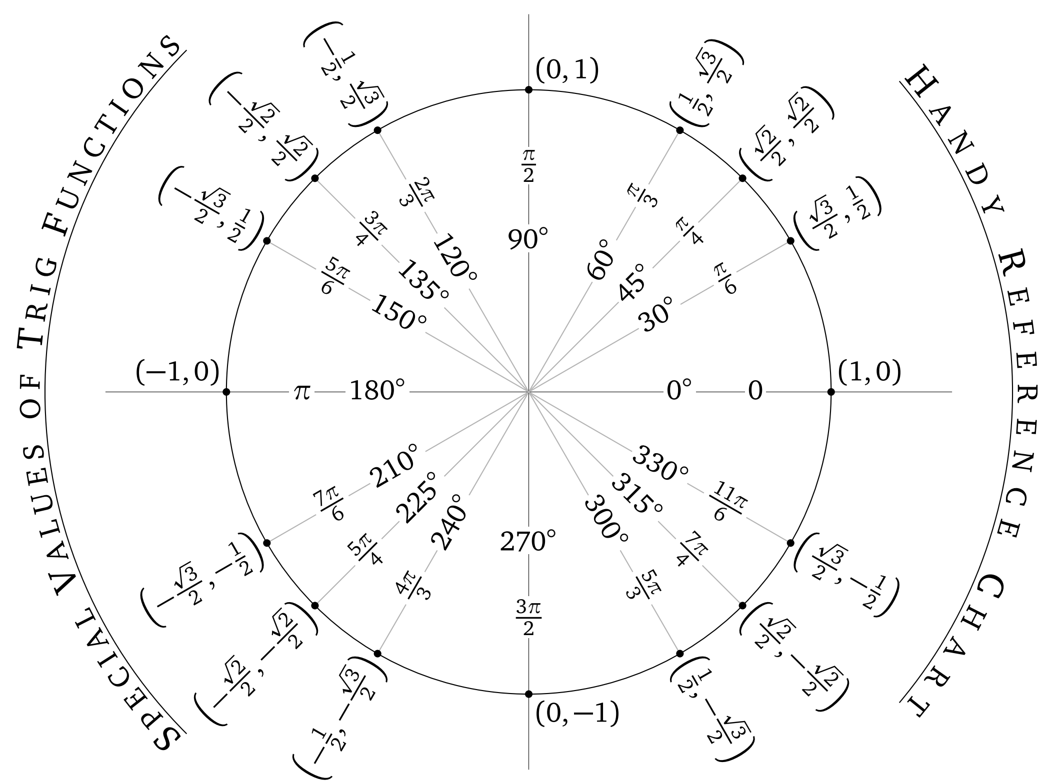
MATH 1553-B
MIDTERM EXAMINATION 3

Name		Section	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



Problem 1.

[2 points each]

In this problem, if the statement is always true, circle **T**; otherwise, circle **F**.

a) **T** **F** If A is row equivalent to B, then A and B have the same eigenvalues.

b) **T** **F** If A is similar to B, then A and B have the same characteristic polynomial.

c) **T** **F** If A is similar to B, then A and B have the same eigenvectors.

d) **T** **F** If A is diagonalizable, then A has n distinct eigenvalues.

e) **T** **F** Every square matrix is diagonalizable if we allow complex eigenvalues and eigenvectors.

Solution.

a) False: for instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) True.

c) False: for instance, if A is diagonalizable, then $A = PDP^{-1}$ for D diagonal. The unit coordinate vectors are eigenvectors of D, but the columns of P are eigenvectors of A.

d) False: for instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

e) False: for instance, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable even over the complex numbers because there is always only one linearly independent eigenvector, namely $\binom{0}{1}$.

Problem 2.

In this problem, you need not explain your answers; just circle the correct one(s).

Let A be an $n \times n$ matrix.

a) [3 points] Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) [3 points] Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

c) [4 points] Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

c) The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix A has a real eigenvalue λ_1 of algebraic multiplicity 2, then it has another real eigenvalue λ_2 of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

Problem 3.

Consider the matrix

$$A = \begin{pmatrix} -1 & -4 & 0 \\ 1 & 3 & 0 \\ 7 & 10 & 2 \end{pmatrix}.$$

a) [4 points] Find the eigenvalues of A, and compute their algebraic multiplicities.

b) [4 points] For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) [2 points] Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, why not?

