution.**
   a) Does the following vector equation have a solution?

   \[
   x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}
   \]

   b) Does the following linear system have a solution?

   \[
   \begin{align*}
   2x - 2y &= 2 \\
   x - 3y &= -4 \\
   3x - y &= 8
   \end{align*}
   \]

   c) As an augmented matrix:

   \[
   \begin{pmatrix}
   2 & -2 & | & 2 \\
   1 & -3 & | & -4 \\
   3 & -1 & | & 8
   \end{pmatrix}
   \]

   e) Row reducing yields

   \[
   \begin{pmatrix}
   1 & 0 & | & 7/2 \\
   0 & 1 & | & 5/2 \\
   0 & 0 & | & 0
   \end{pmatrix}
   \]

   so \( x = 7/2 \) and \( y = 5/2. \)
2. Consider the system of linear equations

\[
\begin{align*}
    x + 2y &= 7 \\
    2x + y &= -2 \\
    -x - y &= 4
\end{align*}
\]

**Question:** Does this system have a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a vector equation.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

**Solution.**

a) As an augmented matrix:

\[
\begin{pmatrix}
    1 & 2 & \vdots & 7 \\
    2 & 1 & \vdots & -2 \\
    -1 & -1 & \vdots & 4
\end{pmatrix}
\]

b) What are the solutions to the following vector equation?

\[
x \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}
\]

c) There exists a solution if and only if \( \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix} \) in the span of \( \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \) and \( \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \).

e) Row reducing yields

\[
\begin{pmatrix}
    1 & 0 & \vdots & 0 \\
    0 & 1 & \vdots & 0 \\
    0 & 0 & \vdots & 1
\end{pmatrix},
\]

so there are no solutions. (This should be obvious from the picture in (d)).
3. Consider the vector equation
\[
\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.
\]

**Question:** Is there a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

**Solution.**

a) As an augmented matrix:
\[
\begin{pmatrix}
2 & -2 & 3 & -5 \\
1 & -1 & 0 & -1 \\
3 & -1 & 4 & -2
\end{pmatrix}
\]

b) What is the solution set of the following linear system?
\[
\begin{align*}
2x - 2y + 3z &= -5 \\
x - y &= -1 \\
3x - y + 4z &= -2
\end{align*}
\]

c) There exists a solution if and only if \( \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix} \) is in \( \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \\ 4 \end{pmatrix} \right\} \).

d) Row reducing yields
\[
\begin{pmatrix}
1 & 0 & 0 & 3/2 \\
0 & 1 & 0 & 5/2 \\
0 & 0 & 1 & -1
\end{pmatrix},
\]
so \( x = 3/2, y = 5/2, \) and \( z = -1. \).
4. Consider the augmented matrix

\[
\begin{pmatrix}
2 & -2 & 2 & 0 \\
1 & -3 & -4 & -9 \\
3 & -1 & 8 & 9 \\
\end{pmatrix}
\]

**Question:** Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) What does this mean in terms of spans?

d) Answer the question using the interactive demo.

e) Answer the question using row reduction.

f) Find a different solution in parts (e) and (d).

**Solution.**

a) What are the solutions to the following vector equation?

\[
x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + z \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix}
\]

b) What is the solution set of the following linear system?

\[
\begin{align*}
2x - 2y + 2z &= 0 \\
x - 3y - 4z &= -9 \\
3x - y + 8z &= 9
\end{align*}
\]

c) There exists a solution if and only if \( \begin{pmatrix} 0 \\ -9 \\ 9 \end{pmatrix} \) is in \( \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix} \right\} \).

e) Row reducing yields

\[
\begin{pmatrix}
1 & 0 & 7/2 & 9/2 \\
0 & 1 & 5/2 & 9/2 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Hence \( z \) is a free variable, so the solution in parametric form is

\[
\begin{align*}
x &= \frac{9}{2} - \frac{7}{2} z \\
y &= \frac{9}{2} - \frac{5}{2} z.
\end{align*}
\]

Taking \( z = 0 \) yields the solution \( x = y = 9/2 \).

f) Taking \( z = 1 \) yields the solution \( x = 1, y = 2 \).
5. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$.

b) The span of any set contains the zero vector.

Solution.

a) This is false. For instance, the vectors

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}
\]

only span the $x$-axis.

b) This is true. We have

\[0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.\]

Aside: the span of the empty set is equal to $\{0\}$, because $0$ is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector $v$, you get $v + (\text{no other summands})$, which is just $v$; and the only vector which gives you $v$ when you add it to $v$, is $0$. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)