MATH 1553-B
FINAL EXAMINATION

Name		Section	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work, unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

KEEP
CALM
AND
ACE YOUR
FINAL EXAMS

Problem 1.

[2 points each]

In this problem, you need not explain your answers.

a) The matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is in reduced row echelon form:

1. True 2. False

b) How many solutions does the linear system corresponding to the augmented matrix $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$ have?

1. Zero.
2. One.
3. Infinity.
4. Not enough information to determine.

c) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Which of the following are equivalent to the statement that T is one-to-one? (Circle all that apply.)

1. A has a pivot in each row.
2. The columns of A are linearly independent.
3. For all vectors v, w in $\mathbf{R}^n$, if $T(v) = T(w)$ then $v = w$.
4. A has n columns.
5. $\operatorname{Nul} A = \{0\}$.

d) Every square matrix has a (real or) complex eigenvalue.

1. True 2. False

e) Let A be an $n \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. Which of the following are equivalent to the statement that A is *not* invertible? (Circle all that apply.)

1. There exists an $n \times n$ matrix B such that $AB = 0$.
2. $\operatorname{rank} A = 0$.
3. $\det(A) = 0$.
4. $\operatorname{Nul} A = \{0\}$.
5. There exist $v \neq w$ in $\mathbf{R}^n$ such that $T(v) = T(w)$.

Solution.

a) 2 (false).

b) 1 (zero). The corresponding system is inconsistent.

c) 2, 3, 5. If A has a pivot in each row, then T is *onto*. Option 3 is the definition of one-to-one.

d) 1 (true). Its characteristic polynomial always has a complex root.

e) 3, 5. Option 1 is always true: take $B = 0$. Option 2 means $A = 0$. Option 5 means T is not one-to-one.

Problem 2.

[2 points each]

In this problem, you need not explain your answers.

a) Let A be an $m \times n$ matrix, and let b be a vector in $\mathbf{R}^m$. Which of the following are equivalent to the statement that $Ax = b$ is consistent? (Circle all that apply.)

1. b is in $\operatorname{Nul} A$.
2. b is in $\operatorname{Col} A$.
3. A has a pivot in every row.
4. The augmented matrix $\left(A \mid b\right)$ has no pivot in the last column.

b) Let $A = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 2 \end{pmatrix}$. For what values of a and b is A diagonalizable? (Circle all that apply.)

1. $a = 1, b = 1$ 2. $a = 2, b = 1$ 3. $a = 1, b = 2$ 4. $a = 0, b = 1$

c) Let W be the subset of $\mathbf{R}^2$ consisting of the x-axis and the y-axis. Which of the following are true? (Circle all that apply.)

1. W contains the zero vector.
2. If v is in W, then all scalar multiples of v are in W.
3. If v and w are in W, then $v + w$ is in W.
4. W is a subspace of $\mathbf{R}^2$.

d) Every subspace of $\mathbf{R}^n$ admits an orthogonal basis:

1. True 2. False

e) Let x and y be nonzero orthogonal vectors in $\mathbf{R}^n$. Which of the following are true? (Circle all that apply.)

1. $x \cdot y = 0$
2. $\|x - y\|^2 = \|x\|^2 + \|y\|^2$
3. $\operatorname{proj}_{\operatorname{Span}\{x\}}(y) = 0$
4. $\operatorname{proj}_{\operatorname{Span}\{y\}}(x) = 0$

Solution.

a) 2, 4. Option 3 means $Ax = b$ is consistent for *every* b.

b) 3, 4.

c) 1, 2. Note that e_1 and e_2 are in W, but $e_1 + e_2$ is not.

d) 1 (true). Take any basis, and apply Gram–Schmidt.

e) 1, 2, 3, 4.

Problem 3.

[2 points each]

Short answer questions: you need not explain your answers.

a) Let A be an $n \times n$ matrix. Write the definition of an eigenvector and an eigenvalue of A.

b) Let A be a positive stochastic matrix with steady state $w = \binom{1/3}{2/3}$. What does $A^n\binom{9}{0}$ converge to as $n \to \infty$?

c) Give an example of a 2×2 matrix that has no (real) eigenvectors.

d) Let W be the span of $(1,1,1,1)$ in $\mathbf{R}^4$. Find a matrix whose null space is $W^\perp$.

e) Let A be a 3×3 matrix that is obtained from the identity matrix by the following row operations: (1) swap rows 1 and 2 (2) add $2\times$row 3 to row 1 (3) multiply row 1 by 3. Write A as a product of elementary matrices. (You need not compute the product.)

