Announcements
September 7

▶ Homework 1.3 is due Friday.
▶ Quiz on Friday: section 1.3.
▶ My office hours only today are moved to 10–11am, right after class.
  ▶ As always, TAs’ office hours are posted on the website.
  ▶ Also there are links to other resources like Math Lab.
Section 1.4

The Matrix Equation $Ax = b$
Matrix $\times$ Vector

Let $A$ be an $m \times n$ matrix ($m$ rows, $n$ columns)

$$A = \begin{pmatrix}
\vdots \\
v_1 & v_2 & \cdots & v_n \\
\vdots 
\end{pmatrix}$$

with columns $v_1, v_2, \ldots, v_n$

**Definition**

The product of $A$ with a vector $x$ in $\mathbb{R}^n$ is the linear combination

$$Ax = \begin{pmatrix}
\vdots \\
v_1 & v_2 & \cdots & v_n \\
\vdots 
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} := x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$  

The output is a vector in $\mathbb{R}^m$.

Note that the number of *columns* of $A$ has to equal the number of *rows* of $x$.

**Example**

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$
Question
Let $v_1, v_2, v_3$ be vectors in $\mathbb{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let $A$ be the matrix with columns $v_1, v_2, v_3$, and let $x$ be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$
Matrix Equations
In General

Let $v_1, v_2, \ldots, v_n$, and $b$ be vectors in $\mathbb{R}^m$. Consider the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = b.$$ 

It is equivalent to the matrix equation

$$Ax = b$$

where

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$ 

Conversely, if $A$ is any $m \times n$ matrix, then

$$Ax = b$$

is equivalent to the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = b$$

where $v_1, \ldots, v_n$ are the columns of $A$, and $x_1, \ldots, x_n$ are the entries of $x$. 
We now have four equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

   \[
   2x_1 + 3x_2 = 7 \\
   x_1 - x_2 = 5
   \]

2. As an augmented matrix:

   \[
   \begin{pmatrix}
   2 & 3 & | & 7 \\
   1 & -1 & | & 5
   \end{pmatrix}
   \]

3. As a vector equation \((x_1v_1 + \cdots + x_nv_n = b)\):

   \[
   x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}
   \]

4. As a matrix equation \((Ax = b)\):

   \[
   \begin{pmatrix}
   2 & 3 \\
   1 & -1
   \end{pmatrix}
   \begin{pmatrix}
   x_1 \\
   x_2
   \end{pmatrix} = \begin{pmatrix}
   7 \\
   5
   \end{pmatrix}
   \]

In particular, all four have the same solution set. We will go back and forth freely between these over and over again, for the rest of the semester.
**Definition**

A **row vector** is a matrix with one row. The product of a row vector of length $n$ and a (column) vector of length $n$ is

\[
(a_1 \cdots a_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := a_1 x_1 + \cdots + a_n x_n.
\]

This is a scalar.

If $A$ is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and $x$ is a vector in $\mathbb{R}^n$, then

\[
Ax = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}
\]

This is a vector in $\mathbb{R}^m$ (again).
Example

\[
\begin{pmatrix}
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
= \begin{pmatrix}
4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\
7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3
\end{pmatrix}
= \begin{pmatrix}
32 \\
50
\end{pmatrix}.
\]

Note this is the same as before:

\[
\begin{pmatrix}
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
= 1 \begin{pmatrix}
4 \\
7
\end{pmatrix} + 2 \begin{pmatrix}
5 \\
8
\end{pmatrix} + 3 \begin{pmatrix}
6 \\
9
\end{pmatrix}
= \begin{pmatrix}
1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\
1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9
\end{pmatrix}
= \begin{pmatrix}
32 \\
50
\end{pmatrix}.
\]

Now you have two ways of computing $Ax$. In the second, you calculate $Ax$ one entry at a time.

Try both and decide which is your favorite!
Spans and Solutions to Equations

Let $A$ be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

Ax = b has a solution $\iff$ there exist $x_1, \ldots, x_n$ such that $A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$ $\iff$ there exist $x_1, \ldots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$ $\iff$ $b$ is a linear combination of $v_1, \ldots, v_n$ $\iff$ $b$ is in the span of the columns of $A$.

Very Important

"if and only if"

The last condition is very geometric.
**Question**

Let \( A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \). Does the equation \( Ax = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) have a solution?

