Math 6421 Homework 4
Due at the beginning of class on Friday, September 18.

In the following exercises, the ground field, if unspecified, is a general algebraically closed field $K$.

(1) Let $X$ be an irreducible affine variety with function field $K(X) = \text{Frac}(A(X))$.
   (a) Let $U_1, U_2$ be a nonempty open subsets of $X$ and let $f_1, f_2, g_1, g_2 \in A(X)$ with $g_i$ nonzero on $U_i$. Show that if $f_1(x)/g_1(x) = f_2(x)/g_2(x)$ for all $x \in U_1 \cap U_2$, then $f_1/g_1 = f_2/g_2$ as elements in $K(X)$.
   (b) Use (a) to define a canonical inclusion $O(U) \hookrightarrow A(X)$ for any nonempty open set $U \subset X$, and prove that these inclusions are compatible with restriction in the obvious way.
   (c) If $U = \bigcup_{i \in I} U_i$ is an open cover, show that $O_X(U) = \bigcap_{i \in I} O_X(U_i)$ in $K(X)$.

(2) Let $X$ be an irreducible affine variety with function field $K(X)$ and let $\varphi \in K(X)$.
   (a) Show that there is a maximal (with respect to inclusion) open subset $U \subset X$ such that $\varphi \in O_X(U)$, identifying $O_X(U)$ with a subring of $K(X)$ as in (1). We call $U$ the domain of definition of $\varphi$.
   (b) Let $\varphi = g/f \in K(A^n)$. Find the domain of definition of $\varphi$.
   (c) Let $\varphi$ be the rational function of Example 3.5 in Gathmann. Find the domain of definition of $\varphi$.
   (d) Suppose $X = V(x^2 + y^2 = 1)$, and let $\varphi = \frac{1-y}{x}$. Find the domain of definition of $\varphi$.

(3) Let $X$ be a Hausdorff topological space, let $x \in X$, and let $A$ be a ring. Define a presheaf $\mathcal{F}$ on $X$ as follows: $\mathcal{F}(U) = A$ if $x \in U$, and $\mathcal{F}(U) = 0$ otherwise, with the obvious transition maps. Prove that $\mathcal{F}$ is a sheaf, and calculate all stalks of $\mathcal{F}$. We call $\mathcal{F}$ a skyscraper sheaf.

(4) Let $X$ be an affine variety and let $Y \subset X$ be a closed subset. For an open set $U \subset X$ let
   $$\mathcal{I}_Y(U) = \{ f \in O_X(U) : f(y) = 0 \text{ for all } y \in Y \cap U \}.$$ 
Prove that $\mathcal{I}_Y(U)$ is an ideal in $O_X(U)$, and that $U \mapsto \mathcal{I}_Y(U)$ is a sheaf of abelian groups.