Solution.

a) An eigenvector of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. An eigenvalue of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

b) $9w = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

c) The matrix for rotation by any angle that is not a multiple of 180 degrees works. For instance,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

d) $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

e) $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 4.

[5 points each]

Let

$$A = \begin{pmatrix} -5 & 1 & -1 \\ -6 & 5 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

a) Compute A^{-1} and $\det(A)$.

b) Solve for x in terms of the variables b_1, b_2, b_3:

$$Ax = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Solution.

a) One way to invert A is to row reduce the augmented matrix $\left(A \mid I\right)$:

$$\left(\begin{array}{ccc|ccc} -5 & 1 & -1 & 1 & 0 & 0 \\ -6 & 5 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 3 & -\frac{5}{2} & \frac{21}{2} \\ 0 & 0 & 1 & -3 & \frac{5}{2} & -\frac{19}{2} \end{array}\right).$$

Hence

$$A^{-1} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & -\frac{5}{2} & \frac{21}{2} \\ -3 & \frac{5}{2} & -\frac{19}{2} \end{pmatrix}.$$

One can simultaneously compute $\det(A)$ by keeping track of the row swaps and the row scaling; the answer is $\det(A) = 2$.

b)

$$x = A^{-1}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & -\frac{5}{2} & \frac{21}{2} \\ -3 & \frac{5}{2} & -\frac{19}{2} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 - b_2 + 4b_3 \\ 3b_1 - \frac{5}{2}b_2 + \frac{21}{2}b_3 \\ -3b_1 + \frac{5}{2}b_2 - \frac{19}{2}b_3 \end{pmatrix}$$

Problem 5.

Consider the matrix
$$A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{pmatrix}.$$

a) [4 points] Find an orthogonal basis for Col$A$.

b) [4 points] Find a QR factorization of A.

c) [2 points] Find a different orthogonal basis for Col$A$. (Reordering and scaling your basis in (a) does not count.)

Solution.

a) Let
$$v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$$
be the columns of A. We will perform Gram–Schmidt on $\{v_1, v_2, v_3\}$. Let
$$\begin{aligned} u_1 & & &= v_1 & &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\ u_2 &= v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 & &= v_2 - 2u_1 & &= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 & &= v_3 - u_1 + u_2 & &= \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}. \end{aligned}$$
Then $\{u_1, u_2, u_3\}$ is an orthogonal basis for Col$A$.

b) First we solve for v_1, v_2, v_3 in terms of u_1, u_2, u_3:
$$v_1 = 1u_1 \qquad v_2 = 2u_1 + 1u_2 \qquad v_3 = 1u_1 - 1u_2 + 1u_3.$$
Therefore,
$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$
Dividing the columns of the first matrix by the length of the column, and multiplying the rows of the second by the same factor, gives the QR factorization
$$A = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \end{pmatrix} \begin{pmatrix} \sqrt{5} & 2\sqrt{5} & \sqrt{5} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{30} \end{pmatrix}.$$

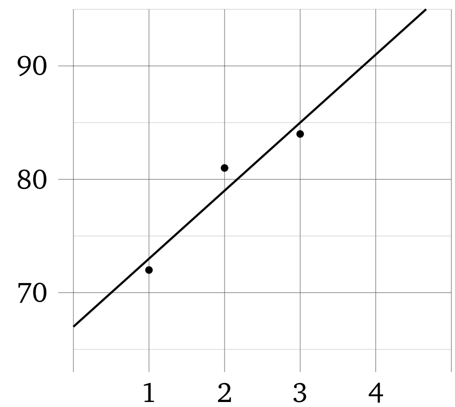
c) The columns of A are linearly independent (otherwise Gram–Schmidt would have produced the zero vector), so $\mathrm{Col} A = \mathbf{R}^3$, and hence $\{e_1, e_2, e_3\}$ is an orthogonal basis.

Problem 6.

Suppose that your roomate Jamie is currently taking Math 1551. Jamie scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Jamie does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) [4 points] The general equation of a line in $\mathbf{R}^2$ is $y = C + Dx$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(1,72)$, $(2,81)$, and $(3,84)$, and then write down the corresponding matrix equation.

b) [4 points] Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below.



