Exercises in Samuel:
Chapter II #4.

Exercises not from the text:

(1) Let $A, B, C$ be rings, with $A$ a subring of $B$ and $B$ a subring of $C$.
   (a) Suppose that $B$ is a finitely generated $A$-module and $C$ is a finitely
       generated $B$-module. Prove that $C$ is a finitely generated $A$-module.
   (b) Suppose that $B$ is a finitely generated $A$-algebra and $C$ is a finitely
       generated $B$-algebra. Prove that $C$ is a finitely generated $A$-algebra.
   (c) Find an example of a ring which is a finitely generated $\mathbb{Q}$-algebra but
       not a finitely generated $\mathbb{Q}$-module.

(2) Let $\zeta = e^{2\pi i / 5}$. Use the proof of Theorem II.1(c $\implies$ a) to find an explicit
    equation of integral dependence for $\zeta + \zeta^2$ over $\mathbb{Z}$.

(3) Let $A$ be a subring of $B$, with $B$ integral over $A$. Prove that $B^\times \cap A = A^\times$.
    Show this is false in general without the integrality hypothesis.

(4) Let $A$ be a subring of $B$, such that the set $B \setminus A$ is closed under multiplication.
    Show that $A$ is integrally closed in $B$.

(5) Let $A$ be a ring, let $G$ be a finite group of automorphisms\(^1\) of $A$, and let
    $A^G = \{x \in A : \forall \sigma \in G, \sigma(x) = x\}$. Prove that $A$ is integral over $A^G$. [Hint:
    if $A$ is a field, this is a basic fact from Galois theory — how is it proved in
    that context?]

\(^1\)An automorphism of $A$ is an isomorphism from $A$ to itself. The set of all automorphisms forms a
   group under composition.