This is the final homework assignment. It is a bit longer because you have two weeks (including Thanksgiving break) to work on it.

1. Let $A$ be a Dedekind domain, let $S \subset A \setminus \{0\}$ be a multiplicatively closed subset, and let $A' = S^{-1}A$. Prove that for any two ideals $a', b' \subset A'$, we have

$$(a' \cap A) \cdot (b' \cap A) = (a'b') \cap A.$$

In other words, contraction is compatible with products in the case of localizations of Dedekind domains.

2. Let $A$ be a Dedekind domain, let $S \subset A \setminus \{0\}$ be a multiplicatively closed subset, and let $A' = S^{-1}A$.

   a) Let $p \subset A$ be a nonzero prime ideal such that $p \cap S = \emptyset$. Generalize [Samuel, Proposition 5.1.5] to prove that for any $n \geq 0$, the natural homomorphism

$$A/p^n \longrightarrow A'/((pA')^n)$$

is an isomorphism.

   b) Let $a \subset A$ be a nonzero contracted ideal. Prove that the natural homomorphism

$$A/a \longrightarrow A'/aA'$$

is an isomorphism.

3. Let $A$ be a Dedekind domain and let $a \subset A$ be a nonzero proper ideal.

   a) Let $S = 1 + a$. Verify that $S$ is a multiplicatively closed subset, and show that for a prime ideal $p \subset A$, we have $p \cap S \neq \emptyset$ if and only if $p + a = A$.

   b) Prove that every ideal of $A/a$ is principal. [Localize at $S = 1 + a$. Show that $a$ is contracted and $S^{-1}A$ is semi-local, then use problem 2(b).]

   c) Prove that $a$ can be generated by two elements. [Apply (2) to the ideal $a/aA$ of $A/aA$ for any nonzero element $a \in a$.]

4. a) Let $A$ be a semi-local Dedekind domain with perfect fraction field $F$, let $K/F$ be a finite extension, and let $B$ be the integral closure of $A$ in $K$. Show $B$ is semi-local.

   b) Suppose that there were finitely many prime numbers. Use (a) to prove that $\mathbb{Z}[\sqrt{-5}]$ is a PID, and derive a contradiction. Thus there are infinitely many prime numbers.$^1$

5. Let $A$ be a Dedekind domain with perfect fraction field $F$. Let $K/F$ be a finite extension of degree $n$, and let $B$ be the integral closure of $A$ in $K$. Let $b, b'$ be nonzero non-zero.

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$^1$Brian Conrad attributes this ridiculous proof to Larry Washington.
ideals of $B$, with $b | b'$. Show that $N_{B/A}(b) | N_{B/A}(b')$, and that if $N_{B/A}(b) = N_{B/A}(b')$ then $b = b'$.

6. With the notation in (5), let $b$ be a nonzero ideal of $B$.

a) Prove that

$$N_{B/A}(b) = \left\{ N_{K/F}(x) \mid x \in b \right\}.$$  

[Localize at a nonzero prime ideal of $A$.]

b) Prove by example that $N_{B/A}(b)$ is not necessarily generated by the norms of a given set of generators for $b$. [Take $b = B$, and suppose that there are two distinct nonzero prime ideals of $B$ which contract to the same ideal of $A$.]