Math 1553 Worksheet 4  
September 16, 2016

Linear Independence: Concept Questions

1. If three vectors \( v_1, v_2, v_3 \) span \( \mathbb{R}^3 \), must those vectors be linearly independent? Why or why not?

   Say \( \{v_1, v_2, v_3\} \) is not linearly independent. Then we can say that at least one of the vectors can be expressed as a linear combination of the others. Without loss of triviality, say \( v_1 = c_1 v_2 + c_2 v_3 \). Then \( v_1 \) is in the span of \( v_2, v_3 \), which can only be a plane (\( \mathbb{R}^2 \)). This is a contradiction with the initial statement. Therefore \( \{v_1, v_2, v_3\} \) must be linearly independent.

2. Which of the following true statements can be checked without row reduction?

   \[ \begin{pmatrix} 3 & 3 & 0 \\ 3 & 10 & 0 \\ 4 & 20 & 0 \end{pmatrix} \]
   a) \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)
   \( \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix} \)
   is linearly independent.

   \[ \begin{pmatrix} 3 & 3 & 0 \\ 3 & 10 & 0 \\ 4 & 20 & 0 \end{pmatrix} \]
   b) \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)
   \( \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix} \)
   is linearly independent.

   \[ \begin{pmatrix} 3 & 3 & 0 \\ 3 & 10 & 0 \\ 4 & 20 & 0 \end{pmatrix} \]
   c) \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)
   \( \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix} \)
   is linearly dependent.

   \[ \begin{pmatrix} 3 & 3 & 0 \\ 3 & 10 & 0 \\ 4 & 20 & 0 \end{pmatrix} \]
   d) \( \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \)
   \( \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix} \)
   is linearly dependent.

   These vectors are in \( \mathbb{R}^3 \) so maximum #
   of vectors that can span \( \mathbb{R}^3 \) is 3. If you have a 4th vector, at least 1 must be dependent
   on the others.

3. How many solutions can the matrix equation \( Ax = b \) have if the columns of \( A \) are
   linearly independent? [Try \( b = 0 \) first.]

<table>
<thead>
<tr>
<th>Columns</th>
<th>Linearly independent?</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) 0</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix} ) → Unique (1 soln)</td>
</tr>
<tr>
<td>b) 1</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix} ) → either unique or no solns</td>
</tr>
<tr>
<td>c) ∞</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix} ) → columns of ( A ) not linearly independent</td>
</tr>
</tbody>
</table>

   \[ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \]

   \[ x = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \]

   a) \( c_1(0) + c_2(0) = (c) \)
Linear Independence: Additive Color Theory

Every color on my computer monitor is a vector in \( \mathbb{R}^3 \) with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.

Given colors \( v_1, v_2, \ldots, v_p \), we can form a “weighted average” of these colors by making a linear combination

\[
v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p
\]

with \( c_1 + c_2 + \cdots + c_p = 1 \). Example:

\[
\frac{1}{2} \begin{pmatrix} 140 \\ 120 \\ 125 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 100 \\ 75 \end{pmatrix} = \begin{pmatrix} 70 \\ 60 \\ 100 \end{pmatrix}
\]

4. Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors? If \( Ax = 0 \) has \( x = 0 \) (trivial), then they are lin. indep.

\[
\begin{pmatrix} 240 & 0 & 0 \\ 140 & 120 & 0 \\ 0 & 100 & 75 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} 10 & 0 & \frac{3}{8} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\( x_3 \) is free, so all three are linearly independent. \( x_1 \) and \( x_2 \) are pivots, so in order.

5. Consider the colors on the right. For which \( h \) is

\[
\left\{ \begin{pmatrix} 180 \\ 100 \\ 116 \\ 130 \end{pmatrix} \right\}
\]

linearly dependent? What does that say about the corresponding color?

\[
h = \begin{pmatrix} 40 \\ 80 \\ 120 \\ 160 \\ 200 \\ 240 \end{pmatrix}
\]

\( V_1, V_2, V_3 \) are linearly independant if the solution to \( Ax = 0 \) is \( \bar{0} \) and linearly dependant if not.

\[
V_1, V_2, V_3
\]

You can mix \( v_1, v_2 \) to get \( v_3 \) by \( v_1 + v_2 = v_3 \). You can mix \( v_2, v_3 \) to get \( v_1 \) by \( v_2 + v_3 = v_1 \). You cant mix \( v_3, v_1 \) to get \( v_2 \).

\[
\frac{4}{3} V_2 - \frac{1}{3} V_1 = V_3
\]

When you mix colors, you cant remove a part of a mixed color (e) blue+yellow = green, green+yellow = blue. You cant remove yellow, its mixed.