Announcements
November 7

▶ Go vote tomorrow.

▶ WeBWorK assignment 5.3 is due on Friday at 6am.

▶ Read the Piazza post about the China Summer Program.

▶ Midterm 3 will take place in recitation on Friday, 11/18.

▶ Office hours: Wednesday 1-2pm, Thursday 3:30-4:30pm, and by appointment, in Skiles 221.

   ▶ As always, TAs’ office hours are posted on the website.
   ▶ Math Lab is also a good place to visit.
Section 5.5

Complex Eigenvalues
In recitation you discussed the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

This matrix has no eigenvectors, as you can see geometrically:

or algebraically:

$$f(\lambda) = \det \left( \frac{1}{\sqrt{2}} - \lambda \quad -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} - \lambda \right) = \lambda^2 - \sqrt{2}\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$
Complex Numbers

It makes us sad that $-1$ has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician’s solution: we’re just not using enough numbers! We’re going to declare by *fiat* that there exists a square root of $-1$.

**Definition**
The number $i$ is defined such that $i^2 = -1$.

Once we have $i$, we have to allow numbers like $a + bi$ for real numbers $a, b$.

**Definition**
A *complex number* is a number of the form $a + bi$ for $a, b$ in $\mathbb{R}$. The set of all complex numbers is denoted $\mathbb{C}$.

We can identify $\mathbb{C}$ with $\mathbb{R}^2$ by $a + bi \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$. So when we draw a picture of $\mathbb{C}$, we draw the plane:

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![Diagram of complex plane](attachment:complex_plane.png)
In the beginning, people only used counting numbers for, well, counting things: 1, 2, 3, 4, 5, ... Then someone had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone’s mind.

Then it occurred to someone that there should be negative numbers to represent a deficit in quantity. That seemed reasonable, until people realized that $10 + (-3)$ would have to equal 7. This is when people started saying, “bah, math is just too hard for me.”

At this point it was inconvenient that you couldn’t divide 2 by 3. Thus fractions (rational numbers) were invented to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 - 2 = 0$.

So what’s so strange about inventing a number $i$ to solve the equation $x^2 + 1 = 0$?
Operations on Complex Numbers

Addition: \((2 - 3i) + (-1 + i) = 1 - 2i\).

Multiplication: \((2 - 3i)(-1 + i) = 2(-1) + 2i + 3i - 3i^2 = -2 + 5i + 3 = 1 + 5i\).

Complex conjugation: \(\overline{a + bi} = a - bi\) is the complex conjugate of \(a + bi\).
Check: \(\overline{z + w} = \overline{z} + \overline{w}\) and \(\overline{zw} = \overline{z} \cdot \overline{w}\).

Absolute value: \(|a + bi| = \sqrt{a^2 + b^2}\). This is a real number.
Note: \((a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2\). So \(|z| = \sqrt{z\overline{z}}\).
Check: \(|zw| = |z| \cdot |w|\).

Division by a nonzero real number: \(\frac{a + bi}{c} = \frac{a}{c} + \frac{b}{c}i\).

Division by a nonzero complex number: \(\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}\).

Example:
\[
\frac{1 + i}{1 - i} = \frac{(1 + i)^2}{1^2 + (-1)^2} = \frac{1 + 2i + i^2}{2} = \frac{1 + 2i - 1}{2} = i.
\]

Real and imaginary part: \(\text{Re}(a + bi) = a \quad \text{Im}(a + bi) = b\).
Any complex number $z = a + bi$ has the form

$$z = |z|(\cos \theta + i \sin \theta).$$

The angle $\theta$ is called the **argument** of $z$. Note the argument of $\bar{z}$ is negative the argument of $z$.

When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|z|(\cos \theta + i \sin \theta) \cdot |w|(\cos \varphi + i \sin \varphi) = |z||w|(\cos(\theta + \varphi) + i \sin(\theta + \varphi)).$$
The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

**Fundamental Theorem of Algebra**

Every polynomial of degree $n$ has exactly $n$ complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is a polynomial of degree $n$, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

**Important**

If $f$ is a polynomial with *real* coefficients, and if $\lambda$ is a root of $f$, then so is $\bar{\lambda}$:

$$0 = f(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

$$= \bar{\lambda}^n + a_{n-1}\bar{\lambda}^{n-1} + \cdots + a_1\bar{\lambda} + a_0 = f(\bar{\lambda}).$$

Therefore complex roots of real polynomials come in *conjugate pairs*.
Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

\[ f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}. \]

For instance, if \( f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1 \) then

\[ \lambda = \frac{\sqrt{2} \pm \sqrt{2}}{2} = \frac{\sqrt{2}}{2} (1 \pm i) = \frac{1 \pm i}{\sqrt{2}}. \]

Note the roots are complex conjugates if \( b, c \) are real.
Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:

\[ f(\lambda) = (\lambda - 2) \left( 5\lambda^2 - 8\lambda + 5 \right) \]

respectively.

