1. Let $K$ be a number field of degree $n = r_1 + 2r_2$. Recall that

$$\rho(K) := \left(\frac{4}{\pi}\right)^{1/2} \frac{n!}{n^b} |D_K|^{1/2}. $$

a) Show that for any integral ideal $a \subset \mathcal{O}_K$, if $a = N(a)$ then $a \mid (a)$.

b) Prove that $C(K)$ is generated by the classes of the nonzero prime ideals $p \subset \mathcal{O}_K$ with $N(p) \leq \rho(K)$.

c) Prove that any prime ideal in (b) is a prime factor of $(p)$ for $p \leq \rho(K)$ prime. Conclude that there are finitely many such prime ideals.

2. Let $x \in \mathbb{C}$ be a root of $f(X) = X^3 + 2X + 1$, and let $K = \mathbb{Q}(x)$. By problem 3 of homework 6 we know $\mathcal{O}_K = \mathbb{Z}[x]$ and $D_K = -59$.

a) Compute $\rho(K)$.

b) Find all prime ideals $p \subset \mathcal{O}_K$ with $N(p) \leq \rho(K)$. It may help to use the isomorphisms

$$\mathbb{Z}[x]_{(p)} \sim \mathbb{Z}[X]_{(p,f)} \sim \mathbb{F}_p[X]_{(f)}$$

for prime numbers $p$.

c) Prove that $#C(K) = 1$. [Look for elements of small norm.]

Thus $\mathbb{Z}[x]$ is a PID.

3. For $K = \mathbb{Q}(\sqrt{3})$ and $K = \mathbb{Q}(\sqrt{5})$, draw the image of $\mathcal{O}_K$ in $\mathbb{R}^2$ under the canonical embedding, and find all four fundamental units in $\mathcal{O}_K$. [Continued on next page...]
4. Let $K$ be a cubic field with $r_1 = r_2 = 1$. We identify $K$ with a subfield of $\mathbb{R}$ via its real embedding.

a) Prove that $\mathcal{O}_K^\times = \{\pm u^n \mid n \in \mathbb{Z}\}$ for some $u \in \mathcal{O}_K^\times$. We call $\pm u, \pm u^{-1}$ the primitive units. Show that there is a unique primitive unit $u > 1$, and that $K = \mathbb{Q}(u)$.

b) Let $x$ be the positive square root of $u$. Show that the conjugates of $u$ are $x^{-1}e^{\pm iy}$ for some real number $y$. Prove that

$$D(1, u, u^2) = -4\left[ \sin(2y) - (x^3 + x^{-3}) \sin(y) \right]^2.$$  

It is a fact\(^1\) that, for fixed $x > 1$, one has

$$\left[ \sin(2y) - (x^3 + x^{-3}) \sin(y) \right]^2 \leq x^6 + 6$$

for all $y \in \mathbb{R}$.

c) Prove that $|D_K| \leq 4u^3 + 24$. [Careful: $D_K \neq D(1, u, u^2)$ in general.]

5. For $K$ in problem 2, compute $\mathcal{O}_K^\times$. [Hint: $X^3 + 2X + 1$ has a root $x \sim -0.45340$.]

---

\(^1\)This is an exercise in single-variable calculus (finding the global maximum of a smooth periodic function), but I will be extremely impressed if you are able to prove it. Give it a try and see how good you really are at Calc I!