Please read all instructions carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- All answers must be justified unless otherwise noted, and all proofs must be written in clear and grammatical English.
- You may cite any theorem, lemma, proposition, etc. proved in class or in the sections we covered in the text, in addition to any assigned homework problem (unless the exam problem itself was assigned on the homework).
- Good luck!
Problem 1. [10 points]

A group $G$ of order 12 contains a conjugacy class of size 4. Prove that the center of $G$ is trivial.

Solution.

Let $x \in G$ have conjugacy class size 4. Its centralizer $Z(x)$ therefore has size 3. Since $x \in Z(x)$ and $x \neq 1$, we have $Z(x) = \langle x \rangle$. But the center $Z(G)$ is contained in $Z(x)$, so either $|Z(G)| = 3$ or $|Z(G)| = 1$. The former case implies $x \in Z(G)$, which is impossible since the conjugacy class of $x$ does not have size 1. Hence $Z(G) = \{1\}$. 
Problem 2.  

Let $S_3$ act on the product set $S = \{1, 2, 3\} \times \{1, 2, 3\}$ diagonally: that is, $\sigma \cdot (i, j) := (\sigma(i), \sigma(j))$. Let $x = (123) \in S_3$ and $y = (12) \in S_3$. Determine the images of $x$ and $y$ under the permutation representation $\varphi : S_3 \to \text{Perm}(S) \cong S_9$. In other words, find the cycle decompositions of $\varphi(x)$ and $\varphi(y)$.

Solution.

Set

\[
\begin{align*}
a &= (1, 1) & b &= (1, 2) & c &= (1, 3) \\
d &= (2, 1) & e &= (2, 2) & f &= (2, 3) \\
g &= (3, 1) & h &= (3, 2) & i &= (3, 3). \\
\end{align*}
\]

We calculate

\[
(123) \cdot a = e \quad (123) \cdot e = i \quad (123) \cdot i = a
\]

and

\[
(123) \cdot b = f \quad (123) \cdot f = g \quad (123) \cdot g = b
\]

and

\[
(123) \cdot c = d \quad (123) \cdot d = h \quad (123) \cdot h = c.
\]

Therefore, $\varphi((123)) = (aei)(bf g)(cdh)$. Similarly,

\[
(12) \cdot a = e \quad (12) \cdot e = a \quad (12) \cdot i = i
\]

and

\[
(12) \cdot b = d \quad (12) \cdot d = b \quad (12) \cdot c = f \quad (12) \cdot f = c \quad (12) \cdot g = h \quad (12) \cdot h = g.
\]

Therefore, $\varphi((12)) = (ae)(bd)(cf)(gh)$. 

Problem 3. \[10 \text{ points; 5 points each}\]

Let \( G = \text{GL}_2(\mathbb{R}) \) act on \( \mathbb{R}^2 \) by left multiplication of a vector by a matrix.

a) Determine the orbits for this action.

b) What is the stabilizer of \((1, 0) \in \mathbb{R}^2\)?

Solution.

a) Since \( M \mathbf{0} = \mathbf{0} \) for all \( M \in \text{GL}_2(\mathbb{R}) \), the orbit of \( \mathbf{0} \) is \( \{\mathbf{0}\} \). On the other hand, given any nonzero \( v \in \mathbb{R}^2 \), there exists \( w \in \mathbb{R}^2 \) such that \( \{v, w\} \) forms a basis of \( \mathbb{R}^2 \). The matrix \( M \) with columns \( v \) and \( w \) is invertible and sends \((1, 0)\) to \( v \), so the orbit of \((1, 0)\) is all of \( \mathbb{R}^2 \setminus \{0\} \). Hence there are two orbits.

b) We have

\[
\begin{bmatrix}
 a & b \\
 c & d \\
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
 a \\
 c \\
\end{bmatrix}
\]

so the stabilizer of \((1, 0)\) consists of all invertible matrices of the form \( \begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix} \).
Problem 4.  

Prove that a group $G$ of order 20 contains exactly four elements of order 5.

Solution.  
An element of order 5 generates a 5-Sylow subgroup of $G$, so we must show that $G$ contains exactly one 5-Sylow subgroup. Indeed, the number of 5-Sylow subgroups divides 4 and is congruent to 1 modulo 5, and the only such number is 1.
Problem 5. [10 points; 5 points each]

Consider the permutation (123) in $S_4$.

a) What is the centralizer of (123)? Justify your answer.

b) Find $\sigma \in S_4$ such that $\sigma(123)\sigma^{-1} = (132)$.

Solution.

a) The conjugacy class of (123) consists of all 3-cycles, namely,

$$(123), (132), (124), (142), (134), (143), (234), (243).$$

Hence the centralizer has size $24/8 = 3$. Since $Z((123))$ contains $\{1, (123), (132)\}$, the containment is an equality.

b) We get (132) from (123) by relabeling 3 to 2 and 2 to 3. Hence we may take $\sigma = (23)$. 
[Scratch work]