Solution.

a) We compute the characteristic polynomial by expanding along the third column:

$$\begin{aligned} f(\lambda) &= \det \begin{pmatrix} -1-\lambda & -4 & 0 \\ 1 & 3-\lambda & 0 \\ 7 & 10 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)\big((-1-\lambda)(3-\lambda)+4\big) \\ &= (2-\lambda)(\lambda^2 - 2\lambda + 1) \\ &= (2-\lambda)(\lambda-1)^2 \end{aligned}$$

The roots are 1 (with multiplicity 2) and 2 (with multiplicity 1).

b) First we compute the 1-eigenspace by solving $(A-I)x = 0$:

$$A - I = \begin{pmatrix} -2 & -4 & 0 \\ 1 & 2 & 0 \\ 7 & 10 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{pmatrix}$$

The parametric vector form of the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \end{pmatrix}$, so a basis for the 0-eigenspace is $\left\{ \begin{pmatrix} -1/2 \\ 1/4 \\ 1 \end{pmatrix} \right\}$.

Next we compute the 2-eigenspace by eyeballing it. Clearly $Ae_3 = 2e_3$ because the third column of A is $2e_3$, so e_3 is an eigenvector with eigenvalue 2. This eigenvalue has algebraic multiplicity 1, so the 2-eigenspace has dimension 1, and therefore a basis for the 2-eigenspace is $\{e_3\}$.

c) We have shown that every eigenvector of A is a multiple of e_3 or $\begin{pmatrix} -1/2 \\ 1/4 \\ 1 \end{pmatrix}$. Hence A does not have 3 linearly independent eigenvectors, so it is not diagonalizable.

Problem 4.

Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) [2 points] Find both complex eigenvalues of A.

b) [2 points] Find an eigenvector corresponding to each eigenvalue.

c) [3 points] Find an invertible matrix P and a rotation-scale matrix C such that $A = PCP^{-1}$.

d) [1 point] By what factor does C scale?

e) [2 points] By what angle does C rotate?

Solution.

a) We compute the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 3\sqrt{3}-1-\lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1-\lambda \end{pmatrix} \\ &= (-1-\lambda+3\sqrt{3})(-1-\lambda-3\sqrt{3}) + (2)(5)(3) \\ &= (-1-\lambda)^2 - 9(3) + 10(3) \\ &= \lambda^2 + 2\lambda + 4. \end{aligned}$$

By the quadratic formula,

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(4)}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i.$$

b) Let $\lambda = -1-\sqrt{3}i$. Then

$$A - \lambda I = \begin{pmatrix} (i+3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i-3)\sqrt{3} \end{pmatrix}.$$

Since $\det(A-\lambda I) = 0$, the second row is a multiple of the first, so a row echelon form of A is

$$\begin{pmatrix} i+3 & -5 \\ 0 & 0 \end{pmatrix}.$$

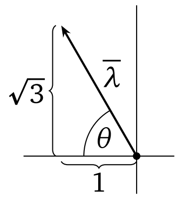
Hence an eigenvector with eigenvalue $-1-\sqrt{3}i$ is $v = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$. It follows that an eigenvector with eigenvalue $-1+\sqrt{3}i$ is $\overline{v} = \begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

c) Using the eigenvalue $\lambda = -1-\sqrt{3}i$ and eigenvector $v = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$, we can take

$$P = \begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix} = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

d) The scaling factor is $|\lambda| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2.$

e) We need to find the argument of $\overline{\lambda} = -1 + \sqrt{3}i$. We draw a picture:



$$\theta = \frac{\pi}{3} \text{ (trig identity)}$$

$$\text{argument} = \pi - \theta = \frac{2\pi}{3}$$

The matrix C rotates by $2\pi/3$.

Problem 5.

In any given year, 10% of city dwellers will move to the country, while 90% will stay in the city. Likewise, 30% of country dwellers will move to the city, while 70% will stay in the country.

a) [3 points] Let x_n be the number of people in the city in year n, and let y_n be the number of people in the country in year n. Find a matrix A such that

$$A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.$$

b) [4 points] Compute the steady state of A.

c) [3 points] If the region (city plus country) starts off with 1,000 residents, about how many people will live in the city 100 years later (assuming the total population stays constant)?

Solution.

a) If $A = \begin{pmatrix} .9 & .3 \\ .1 & .7 \end{pmatrix}$, then $A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} .9x_n + .3y_n \\ .1x_n + .7y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$.

b) First we find an eigenvector with eigenvalue 1:

$$A - I = \begin{pmatrix} -.1 & .3 \\ .1 & -.3 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}.$$

An eigenvector with eigenvalue 1 is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, so the steady state is $\frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$.

c) The population distribution equals approximately 1,000 times the steady state:

$$1000\begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 750 \\ 250 \end{pmatrix}.$$

[Scratch work]