The first midterm is **this Friday**, during recitation. It covers Chapter 1, sections 1–5 and 7–9.

Homework 1.9 is due Friday at 6am.

Homework 2.1 is now due **Monday at 6am**.

Solutions to the practice midterm have been posted on the website.

Other midterm details and study tips have been posted to Piazza.

**Extra office hours:** today, 1–3pm; tomorrow, 2:30–4:30pm.

As always, TAs’ office hours are posted on the website.

Math Lab is also a good place to visit.
Review for Midterm 1

Selected Topics
We have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:
   
   \[ 2x_1 + 3x_2 = 7 \]
   \[ x_1 - x_2 = 5 \]

2. As an augmented matrix:
   
   \[
   \begin{pmatrix}
   2 & 3 & | & 7 \\
   1 & -1 & | & 5
   \end{pmatrix}
   \]

3. As a vector equation \((x_1 v_1 + \cdots + x_n v_n = b)\):
   
   \[ x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \]

4. As a matrix equation \((Ax = b)\):
   
   \[
   \begin{pmatrix}
   2 & 3 \\
   1 & -1
   \end{pmatrix}
   \begin{pmatrix}
   x_1 \\
   x_2
   \end{pmatrix}
   =
   \begin{pmatrix}
   7 \\
   5
   \end{pmatrix}
   \]

In particular, *all four have the same solution set*.

The system of equations is **consistent** if it has at least one solution; it is **inconsistent** otherwise.
The span of vectors $v_1, v_2, \ldots, v_n$ is the set of all linear combinations of these vectors:

$$\text{Span}\{v_1, v_2, \ldots, v_n\} = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, a_2, \ldots, a_n \text{ in } \mathbb{R}\}.$$  

Theorem

Let $v_1, v_2, \ldots, v_n$, and $b$ be vectors in $\mathbb{R}^m$, and let $A$ be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following are equivalent:

1. $Ax = b$ is consistent.
2. $(A | b)$ does not have a pivot in the last column.
3. $b$ is in $\text{Span}\{v_1, v_2, \ldots, v_n\}$ (the span of the columns of $A$).

In this case, a solution to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

gives the linear combination $x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$. 

either they're all true, or they're all false, for the given vectors.
Definition
A transformation (or function or map) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule $T$ that assigns to each vector $x$ in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

Picture and vocabulary words:

It is **one-to-one** if different vectors in the domain go to different vectors in the codomain: $x \neq y \implies T(x) \neq T(y)$.

It is **onto** if every vector in the codomain is $T(x)$ for some $x$. In other words, the range equals the codomain.
Linear Transformations

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if it satisfies:

$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

for every $u, v$ in $\mathbb{R}^n$ and every $c$ in $\mathbb{R}$.

If $A$ is an $m \times n$ matrix, you get a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(x) = Ax.$$

Conversely, if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(x) = Ax$, where $A$ is the $m \times n$ matrix

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}.$$

As always, $e_1, e_2, \ldots, e_n$ are the unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$
Linear Transformations and Matrices

Let $A$ be an $m \times n$ matrix and let $T$ be the linear transformation $T(x) = Ax$.

- The domain of $T$ is $\mathbb{R}^n$. (Inputs are vectors with $n$ entries.)
- The codomain of $T$ is $\mathbb{R}^m$. (Outputs are vectors with $m$ entries.)
- The range of $T$ is span of the columns of $A$.
  (This is the set of all $b$ in $\mathbb{R}^m$ such that $Ax = b$ has a solution.)

Example

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \quad T(x) = Ax$$

- The domain of $T$ is $\mathbb{R}^2$.
- The codomain of $T$ is $\mathbb{R}^3$.
- The range of $T$ is

$$\text{Span} \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$
When the Span is Everything

**Theorem**
Let $A$ be an $m \times n$ matrix, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear transformation $T(x) = Ax$. The following are equivalent:

1. $T$ is onto.
2. $T(x) = b$ has a solution for every $b$ in $\mathbb{R}^m$.
3. $Ax = b$ is consistent for every $b$ in $\mathbb{R}^m$.
4. The columns of $A$ span $\mathbb{R}^m$.
5. $A$ has a pivot in each row.

**Moral:** If $A$ has a pivot in each row then its reduced row echelon form looks like this:

\[
\begin{pmatrix}
1 & 0 & * & 0 & * \\
0 & 1 & * & 0 & * \\
0 & 0 & 0 & 1 & *
\end{pmatrix}
\]

and $(A | b)$ reduces to this:

\[
\begin{pmatrix}
1 & 0 & * & 0 & * & * \\
0 & 1 & * & 0 & * & * \\
0 & 0 & 0 & 1 & * & *
\end{pmatrix}
\]

There’s no $b$ that makes it inconsistent, so there’s always a solution.