**Columns of \( A \):**

\[
\begin{align*}
  v & = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\
  w & = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\end{align*}
\]

**Solution vector:**

\[
  b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

Is \( b \) contained in the span of the columns of \( A \)?
It sure doesn’t look like it.
Question

Let \( A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \). Does the equation \( Ax = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) have a solution?

Answer: Let’s check by solving the matrix equation using row reduction. The first step is to put the system into an augmented matrix.

\[
\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

The last equation is \( 0 = 1 \), so the system is *inconsistent*. In other words, the matrix equation

\[
\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\]

has no solution, as we guessed.
Example

Question
Let \( A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \). Does the equation \( Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \) have a solution?

Columns of \( A \):
\[
\begin{align*}
v &= \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\
w &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}
\end{align*}
\]

Solution vector:
\[
b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}
\]

Is \( b \) contained in the span of the columns of \( A \)?
It looks like it. Can you see what \( x \) is from the grid on \( \text{Span}\{v, w\} \)?
Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$
\begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} \quad \text{row reduce} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}
$$

This gives us

$$x = 1 \quad y = -1.$$

In other words,

$$1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{or} \quad A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. $$
Which of the following true statements can be checked by eye-balling them, *without* row reduction?

<table>
<thead>
<tr>
<th>Option</th>
<th>Statement</th>
<th>Basis Vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.</td>
<td>((0, 1, 2, 0)) is in the span of ((3, 10, -2)), ((0, 10, -2)), ((-1)).</td>
<td></td>
</tr>
<tr>
<td>B.</td>
<td>((0, 1, 2, 0)) is in the span of ((3, 5, 6)), ((3, 7, 8)), ((-1)).</td>
<td></td>
</tr>
<tr>
<td>C.</td>
<td>((0, 1, 2, 0)) is in the span of ((3, 1, 0)), ((3, 4, \sqrt{2})), ((-1)).</td>
<td></td>
</tr>
<tr>
<td>D.</td>
<td>((0, 1, 2, 0)) is in the span of ((5, 6, 3)), ((7, 8, 3)), ((-1)).</td>
<td></td>
</tr>
</tbody>
</table>
Here are criteria for a linear system to always have a solution.

**Theorem**
Let \( A \) be an \( m \times n \) (non-augmented) matrix. The following are equivalent (either they're all true, or they're all false):

1. \( Ax = b \) has a solution for all \( b \) in \( \mathbb{R}^m \).
2. The span of the columns of \( A \) is all of \( \mathbb{R}^m \).
3. \( A \) has a pivot in each row.

Why is (1) the same as (2)? Why is (1) the same as (3)? If \( A \) has a pivot in each row then its reduced row echelon form looks like this:

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast \\
0 & 1 & \ast & 0 & \ast \\
0 & 0 & 0 & 1 & \ast
\end{pmatrix}
\]

and \( (A | b) \) reduces to this:

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast & | \\
0 & 1 & \ast & 0 & \ast & | \\
0 & 0 & 0 & 1 & \ast & | \\
\end{pmatrix}.
\]

There's no \( b \) that makes it inconsistent, so there's always a solution. If \( A \) doesn't have a pivot in each row, then its reduced form looks like this:

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast \\
0 & 1 & \ast & 0 & \ast \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and this can be made inconsistent:

\[
\begin{pmatrix}
1 & 0 & \ast & 0 & \ast & | 0 \\
0 & 1 & \ast & 0 & \ast & | 0 \\
0 & 0 & 0 & 0 & 0 & | 16
\end{pmatrix}.
\]
Properties of the Matrix–Vector Product

Let \( c \) be a scalar, \( u, v \) be vectors, and \( A \) a matrix.

\[ A(u + v) = Au + Av \]
\[ A(cv) = cAv \]

See Lay, §1.4, Theorem 5.

For instance, \( A(3u - 7v) = 3Au - 7Av \).

**Consequence:** If \( u \) and \( v \) are solutions to \( Ax = 0 \), then so is every vector in \( \text{Span}\{u, v\} \). Why?

\[
\begin{align*}
Au &= 0 \\
Av &= 0
\end{align*}
\] \[ A(xu + yv) = xAu + yAv = x0 + y0 = 0. \]

(Here 0 means the zero vector.)

**Important**

The set of solutions to \( Ax = 0 \) is a span.