Announcements
Wednesday, August 30

- WeBWorK due on **Friday** at 11:59pm.
- The first quiz is on Friday, during recitation. It covers **through Monday’s material**.
  - Quizzes mostly test your understanding of the homework.
  - Quizzes last 10 minutes. Books, calculators, etc. are not allowed.
  - There will generally be a quiz every Friday when there’s no midterm.
  - Check the schedule if you want to know what will be covered.

- My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.

- Your TAs have office hours too. You can go to any of them. Details on the website.

- Many other resources are also contained in the “Help” tab of the master website. This includes Math Lab (not to be confused with MyMathLab), a free one-on-many tutoring service, open for many hours most days, provided by the School of Math.
Theorem
Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We’ll give an algorithm, called row reduction, which demonstrates that every matrix is row equivalent to at least one matrix in reduced row echelon form.

Note: Like echelon forms, the row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

The uniqueness statement is interesting—it means that, no matter how you row reduce, you always get the same matrix in reduced row echelon form. (Assuming you only do the three legal row operations.) (And you don’t make any arithmetic errors.)

Maybe you can figure out why it’s true!
Row Reduction Algorithm

Step 1a  Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).
Step 1b  Scale 1st row so that its leading entry is equal to 1.
Step 1c  Use row replacement so all entries below this 1 are 0.
Step 2a  Swap the 2nd row with a lower one so that the leftmost nonzero entry is in 2nd row.
Step 2b  Scale 2nd row so that its leading entry is equal to 1.
Step 2c  Use row replacement so all entries below this 1 are 0.
Step 3a  Swap the 3rd row with a lower one so that the leftmost nonzero entry is in 3rd row.

etc.

Last Step  Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Example

\[
\begin{pmatrix}
0 & -7 & -4 & | & 2 \\
2 & 4 & 6 & | & 12 \\
3 & 1 & -1 & | & -2 \\
\end{pmatrix}
\]
Row Reduction

Example

\[
\begin{pmatrix}
0 & -7 & -4 & 2 \\
2 & 4 & 6 & 12 \\
3 & 1 & -1 & -2 \\
\end{pmatrix}
\]
Row Reduction
Example, continued

\[
\begin{pmatrix}
1 & 2 & 3 & 6 \\
0 & -5 & -10 & -20 \\
0 & -7 & -4 & 2
\end{pmatrix}
\]

Step 2a: This is already nonzero.

Step 2b: Scale to make this 1.

(There are no fractions because of the optional step before.)

\[R_2 = R_2 \div -5\]

\[
\begin{pmatrix}
1 & 2 & 3 & 6 \\
0 & -5 & -10 & -20 \\
0 & -7 & -4 & 2
\end{pmatrix}
\]

Step 2c: Add 7 times the second row to clear this.

\[R_3 = R_3 + 7R_2\]

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

\[
\begin{pmatrix}
1 & * & * & * \\
0 & * & * & * \\
0 & * & * & * \\
\end{pmatrix}
\]

“Active” row
Row Reduction
Example, continued

\[
\begin{pmatrix}
1 & 2 & 3 & 6 \\
0 & 1 & 2 & 4 \\
0 & 0 & 10 & 30 \\
\end{pmatrix}
\]

Step 3a: This is already nonzero.

Step 3b: Scale to make this 1.

\[ R_3 = R_3 \div 10 \]

\[
\begin{pmatrix}
1 & 2 & 3 & 6 \\
0 & 1 & 2 & 4 \\
0 & 0 & 1 & 3 \\
\end{pmatrix}
\]

Note: Step 3 never messes up the columns to the left.

Note: The matrix is now in row echelon form!

Last step: Add multiples of the third row to clear these.

\[ R_2 = R_2 - 2R_3 \]

\[ R_1 = R_1 - 3R_3 \]

Last step: Add \(-2\) times the third row to clear this.

\[ R_1 = R_1 - 2R_2 \]
Success! The reduced row echelon form is
\[
\begin{pmatrix}
1 & 0 & 0 & | & 1 \\
0 & 1 & 0 & | & -2 \\
0 & 0 & 1 & | & 3 \\
\end{pmatrix}
\]
\[
\implies
\begin{cases}
x &= 1 \\
y &= -2 \\
z &= 3
\end{cases}
\]
Recap

<table>
<thead>
<tr>
<th>Get a 1 here</th>
<th>Clear down</th>
<th>Get a 1 here</th>
<th>Clear down</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\star & \star & \star & \star \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & \star & \star & \star \\
0 & \star & \star & \star \\
0 & \star & \star & \star \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & \star & \star & \star \\
0 & \star & \star & \star \\
\end{pmatrix}
\] |

(maybe these are already zero)

<table>
<thead>
<tr>
<th>Get a 1 here</th>
<th>Clear down</th>
<th>Matrix is in REF</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & \star \\
0 & 0 & 0 & \star \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & \star \\
0 & 0 & 0 & \star \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] |

<table>
<thead>
<tr>
<th>Clear up</th>
<th>Clear up</th>
<th>Matrix is in RREF</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
1 & \star & \star & \star \\
0 & 1 & \star & \star \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & \star & \star & 0 \\
0 & 1 & \star & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & \star & 0 \\
0 & 1 & \star & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\] |