$y = 67 + 6x$

c) [2 points] What score does this line predict for the fourth (final) exam?

Solution.

a) If $y = C + Dx$ were satisfied by all three points, then we would have

$$\begin{aligned} 72 &= C + D(1) \\ 81 &= C + D(2) \\ 84 &= C + D(3) \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix}.$$

b) The least squares problem is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 237 \\ 486 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 6 & 237 \\ 6 & 14 & 486 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 67 \\ 0 & 1 & 6 \end{array}\right).$$

Hence $C = 67$ and $D = 6$.

c) $67 + 6(4) = 91\%$

Problem 7.

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace $W = \mathrm{Span}\{v_1, v_2, v_3, v_4\}$.

a) [2 points] Find a linear dependence relation among v_1, v_2, v_3, v_4.

b) [3 points] What is the dimension of W?

c) [3 points] Which subsets of $\{v_1, v_2, v_3, v_4\}$ form a basis for W?

d) [2 points] Choose a basis $\mathcal{B}$ for W from (c), and find the $\mathcal{B}$-coordinates of the vector $w = (0, 0, 4, 0)$.

[*Hint:* it is helpful, but not necessary, to use the fact that $\{v_1, v_2, v_3\}$ is orthogonal.]

Solution.

a) We know that $\{v_1, v_2, v_3\}$ is linearly independent, because it is an orthogonal set. Hence v_4 must be a linear combination of v_1, v_2, v_3, i.e., v_4 is in $\mathrm{Span}\{v_1, v_2, v_3\}$. We can compute the coordinates using dot products:

$$v_4 = \frac{v_4 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v_4 \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{v_4 \cdot v_3}{v_3 \cdot v_3} v_3 = v_1 + v_2 + v_3.$$

Hence $v_1 + v_2 + v_3 - v_4 = 0$ is a linear dependence relation.

b) We know that $\{v_1, v_2, v_3\}$ is linearly independent, and it spans W because v_4 is in $\mathrm{Span}\{v_1, v_2, v_3\}$. Thus $\dim(W) = 3$.

c) Any set of three vectors from $\{v_1, v_2, v_3, v_4\}$ spans W, because the fourth is a linear combination of the other three (from $v_1 + v_2 + v_3 - v_4 = 0$). Hence any three vectors in $\{v_1, v_2, v_3, v_4\}$ is a basis for W.

d) We choose the basis $\mathcal{B} = \{v_1, v_2, v_3\}$. Then

$$[w]_{\mathcal{B}} = \left(\frac{w \cdot v_1}{v_1 \cdot v_1}, \frac{w \cdot v_2}{v_2 \cdot v_2}, \frac{w \cdot v_3}{v_3 \cdot v_3} \right) = (1,\, 1,\, -1).$$

Alternatively, you can ignore the fact that $\{v_1, v_2, v_3\}$ is orthogonal and use row reduction in (a), (b), and (d), but this requires more work.

Problem 8.

Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ -1 & -3 & -4 & 2 \\ 5 & 15 & 1 & 9 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 14 \end{pmatrix}.$$

a) [3 points] Find the parametric vector form of the solution set of $Ax = b$.

b) [2 points] Find a basis for $\mathrm{Nul}A$.

c) [2 points] What are $\dim(\mathrm{Nul}A)$ and $\dim((\mathrm{Nul}A)^\perp)$?

d) [3 points] Find a basis for $(\mathrm{Nul}A)^\perp$.

Solution.

a) Row reducing the augmented matrix $\begin{pmatrix} A & b \end{pmatrix}$ yields

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_2 and x_4 are free. The parametric form and the parametric vector form of the general solution are

$$\begin{array}{rcl} x_1 + 3x_2 \qquad + 2x_4 &=& 3 \\ x_2 &=& x_2 \\ x_3 - \ x_4 &=& -1 \\ x_4 &=& x_4 \end{array} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

b) We can read off the null space from the parametric vector form; a basis is

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

c) From (b) we see $\dim(\mathrm{Nul}A) = 2$. Since $\mathrm{Nul}A$ is a subspace of $\mathbf{R}^4$, we have

$$\dim((\mathrm{Nul}A)^\perp) = 4 - \dim(\mathrm{Nul}A) = 2.$$

d) Recall that $(\mathrm{Nul}A)^\perp = \mathrm{Row}A$. The first two rows of A are not multiples of each other, so they are linearly independent. We know already that $\dim((\mathrm{Nul}A)^\perp) = 2$, so the first two rows of A form a basis:

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -4 \\ 2 \end{pmatrix} \right\}.$$

Problem 9.

Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -10 & 7 \end{pmatrix}.$$

a) [2 points] Compute the characteristic polynomial of A.

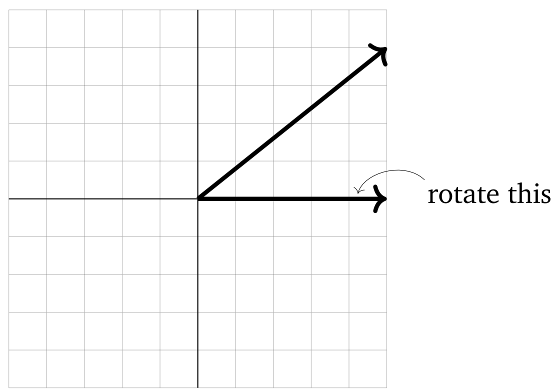
b) [2 points] The complex number $\lambda = 5 - 4i$ is an eigenvalue of A. What is the other eigenvalue? Produce eigenvectors for both eigenvalues.

c) [3 points] Find an invertible matrix P and a rotation-scaling matrix C such that

$$A = PCP^{-1}.$$

d) [1 point] By what factor does C scale?

e) [2 points] What ray does C rotate the positive x-axis onto? Draw it below.



Solution.

a) $f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 10\lambda + 41.$

b) The other eigenvalue is $\overline{\lambda} = 5 + 4i$.

$$A - \lambda I = \begin{pmatrix} -2+4i & 2 \\ \star & \star \end{pmatrix} \xrightarrow{\text{eigenvector}} v = \begin{pmatrix} 2 \\ 2-4i \end{pmatrix}.$$

Hence an eigenvector for $\overline{\lambda}$ is $\overline{v} = \left(\begin{smallmatrix} 2 \\ 2+4i \end{smallmatrix}\right)$.

c) We can take

$$P = \begin{pmatrix} \mathrm{Re}\, v & \mathrm{Im}\, v \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 4 & 5 \end{pmatrix}.$$

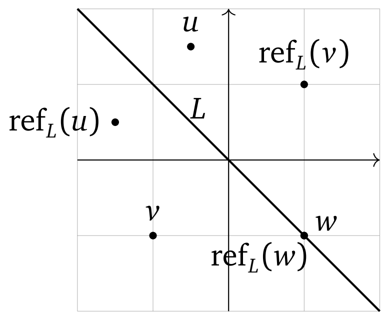
d) C scales by $|\lambda| = \sqrt{5^2 + 4^2} = \sqrt{41}$.

Problem 10.

Let L be a line through the origin in $\mathbf{R}^2$. The **reflection over** L is the linear transformation $\mathrm{ref}_L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\mathrm{ref}_L(x) = x - 2x_{L^\perp} = 2\,\mathrm{proj}_L(x) - x.$$

a) [3 points] Draw (and label) $\mathrm{ref}_L(u)$, $\mathrm{ref}_L(v)$, and $\mathrm{ref}_L(w)$ in the picture below. [*Hint:* think geometrically]



In what follows, L does not necessarily refer to the line pictured above.

b) [2 points] If A is the matrix for ref_L, what is A^2?

c) [3 points] What are the eigenvalues and eigenspaces of A?

d) [2 points] Is A diagonalizable? If so, what diagonal matrix is it similar to?

Solution.

b) Reflecting over L twice brings you back to where you started. Hence $\mathrm{ref}_L \circ \mathrm{ref}_L$ is the identity transformation, so $A^2 = I$.

Alternatively, since $\mathrm{ref}_L(x) = 2\,\mathrm{proj}_L(x) - x$, the matrix for ref_L is $2B - I$, where B is the matrix for proj_L. Hence

$$A^2 = (2B - I)^2 = 4B^2 - 4B + I = 4B - 4B + I = I.$$

c) Anything in L is fixed by ref_L, so 1 is an eigenvalue, and L is the 1-eigenspace. If x is in $L^\perp$ then $\mathrm{ref}_L(x) = -x$, so -1 is an eigenvalue, and $L^\perp$ is the (-1)-eigenspace. There cannot be any more eigenvalues or eigenvectors.

d) Yes: it is similar to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

[Scratch work]

[Scratch work]