• There are 5 problems on this exam. Please solve at least 4 of them. If you solve all 5, I'll count the highest 4 scores toward your grade.

• Each problem is worth 20 points, for a maximum score of 80. There are five points of extra credit available on Problem 4.

• The exam is due on **Thursday, April 30, before 5pm**. You can either email me your solutions, or slip them under my office door.

• You may use your course notes and completed homework assignments, the textbook, and a graphing calculator. No other aids are permitted, and you are not allowed to discuss the problems with your classmates.

• All answers must be justified unless otherwise noted, and all proofs must be written in clear and grammatical English.

• You may cite any theorem, lemma, proposition, etc. proved in class or in the sections we covered in the text, in addition to any assigned homework problem.

• Good luck, and **start early**!
Problem 1.

Prove that the roots of the polynomial \( x^5 - 4x - 1 \in \mathbb{Q}[x] \) are not solvable by radicals.
Problem 2.

Let $d < 0$ be a squarefree integer which is congruent to 1 modulo 4. Let $\delta = \sqrt{d}$, $\eta = \frac{1}{2}(1 + \delta)$, and $h = \frac{1}{4}(1 - d)$, and let

$$f(x) = (x - \eta)(x - \overline{\eta}) = x^2 - x + h,$$

the minimal polynomial for $\eta$. Let $R = \mathbb{Z}[\eta]$, and suppose that $R$ is a unique factorization domain. Thus we know from Heegner’s theorem that $d \in \{-3, -7, -11, -19, -43, -67, -163\}$, but you may not use this fact as we haven’t proven it.

i. Prove that $N(\eta) = h$ and that $\eta$ has minimal norm among all elements of $R \setminus \mathbb{Z}$. [Draw a picture.]

ii. Prove that every prime integer $p < h$ is prime in $R$.

iii. Let $m$ be a positive integer with $m < h$. Prove that $f(m)$ is a prime integer. [Hint: first show $f(m) < h^2$.]

In particular, taking $d = -163$, the minimal polynomial is $f(x) = x^2 - x + 41$, and the forty values $f(1), f(2), f(3), \ldots, f(40)$ are all prime numbers!
Problem 3.

Let $\delta = \sqrt{-7}$ and let $R = \mathbb{Z}[\delta]$, the quadratic integer ring in $\mathbb{Q}(\delta)$. Calculate the class group of $\mathbb{Q}(\delta)$, and give representatives for all of the ideal classes.
Problem 4.

Let \( f(x) \in \mathbb{Q}[x] \) be an irreducible quartic polynomial with exactly two real roots, let \( K \subset \mathbb{C} \) be its splitting field, and let \( G = \text{Gal}(K/\mathbb{Q}) \leq S_4 \) be its Galois group.

i. Prove that \( G \) contains a transposition.

ii. Prove that \( G \) is \( S_4 \) or \( D_4 \).

iii. Find an example of such \( f \) where \( G = D_4 \). [Hint: we saw one in class during an extended example.]

iv. (Extra credit) Find an example of such \( f \) where \( G = S_4 \).
Problem 5.

Let $d = -23$, let $\delta = \sqrt{-23}$, let $\eta = \frac{1}{2}(1 + \delta)$, and let $R = \mathbb{Z}[\eta]$, the quadratic integer ring in $\mathbb{Q}(\delta)$.

i. Prove that $(2) = P\overline{P}$ for $P = (2, \eta)$.

ii. Prove that $P$ is not principal but $P^3$ is principal. [Hint: $N(1 + \eta) = 8$.]

iii. Prove that $\text{Cl}(\mathbb{Q}(\delta)) \cong C_3$.

iv. Prove that the cube of every fractional ideal in $\mathbb{Q}(\delta)$ is principal.