Definition. Let $X \to S$ be an $S$-scheme, and let $S' \to S$ be a morphism. We call $X' := X \times_S S'$ the base change of $X$ to $S'$; it is an $S'$-scheme via the second projection.

This is a psychologically useful bit of alternate terminology for a fiber product. The following lemmas about fiber products (proved in class) will be needed in the problems below.

Lemma 1. Let $f : Y \to X$ be a morphism of $S$-schemes and let $S' \to S$ be a morphism. Then the square

$$
\begin{array}{ccc}
Y \times_S S' & \xrightarrow{(f, \text{Id}_{S'})} & X \times_S S' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f} & X 
\end{array}
$$

is Cartesian.

Lemma 2 (Fiber products are compatible with change of base). Let $X$ and $Y$ be $S$-schemes, and let $S' \to S$ be a morphism. Let $X' = X \times_S S'$ and $Y' = Y \times_S S'$ be the base changes. Then the square

$$
\begin{array}{ccc}
X' \times_S Y' & \xrightarrow{(f, g)} & S' \\
\downarrow & & \downarrow \\
X \times_S Y & \xrightarrow{f} & S
\end{array}
$$

is Cartesian, where $f : X' \to X$ and $g : Y' \to Y$ are projection onto the first coordinate.
1. \(a)\) Let \(f : X \to S\) be a morphism of schemes, let \(s \in S\), and define \(X_s := \text{Spec} (\kappa(s)) \times_S X\), where \(\kappa(s)\) is the residue field of \(s\) (cf. homework 7, problem 2). Prove that the projection \(X_s \to X\) induces a homeomorphism of the topological space underlying \(X_s\) onto \(f^{-1}(s)\) in its induced topology. Hence \(f^{-1}(s)\) naturally has the structure of a scheme; we call it the fiber over \(s\). [Note that \(f^{-1}(s)\) need not be closed in \(X\) if \(\{s\}\) is not closed in \(S\).]

\(b)\) Now let \(X\) and \(Y\) be \(S\)-schemes, and let \(s \in S\). Prove that \((X \times_S Y)_s = X_s \times_{\text{Spec} \kappa(s)} Y_s\). In other words, the fiber product of schemes is fiberwise a fiber product over a field.

2. Let \(X\) and \(Y\) be \(S'\)-schemes, and let \(S' \to S\) be a morphism. Define a natural morphism \(f : X \times_{S'} Y \to X \times_S Y\), and prove that \(f\) is a closed immersion.

3. For a scheme \(Z\) we let \(|Z|\) denote the topological space underlying \(Z\). Let \(X\) and \(Y\) be \(S\)-schemes.

\(a)\) Define a natural map of sets 
\[f : |X \times_S Y| \to |X| \times_{|S|} |Y|,\]
where the right hand side is the fiber product in the category of sets.

\(b)\) Prove that \(f\) is surjective. [Use homework 7, problem 2. If \(k_1\) and \(k_2\) are field extensions of \(k\), why is \(k_1 \otimes_k k_2\) not the zero ring?]

\(c)\) Give an example to show that \(f\) may fail to be injective.

4. Let \(\mathcal{P}\) be a property of morphisms of schemes such that:
  (1) a closed immersion has \(\mathcal{P}\);
  (2) a composition of two morphisms having \(\mathcal{P}\) has \(\mathcal{P}\); and
  (3) \(\mathcal{P}\) is stable under base extension, in that if \(X \to S\) has \(\mathcal{P}\) and \(S' \to S\) is a morphism then the base change \(X \times_S S' \to S'\) has \(\mathcal{P}\).

Prove that:

\(a)\) A product of morphisms having \(\mathcal{P}\) has \(\mathcal{P}\): that is, if \(f_i : X_i \to Y_i\) for \(i = 1, 2\) are morphisms of \(S\)-schemes and \(f_1, f_2\) have \(\mathcal{P}\), then \((f_1, f_2) : X_1 \times_S X_2 \to Y_1 \times_S Y_2\) has \(\mathcal{P}\).

\(b)\) If \(X \to S\) and \(Y \to S\) have \(\mathcal{P}\) then \(X \times_S Y \to S\) has \(\mathcal{P}\).

\(c)\) If \(f : Y \to X\) and \(g : X \to S\) are two morphisms, and if \(g \circ f\) has \(\mathcal{P}\) and \(g\) is separated, then \(f\) has \(\mathcal{P}\). [First show that the graph morphism \(\Gamma_f = \text{Id}_Y \times f : Y \to Y \times_S X\) is obtained by base extension from the diagonal morphism \(\Delta : X \to X \times_S X\). Also see homework 8, problem 3.]

5. Let \(X\) and \(Y\) be separated \(S\)-schemes and let \(f : X \to Y\) be a morphism. Show that \(f\) is separated.