Math 4803/8803 Homework 9
Due at the beginning of class on Wednesday, October 28.

1. Prove that a Dedekind domain $A$ is a principal ideal domain if and only if it is a unique factorization domain. [To prove $\iff$, you actually only need to assume that any nonzero element $a$ is a product of prime elements.]

2. Let $A$ be a Dedekind domain with fraction field $K$ and let $a, b \subset K$ be fractional ideals, with duals $a^*, b^*$. Prove, without doing any computations, that $(ab)^* = a^*b^*$ and that $a^{**} = a$.

3. Let $K$ be a number field, let $p \subset \mathcal{O}_K$ be a nonzero prime ideal, and let $p$ be the prime number such that $p\mathbb{Z} = p \cap \mathbb{Z}$. Prove that the field $k = \mathcal{O}_K / p$ is a finite extension field of $\mathbb{F}_p$, and that $N(p) = \left[ k : \mathbb{F}_p \right]$.

4. Let $K = \mathbb{Q} (\sqrt{d})$ be a quadratic number field. Let $\tau \in \text{Gal}(K/\mathbb{Q})$ be the automorphism sending $\sqrt{d}$ to $-\sqrt{d}$, for $x \in K$ set $\overline{x} := \tau(x)$, and for an ideal $a \subset \mathcal{O}_K$ set $\overline{a} = \tau(a)$. It is a fact that $aa = (n)$ for a unique positive integer $n$. Prove that $n = N(a)$.

5. Let $A$ be any ring.
   a) Let $M$ be an $A$-module, let $a$ be an ideal contained in $\text{Ann}(M)$, let $\overline{A} = A / a$, and let $\pi : A \to \overline{A}$ be the quotient homomorphism. Prove that $M$ is an $\overline{A}$-module via the rule $\pi(a)m := am$.
   b) Let $M$ be an $A$-module and let $a \subset A$ be an ideal. Show that $a$ is contained in $\text{Ann}(M/aM)$. In particular, if $a$ is a maximal ideal then $M/aM$ is a vector space over $A/a$.
   c) Let $A$ be an integral domain, let $p \subset A$ be a maximal ideal, and suppose that $\dim_A (p/p^2) \geq 2$. Prove that $A$ is not Dedekind.
   d) Use (c) to give another proof that $K[X^2, X^3]$ is not Dedekind.
   e) Now let $A$ be a Dedekind domain, let $p \subset A$ be a nonzero prime ideal, and suppose that $[p]^2 = [A]$ in $C(A)$. Prove that $p$ can be generated by two elements.

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1The Zariski tangent space at the prime ideal $p$ is defined to be the vector space $p/p^2$. In geometric language, the condition $\dim_A (p/p^2) \geq 2$ says that $A$ is not nonsingular of dimension one at $p$.

2In fact, any ideal of a Dedekind domain can be generated by two elements, but this requires more commutative algebra background to prove. See Exercise 9.7 in Atiyah–MacDonald’s *Commutative Algebra*. 