Please read all instructions carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!
Problem 1. [2 points each]

In this problem, if the statement is always true, circle T; otherwise, circle F.

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- a) If $A$ is row equivalent to $B$, then $A$ and $B$ have the same eigenvalues.
- b) If $A$ is similar to $B$, then $A$ and $B$ have the same characteristic polynomial.
- c) If $A$ is similar to $B$, then $A$ and $B$ have the same eigenvectors.
- d) If $A$ is diagonalizable, then $A$ has $n$ distinct eigenvalues.
- e) Every square matrix is diagonalizable if we allow complex eigenvalues and eigenvectors.
Problem 2.

In this problem, you need not explain your answers; just circle the correct one(s).

Let $A$ be an $n \times n$ matrix.

a) [3 points] Which one of the following statements is correct?
   1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.
   2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.
   3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.
   4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

b) [3 points] Which one of the following statements is not correct?
   1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.
   2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.
   3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.
   4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

c) [4 points] Which of the following $3 \times 3$ matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)
   1. A matrix with three distinct real eigenvalues.
   2. A matrix with one real eigenvalue.
   3. A matrix with a real eigenvalue $\lambda$ of algebraic multiplicity 2, such that the $\lambda$-eigenspace has dimension 2.
   4. A matrix with a real eigenvalue $\lambda$ such that the $\lambda$-eigenspace has dimension 2.
Problem 3.

Consider the matrix

\[
A = \begin{pmatrix}
-1 & -4 & 0 \\
1 & 3 & 0 \\
7 & 10 & 2
\end{pmatrix}.
\]

a) [4 points] Find the eigenvalues of \( A \), and compute their algebraic multiplicities.

b) [4 points] For each eigenvalue of \( A \), find a basis for the corresponding eigenspace.

c) [2 points] Is \( A \) diagonalizable? If so, find an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PD^1 \). If not, why not?
Problem 4.

Consider the matrix

\[ A = \begin{pmatrix}
3\sqrt{3} - 1 & -5\sqrt{3} \\
2\sqrt{3} & -3\sqrt{3} - 1
\end{pmatrix} \]

a) [2 points] Find both complex eigenvalues of \( A \).

b) [2 points] Find an eigenvector corresponding to each eigenvalue.

c) [3 points] Find an invertible matrix \( P \) and a rotation-scale matrix \( C \) such that \( A = PCP^{-1} \).

d) [1 point] By what factor does \( C \) scale?

e) [2 points] By what angle does \( C \) rotate?
Problem 5.

In any given year, 10% of city dwellers will move to the country, while 90% will stay in the city. Likewise, 30% of country dwellers will move to the city, while 70% will stay in the country.

a) [3 points] Let \( x_n \) be the number of people in the city in year \( n \), and let \( y_n \) be the number of people in the country in year \( n \). Find a matrix \( A \) such that

\[
A \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.
\]

b) [4 points] Compute the steady state of \( A \).

c) [3 points] If the region (city plus country) starts off with 1,000 residents, about how many people will live in the city 100 years later (assuming the total population stays constant)?
[Scratch work]