

Announcements

August 29

- ▶ Homeworks 1.1 and 1.2 are due Friday.
- ▶ The scores for the warmup set have been posted. They don't affect your grade, but you should check that your score was entered correctly.
- ▶ The first quiz is on Friday, during recitation.
 - ▶ Quizzes mostly test your understanding of the homework.
 - ▶ There will generally be a quiz every Friday when there's no midterm.
 - ▶ Check the schedule if you want to know what will be covered.

Recall from Last Time

A matrix is in **reduced row echelon form** if it looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The **leading entries** are called the **pivots**. To get a matrix into reduced row echelon form, you do **row reduction**:

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Step 1b Scale 1st row so that its leading entry is equal to 1.

Step 1c Use row replacement so all entries above and below this 1 are 0.

Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row; uncover 1st row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries above and below this 1 are 0.

Step 3a Cover the first two rows, swap the 3rd row with a lower one so that the leftmost nonzero (uncovered) entry is in 3rd row; uncover first two rows.

etc.

Example

The linear system

$$2x + 10y = -1$$

$$3x + 15y = 2$$

Let's row reduce it:

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array} \right)$$

gives rise to the matrix

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array} \right).$$

divide 1st by 2

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$$\left( \begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 3 & 15 & 2 \end{array} \right)$$

subtract  $3 \times 1$ st from 2nd

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$$\left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & \frac{5}{2} \end{array} \right)$$

multiply 2nd by $\frac{2}{5}$

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$$\left( \begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & 1 \end{array} \right)$$

add  $\frac{1}{2} \times 2$ nd to 1st

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$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

This corresponds to the *inconsistent* system

$$x + 5y = 0$$

$$0 = 1$$

Inconsistent Matrices

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

In other words, it is inconsistent if and only if *the last column is a pivot column*.

Another Example

The linear system

$$2x + y + 12z = 1$$

$$x + 2y + 9z = -1$$

gives rise to the matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right).$$

Let's row reduce it:

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right)$$

swap 1st and 2nd
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$$\left( \begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array} \right)$$

- 2×1st from 2nd  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array} \right)$$

divide 2nd by -3
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$$\left( \begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array} \right)$$

- 2× 2nd from 1st  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right)$$

This corresponds to the linear system

$$x + 5z = 1$$

$$y + 2z = -1$$

Another Example

Continued

The system

$$x + 5z = 1$$

$$y + 2z = -1$$

comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:

$$x = 1 - 5z$$

$$y = -1 - 2z$$

For any value of z , there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

So we have found the solution set: it is all values

$$(x, y, z) = (1 - 5z, -1 - 2z, z) \quad \text{for } z \text{ any real number.}$$

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables x_1, \dots, x_n . Let M be the reduced row echelon form of the matrix for this system. We say that x_i is a **free variable** if its corresponding column in M is *not* a pivot column.

Important

You can choose *any value* for the free variables in a (consistent) linear system.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array} \right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array} \right)$$

the free variables are x_2 and x_4 .

What about the last column? It's not a pivot column either...

One More Example

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array} \right)$$

The free variables are x_2 and x_4 : they are the ones whose columns are *not* pivot columns.

This translates into the system of equations

$$\begin{aligned} x_1 &+ 3x_4 = 2 \\ x_3 + x_4 &= -1 \end{aligned}$$

Rewriting,

$$\begin{aligned} x_1 &= 2 - 3x_4 \\ x_3 &= -1 - 4x_4. \end{aligned}$$

What happened to x_2 ? What is it allowed to be? Anything! The general solution is

$$(x_1, x_2, x_3, x_4) = (2 - 3x_4, x_2, -1 - 4x_4, x_4)$$

for any values of x_2 and x_4 . This is called the **parametric form** of the solution.

Poll

Is it possible for a system of linear equations to have exactly two solutions?

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.

In this case, the system is **inconsistent**. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

2. Every column except the last column is a pivot column.

In this case, the system has a **unique solution**. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array} \right)$$

3. The last column is not a pivot column, and some other column isn't either.

In this case, the system has **infinitely many** solutions, corresponding to the infinitely many possible values of the free variables. Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array} \right)$$