MATH 1553-B
MIDTERM EXAMINATION 2

Name	

1	2	3	4	5	6	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 60 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1. [2 points each]

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** An invertible matrix is a product of elementary matrices.

b) **T** **F** There exists a 3×5 matrix (3 rows, 5 columns) with rank 4.

c) **T** **F** There exists a 3×5 matrix whose null space has dimension 4.

d) **T** **F** If the columns of an $n \times n$ matrix A span $\mathbf{R}^n$, then A is invertible.

e) **T** **F** The solution set of a consistent matrix equation $Ax = b$ is a subspace.

Solution.

a) True: if A is invertible, then it is row equivalent to the identity matrix I. This means there is a sequence of elementary matrices $E_1, E_2, \ldots, E_r$ such that

$$E_r E_{r-1} \cdots E_2 E_1 A = I \quad \text{and hence} \quad A = E_1^{-1} E_2^{-1} \cdots E_{r-1}^{-1} E_r^{-1}.$$

b) False: the rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

c) True: for instance, the null space of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has dimension 4.

d) True: this is one of the conditions of the Invertible Matrix Theorem.

e) False: this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

Problem 2.

[5 points each]

Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

a) Find A^{-1}.

b) Solve for x in $Ax = \binom{a}{b}$.

Solution.

a) We have $\det(A) = -2$. Using the general formula for the inverse of a 2×2 matrix, we have:

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}.$$

b) $x = A^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 4a - 3b \\ -2a + b \end{pmatrix}.$

Problem 3.

Let

$$A=\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

a) [3 points] Compute $\det(A)$.

b) [3 points] Compute $\det(B)$.

c) [2 points] Compute $\det(AB)$.

d) [2 points] Compute $\det(A^2B^{-1}AB^2)$.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) This is a complicated matrix without a lot of zeros, so we compute the determinant by row reduction. After one row swap and several row replacements, we reduce to the matrix

$$\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The determinant of this matrix is -21, so the determinant of the original matrix is 21.

c) $\det(AB) = \det(A)\det(B) = (-2)(21) = -42.$

d) $\det(A^2B^{-1}AB^2) = \det(A)^2\det(B)^{-1}\det(A)\det(B)^2 = \det(A)^3\det(B) = (-2)^3(21) = -168.$

Problem 4.

Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

a) [4 points] Find a basis $\mathcal{B}$ for $\mathrm{Nul}A$.

b) [2 points each] For each of the following vectors v, decide if v is in $\mathrm{Nul}A$, and if so, find $[x]_{\mathcal{B}}$:

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Solution.

a) We compute the parametric vector form for the general solution of $Ax = 0$:

$$\begin{cases} x_1 = -2x_2 + x_4 \\ x_2 = x_2 \\ x_3 = 2x_4 \\ x_4 = x_4 \end{cases} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

b) First we note that if

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

then $c_1 = b$ and $c_2 = d$. This makes it easy to check whether a vector is in $\mathrm{Nul}A$, and to compute the $\mathcal{B}$-coordinates.

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \neq 3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \text{not in } \mathrm{Nul}A.$$

$$\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \left[\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix}-1\\1\\2\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+\begin{pmatrix}1\\0\\2\\1\end{pmatrix}\implies\left[\begin{pmatrix}-1\\1\\2\\1\end{pmatrix}\right]_{\mathcal{B}}=\begin{pmatrix}1\\1\end{pmatrix}.$$

Problem 5.

Consider the matrix A and its reduced row echelon form from the previous problem:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

a) [4 points] Find a basis for $\operatorname{Col} A$.

b) [3 points] What are $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$?

c) [3 points] Find a different basis for $\operatorname{Col} A$. (Reordering your answer from (a) does not count.) Justify your answer.

Solution.

a) A basis for the column space is given by the pivot columns of A:

$$\left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \end{pmatrix} \right\}.$$

b) The rank is the dimension of the column space, which is 2. The rank plus the dimension of the null space equals the number of columns, so the null space has dimension 2 as well (as we know from the previous problem).

c) The column space is a 2-dimensional subspace of $\mathbf{R}^2$. Thus $\operatorname{Col} A = \mathbf{R}^2$, so any basis for $\mathbf{R}^2$ works. For example, we can use the standard basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Problem 6. [2 points each]

Which of the following are subspaces of $\mathbf{R}^4$ and **why?**

a) $\mathrm{Span}\left\{\begin{pmatrix}1\\0\\3\\2\end{pmatrix}, \begin{pmatrix}-2\\7\\9\\13\end{pmatrix}, \begin{pmatrix}144\\0\\0\\1\end{pmatrix}\right\}$

b) $\mathrm{Nul}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $\mathrm{Col}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

d) $V = \left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; xy = zw\right\}$

e) The range of a linear transformation with codomain $\mathbf{R}^4$.

Solution.

a) This is a subspace of $\mathbf{R}^4$: it is a span of three vectors in $\mathbf{R}^4$, and any span is a subspace.

b) This is not a subspace of $\mathbf{R}^4$; it is a subspace of $\mathbf{R}^3$.

c) This is a subspace of $\mathbf{R}^4$: it is the span of three vectors in $\mathbf{R}^4$.

d) This is not a subspace of $\mathbf{R}^4$: it is not closed under addition. For instance,

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \text{ are in } V\text{, but } \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} \text{ is not.}$$

e) This is a subspace of $\mathbf{R}^4$: it is the column space of the associated $4\times ?$ matrix, and any column space is a subspace.

[Scratch work]