**Refer:** slides for 9/7, 9/21.
Linear Independence

A set of vectors \( \{v_1, v_2, \ldots, v_n\} \) is **linearly independent** if

\[
a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{only when} \quad a_1 = a_2 = \cdots = a_n = 0.
\]

Otherwise they are **linearly dependent**, and an equation

\[
a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{with some} \quad a_i \neq 0
\]

is a **linear dependence relation**.

**Theorem**

Let \( v_1, v_2, \ldots, v_n \) be vectors in \( \mathbb{R}^m \), and let \( A \) be the \( m \times n \) matrix with columns \( v_1, v_2, \ldots, v_n \). The following are equivalent:

1. The set \( \{v_1, v_2, \ldots, v_n\} \) is linearly independent.
2. For every \( i \) between 1 and \( n \), \( v_i \) is not in \( \text{Span}\{v_1, v_2, \ldots, v_{i-1}\} \).
3. \( Ax = 0 \) only has the trivial solution.
4. \( A \) has a pivot in every column.

If the vectors are linearly dependent, a nontrivial solution to the matrix equation

\[
A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0
\]

gives the linear dependence relation

\[
x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0.
\]
Theorem
Let $A$ be an $m \times n$ matrix, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear transformation $T(x) = Ax$. The following are equivalent:

1. $T$ is one-to-one.
2. $T(x) = b$ has one or zero solutions for every $b$ in $\mathbb{R}^m$.
3. $Ax = b$ has a unique solution or is inconsistent for every $b$ in $\mathbb{R}^m$.
4. $Ax = 0$ has a unique solution.
5. The columns of $A$ are linearly independent.
6. $A$ has a pivot in each column.

Moral: If $A$ has a pivot in each column then its reduced row echelon form looks like this:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and } (A | b)
\text{ reduces to this: }
\begin{pmatrix}
1 & 0 & 0 & \ast \\
0 & 1 & 0 & \ast \\
0 & 0 & 1 & \ast \\
0 & 0 & 0 & \ast
\end{pmatrix}.
$$

This can be inconsistent, but if it is consistent, it has a unique solution.

Refer: slides for 9/12, 9/14, 9/21.
To find the solution set to $Ax = b$, first form the augmented matrix $(A | b)$, then row reduce.

$$
\begin{pmatrix}
1 & 3 & 0 & 4 & 0 & 2 \\
0 & 0 & 1 & -1 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & -7
\end{pmatrix}
$$

This translates into

$$
\begin{align*}
x_1 + 3x_2 + x_4 &= 2 \\
x_3 - x_4 &= 3 \\
x_5 &= -7
\end{align*}
$$

The variables correspond to the non-augmented columns of the matrix. The free variables correspond to the non-augmented columns without pivots. Move the free variables to the other side, get the parametric form:

$$
\begin{align*}
x_1 &= 2 - 3x_2 - x_4 \\
x_3 &= 3 + x_4 \\
x_5 &= -7
\end{align*}
$$

This is a solution for every value of $x_3$ and $x_4$. 
Parametric Vector Form of Solution Sets

Parametric form:

\[
\begin{align*}
  x_1 &= 2 - 3x_2 - x_4 \\
  x_3 &= 3 + x_4 \\
  x_5 &= -7
\end{align*}
\]

Now collect all of the equations into a vector equation:

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.
\]

This is the parametric vector form of the solution set. This means that the

\[
\text{(solution set)} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.
\]
Homogeneous and Non-Homogeneous Equations

The equation $Ax = b$ is called **homogeneous** if $b = 0$, and **non-homogeneous** otherwise. A homogeneous equation always has the **trivial solution** $x = 0$:

$$A0 = 0.$$ 

The solution set to a homogeneous equation is always a span:

$$(\text{solutions to } Ax = 0) = \text{Span}\{v_1, v_2, \ldots, v_r\}$$

where $r$ is the number of free variables. The solution set to a consistent non-homogeneous equation is

$$(\text{solutions to } Ax = b) = p + \text{Span}\{v_1, v_2, \ldots, v_r\}$$

where $p$ is a **specific solution** (i.e. some vector such that $Ap = b$), and $\text{Span}\{v_1, \ldots, v_r\}$ is the solution set to the homogeneous equation $Ax = 0$. This is a **translate of a span**.

Both expressions can be read off from the parametric vector form.