1. Let $K$ be a number field. Show that $\mathcal{O}_K$ is a PID if and only if every nonzero ideal $a \subset \mathcal{O}_K$ contains an element $x$ such that $|N_{K/Q}(x)| = N(a)$.

2. Let $x \in \mathbb{C}$ be a root of $X^3 + 10X + 1$ and let $y \in \mathbb{C}$ be a root of $X^3 - 8X + 15$. Let $K = \mathbb{Q}(x)$ and $L = \mathbb{Q}(y)$.
   a) Show $D_K = D_L$, and compute $\mathcal{O}_K$ and $\mathcal{O}_L$.
   a) Show that 17 is inert in $K$, but that 17 splits completely in $L$.
   a) Use (b) to prove $K \ncong L$.
   Hence cubic fields with the same discriminant need not be isomorphic. (In contrast, quadratic fields are classified by their discriminant.)

3. Prove that $\mathbb{Z}[\sqrt{2}]$ is a PID.

4. Let $A$ be a Dedekind domain with fraction field $F$, let $K/F$ be a finite separable extension, and let $B$ be the integral closure of $A$ in $K$. Prove that
   
   $$\mathcal{D}_{B/A} = \left\{ D(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in B \right\}.$$ 

   [Localize at a prime ideal of $A$.]

5. Let $K = \mathbb{Q}(\sqrt{-6})$. Determine which prime numbers $p$ split, ramify, and remain inert in $\mathcal{O}_K$. Your answer should only involve the congruence class of $p$ modulo 24 for $p > 3$. [Hint: use quadratic reciprocity.]