MATH 4108
FINAL EXAMINATION

Name	

1	2	3	4	5	Total

- There are 5 problems on this exam. Please solve **at least 4** of them. If you solve all 5, I'll count the highest 4 scores toward your grade.
- Each problem is worth 20 points, for a maximum score of 80. There are five points of extra credit available on Problem 4.
- The exam is due on **Thursday, April 30, before 5pm**. You can either email me your solutions, or slip them under my office door.
- You may use your course notes and completed homework assignments, the textbook, and a graphing calculator. No other aids are permitted, and you are **not** allowed to discuss the problems with your classmates.
- All answers must be justified unless otherwise noted, and all proofs must be written in clear and grammatical English.
- You may cite any theorem, lemma, proposition, etc. proved in class or in the sections we covered in the text, in addition to any assigned homework problem.
- Good luck, and **start early**!

Problem 1.

Prove that the roots of the polynomial $x^5 - 4x - 1 \in \mathbf{Q}[x]$ are not solvable by radicals.

Problem 2.

Let $d < 0$ be a squarefree integer which is congruent to 1 modulo 4. Let $\delta = \sqrt{d}$, $\eta = \frac{1}{2}(1+\delta)$, and $h = \frac{1}{4}(1-d)$, and let

$$f(x) = (x-\eta)(x-\overline{\eta}) = x^2 - x + h,$$

the minimal polynomial for η. Let $R = \mathbf{Z}[\eta]$, and *suppose that R is a unique factorization domain*. Thus we know from Heegner's theorem that $d \in \{-3, -7, -11, -19, -43, -67, -163\}$, but you may not use this fact as we haven't proven it.

i. Prove that $N(\eta) = h$ and that η has minimal norm among all elements of $R \setminus \mathbf{Z}$. [Draw a picture.]

ii. Prove that every prime integer $p < h$ is prime in R.

iii. Let m be a positive integer with $m < h$. Prove that $f(m)$ is a prime integer. [Hint: first show $f(m) < h^2$.]

In particular, taking $d = -163$, the minimal polynomial is $f(x) = x^2 - x + 41$, and the forty values $f(1), f(2), f(3), \ldots, f(40)$ are all prime numbers!

Problem 3.

Let $\delta = \sqrt{-17}$ and let $R = \mathbf{Z}[\delta]$, the quadratic integer ring in $\mathbf{Q}(\delta)$. Calculate the class group of $\mathbf{Q}(\delta)$, and give representatives for all of the ideal classes.

Problem 4.

Let $f(x) \in \mathbf{Q}[x]$ be an irreducible quartic polynomial with exactly two real roots, let $K \subset \mathbf{C}$ be its splitting field, and let $G = \mathrm{Gal}(K/\mathbf{Q}) \leq S_4$ be its Galois group.

i. Prove that G contains a transposition.

ii. Prove that G is S_4 or D_4.

iii. Find an example of such f where $G = D_4$. [Hint: we saw one in class during an extended example.]

iv. (**Extra credit**) Find an example of such f where $G = S_4$.

Problem 5.

Let $d = -23$, let $\delta = \sqrt{-23}$, let $\eta = \frac{1}{2}(1 + \delta)$, and let $R = \mathbf{Z}[\eta]$, the quadratic integer ring in $\mathbf{Q}(\delta)$.

i. Prove that $(2) = P\overline{P}$ for $P = (2, \eta)$.

ii. Prove that P is not principal but P^3 is principal. [Hint: $N(1 + \eta) = 8$.]

iii. Prove that $\mathrm{Cl}(\mathbf{Q}(\delta)) \cong C_3$.

iv. Prove that the cube of every fractional ideal in $\mathbf{Q}(\delta)$ is principal.