

Announcements

August 24

- ▶ Warmup homework set due Friday.
- ▶ Don't forget to register for Piazza using your T-Square email address.
 - ▶ You can change your email address by clicking on the gear icon in the top-right corner of the Piazza window and selecting "Account/Email Settings".
- ▶ Recitations meet on Friday, at the same time — but not here (see the website).
- ▶ My office is Skiles 221 and my office hours are today, 1–2pm and tomorrow, 3:30–4:30pm.

Chapter 1

Linear Equations

Section 1.1

Systems of Linear Equations

Solving Systems of Equations

Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

This is the kind of problem we'll talk about today.

- ▶ A **solution** is a list of numbers x, y, z, \dots that make *all* of the equations true.
- ▶ The **solution set** is the collection of all solutions.
- ▶ **Solving** the system means finding the solution set.

What is a *systematic* way to solve a system of equations?

Solving Systems of Equations

Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

What strategies do you know?

- ▶ Substitution
- ▶ Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

Solving Systems of Equations

Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Elimination method: in what ways can you manipulate the equations?

- ▶ Multiply an equation by a nonzero number (scale).
- ▶ Add one equation to another.
- ▶ Swap two equations.

Solving Systems of Equations

Example

Solve the system of equations

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Multiply first by -3

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$3x + y - z = -2$$

Add first to third

~~~~~→

$$-3x - 6y - 9z = -18$$

$$2x - 3y + 2z = 14$$

$$-5y - 10z = -20$$

Now I've eliminated x from the last equation!

...but there's a long way to go still. Can we make our lives easier?

Solving Systems of Equations

Better notation

It sure is a pain to have to write x, y, z , and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{array}{l} x + 2y + 3z = 6 \\ 2x - 3y + 2z = 14 \\ 3x + y - z = -2 \end{array} \quad \begin{array}{c} \text{becomes} \\ \text{~~~~~} \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- ▶ Multiply all entries in a row by a nonzero number (scale).
- ▶ Add (a multiple of) each entry of one row to the corresponding entry in another.
- ▶ Swap two rows.

Row Operations

Example

Solve the system of equations

$$\begin{aligned}x + 2y + 3z &= 6 \\2x - 3y + 2z &= 14 \\3x + y - z &= -2\end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{array}{lcl} x & = & A \\ y & = & B \\ z & = & C \end{array} \quad \text{or in matrix form,} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array} \right)$$

So we need to do row operations that make the start matrix look like the end one. Strategy: fiddle with it so we only have ones and zeros.

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

2nd $- 2 \times 1$ st
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

3rd  $- 3 \times 1$ st  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

swap 2nd and 3rd
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

divide 2nd by  $-5$   
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

1st $- 2 \times 2$ nd
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

3rd  $+ 7 \times 2$ nd  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

divide 3rd by 10
~~~~~>

1st + 3rd  
~~~~~>

2nd - 2×3rd
~~~~~>

translates into  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$x = 1$$

$$y = -2$$

$$z = 3$$

Success! Check:

$$x + 2y + 3z = 6$$

$$2x - 3y + 2z = 14 \quad \text{substitute solution} \quad \text{~~~~~>}$$

$$3x + y - z = -2$$

$$1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

$$2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 = 14$$

$$3 \cdot 1 + (-2) - 3 = -2$$

So easy a baby can do it!



(Okay, so it's not *that* easy — this was just a weak excuse to show you a picture of my son.)

Row Equivalence

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

A Bad Example

Example

Solve the system of equations

$$x + y = 2$$

$$3x + 4y = 5$$

$$4x + 5y = 9$$

Let's try doing row operations:

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array} \right) \xrightarrow{\text{2nd} - 3 \times \text{1st}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array} \right)$$

$$\xrightarrow{\text{3rd} - 4 \times \text{1st}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

$$\xrightarrow{\text{3rd} - \text{2nd}} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right)$$

A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array} \right) \begin{array}{l} \text{translates into} \\ \text{~~~~~} \end{array} \begin{array}{l} x + y = 2 \\ y = -1 \\ \textcolor{red}{0 = 2} \end{array}$$

In other words, the original equations

$$\begin{array}{ll} x + y = 2 & \\ 3x + 4y = 5 & \text{have the same solutions as} \\ 4x + 5y = 9 & \end{array} \quad \begin{array}{l} x + y = 2 \\ y = -1 \\ 0 = 2 \end{array}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

Section 1.2

Row Reduction and Echelon Forms

Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a “solved” matrix. What do we mean by “solved”?

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the **right** of the leading entry of the row above.
3. Below a leading entry of a row, all entries are **zero**.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A **pivot** is the first nonzero entry of a row of a matrix in row echelon form.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! Stay tuned.

Reduced Row Echelon Form

Continued

Why is this the “solved” version of the matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

is in reduced row echelon form. It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3 \end{aligned}$$

which is clearly the solution.

But what happens if there are fewer pivots than rows? ... parametrized solution set (later).

Reduced Row Echelon Form

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $(0 \ 1 \ 0 \ 0)$ E. $(0 \ 1 \ 8 \ 0)$

F. $\begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Reduced Row Echelon Form

Theorem

Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm, called **row reduction**, which demonstrates that every matrix is row equivalent to *at least one* matrix in reduced row echelon form.

The uniqueness statement is interesting — maybe you can figure out why it's true!

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Step 1b Scale 1st row so that its leading entry is equal to 1.

Step 1c Use row replacement so all entries above and below this 1 are 0.

Step 2a Cover the first row, swap the 2nd row with a lower one so that the leftmost nonzero (uncovered) entry is in 2nd row; uncover 1st row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries above and below this 1 are 0.

Step 3a Cover the first two rows, swap the 3rd row with a lower one so that the leftmost nonzero (uncovered) entry is in 3rd row; uncover first two rows.

etc.

Example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

Row Reduction Algorithm

Example

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

2nd $- 2 \times 1$ st
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array} \right)$$

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$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array} \right)$$

swap 2nd and 3rd
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

divide 2nd by  $-5$   
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$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

1st $- 2 \times 2$ nd
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$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array} \right)$$

3rd  $+ 7 \times 2$ nd  
~~~~~>

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

Row Reduction Algorithm

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array} \right)$$

divide 3rd by 10
~~~~~>

1st + 3rd  
~~~~~>

2nd - 2×3rd
~~~~~>

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$