Profit?
The linear system
\[
2x + 10y = -1
3x + 15y = 2
\]
gives rise to the matrix
\[
\begin{pmatrix}
2 & 10 & -1 \\
3 & 15 & 2
\end{pmatrix}
\]
Let’s row reduce it: [interactive row reducer]

The row reduced matrix
\[
\begin{pmatrix}
1 & 5 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
corresponds to the *inconsistent* system
\[
x + 5y = 0
0 = 1.
\]
Inconsistent Matrices

**Question**
What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

```latex
\begin{bmatrix}
1 & 0 & \star \\
\star & \star & \star \\
0 & 1 & \star \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
```

An augmented matrix corresponds to an inconsistent system of equations if and only if the last (i.e., the augmented) column is a pivot column.
Another Example

The linear system
\[
\begin{align*}
2x + y + 12z &= 1 \\
-x + 2y + 9z &= -1
\end{align*}
\]
gives rise to the matrix
\[
\begin{pmatrix}
2 & 1 & 12 \\
1 & 2 & 9 \\
-1 & & 
\end{pmatrix}
\]
Let’s row reduce it: [interactive row reducer]

\[
\begin{pmatrix}
2 & 1 & 12 \\
1 & 2 & 9 \\
-1 & & 
\end{pmatrix}
\]

The row reduced matrix
\[
\begin{pmatrix}
1 & 0 & 5 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
corresponds to the linear system
\[
\begin{align*}
x + 5z &= 1 \\
y + 2z &= -1
\end{align*}
\]
The system
\[
\begin{align*}
x + 5z &= 1 \\
y + 2z &= -1
\end{align*}
\]
comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:
\[
\begin{align*}
x &= 1 - 5z \\
y &= -1 - 2z
\end{align*}
\]

For any value of \( z \), there is exactly one value of \( x \) and \( y \) that makes the equations true. But \( z \) can be anything we want!

So we have found the solution set: it is all values \( x, y, z \) where
\[
\begin{align*}
x &= 1 - 5z \\
y &= -1 - 2z \quad \text{for } z \text{ any real number.} \\
(z &= \quad z)
\end{align*}
\]

This is called the **parametric form** for the solution. [interactive picture]
Free Variables

Definition
Consider a consistent linear system of equations in the variables $x_1, \ldots, x_n$. Let $A$ be a row echelon form of the matrix for this system.

We say that $x_i$ is a free variable if its corresponding column in $A$ is not a pivot column.

Important

1. You can choose any value for the free variables in a (consistent) linear system.
2. Free variables come from columns without pivots in a matrix in row echelon form.

In the previous example, $z$ was free because the reduced row echelon form matrix was

\[
\begin{pmatrix}
1 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\
\end{pmatrix} \quad \begin{pmatrix}
4 \\ -1 \\
\end{pmatrix}.
\]

In this matrix:

\[
\begin{pmatrix}
1 & \star & 0 & \star \\ 0 & 0 & 1 & \star \\
\end{pmatrix}
\]

the free variables are $x_2$ and $x_4$. (What about the last column?)
One More Example

The reduced row echelon form of the matrix for a linear system in $x_1, x_2, x_3, x_4$ is

$$
\begin{pmatrix}
1 & 0 & 0 & 3 & | & 2 \\
0 & 0 & 1 & 4 & | & -1
\end{pmatrix}
$$

The free variables are $x_2$ and $x_4$: they are the ones whose columns are not pivot columns.

This translates into the system of equations

$$
\begin{align*}
x_1 + 3x_4 &= 2 \\
x_3 + 4x_4 &= -1
\end{align*}
\implies
\begin{align*}
x_1 &= 2 - 3x_4 \\
x_3 &= -1 - 4x_4
\end{align*}
$$

What happened to $x_2$? What is it allowed to be? Anything! The general solution is

for any values of $x_2$ and $x_4$.

The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.
The linear system

\[ x + y + z = 1 \]

has matrix form

This is in reduced row echelon form. The free variables are \( y \) and \( z \). The parametric form of the general solution is

Rearranging:

\[ (x, y, z) = (1 - y - z, y, z), \]

where \( y \) and \( z \) are arbitrary real numbers. This was an example in the second lecture!
Poll

Is it possible for a system of linear equations to have exactly two solutions?
Summary

There are three possibilities for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is inconsistent. There are zero solutions, i.e. the solution set is empty. Picture:
   \[
   \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   \end{pmatrix}
   \]

2. Every column except the last column is a pivot column.
   In this case, the system has a unique solution. Picture:
   \[
   \begin{pmatrix}
   1 & 0 & 0 & ∗ \\
   0 & 1 & 0 & ∗ \\
   0 & 0 & 1 & ∗ \\
   \end{pmatrix}
   \]

3. The last column is not a pivot column, and some other column isn’t either.
   In this case, the system has infinitely many solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:
   \[
   \begin{pmatrix}
   1 & ∗ & 0 & ∗ & ∗ \\
   0 & 0 & 1 & ∗ & ∗ \\
   \end{pmatrix}
   \]