For instance, if \( f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10 \), then 
\[ f(\lambda) = (\lambda - 2) \left( 5\lambda^2 - 8\lambda + 5 \right) \]
since \( f(2) = 0 \). Using the quadratic formula, the second polynomial has a root when

\[ \lambda = \frac{8 \pm \sqrt{64 - 100}}{10} = \frac{4 \pm \sqrt{-36}}{5} = \frac{4 \pm 3i}{5}. \]

Therefore,

\[ f(\lambda) = 5(\lambda - 2) \left( \lambda - \frac{4 + 3i}{5} \right) \left( \lambda - \frac{4 - 3i}{5} \right). \]
The characteristic polynomial of

\[ A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

is \( f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1 \). This has two complex roots.

Does \( A \) have any eigenvectors? If so, what are they?
Every matrix is guaranteed to have complex eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

has eigenvalues $\lambda = \frac{1 \pm i}{\sqrt{2}}$.

Let’s compute an eigenvector for $\lambda = (1 + i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - (1 + i) & -1 \\ 1 & 1 - (1 + i) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is $i$ times the first, so we row reduce:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1 - i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. 

A Matrix with an Eigenvector
For $A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, 

the eigenvalue $\frac{1 + i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

the eigenvalue $\frac{1 - i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Do you notice a pattern?

**Fact**
Let $A$ be a real square matrix. If $\lambda$ is an eigenvalue with eigenvector $\nu$, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{\nu}$.

**Why?**

$$A\nu = \lambda \implies A\overline{\nu} = \overline{A\nu} = \overline{\lambda\nu} = \overline{\lambda}\overline{\nu}.$$
A 3 × 3 Example

Find the eigenvalues and eigenvectors of

\[
A = \begin{pmatrix}
\frac{4}{5} & -\frac{3}{5} & 0 \\
\frac{3}{5} & \frac{4}{5} & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

The characteristic polynomial is

\[
f(\lambda) = \det \left( \begin{pmatrix}
\frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\
\frac{3}{5} & \frac{4}{5} - \lambda & 0 \\
0 & 0 & 2 - \lambda
\end{pmatrix} \right) = (2 - \lambda) \left( \lambda^2 - \frac{8}{5} \lambda + 1 \right).
\]

We computed the roots of this polynomial (times 5) before:

\[
\lambda = 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.
\]

We eyeball an eigenvector with eigenvalue 2 as \((0, 0, 1)\).
To find the other eigenvectors, we row reduce:

\[
A - \frac{4 + 3i}{5} I = \begin{pmatrix}
-\frac{3}{5}i & -\frac{3}{5}i & 0 \\
-\frac{3}{5}i & -\frac{3}{5}i & 0 \\
0 & 0 & 2 - \frac{4 + 3i}{5}
\end{pmatrix}
\]

scale rows

\[
\begin{pmatrix}
-i & -1 & 0 \\
1 & -i & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The second row is \(i\) times the first:

row replacement

\[
\begin{pmatrix}
i & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

divide by \(-i\), swap

\[
\begin{pmatrix}
1 & -i & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

The parametric form is \(x = iy, z = 0\), so an eigenvector is \(\begin{pmatrix}i \\ 1 \\ 0 \end{pmatrix}\). Therefore, an eigenvector with conjugate eigenvalue \(\frac{4 - 3i}{5}\) is \(\begin{pmatrix}-i \\ 1 \\ 0 \end{pmatrix}\).
Theorem
Let $A$ be a $2 \times 2$ matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let $v$ be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} \text{Re } v & \text{Im } v \\ \text{Re } v & \text{Im } v \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$ 

If $a + bi = r(\cos \theta + i \sin \theta)$ then

$$C = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

is a composition of rotation by $-\theta$ and scaling by $r$.

A $2 \times 2$ matrix with complex eigenvalue $\lambda$ is similar to (rotation by the argument of $\lambda$) composed with (scaling by $|\lambda|$). This is multiplication by $\lambda$ in $\mathbb{C} \sim \mathbb{R}^2$. 