Recent Advances in Wavelet Analysis

Edited by

Larry L. Schumaker
Glenn Webb

Department of Mathematics
Vanderbilt University
Nashville, Tennessee

ACADEMIC PRESS, INC.

Harcourt Brace & Company, Publishers

Boston San Diego New York
London Sydney Tokyo Toronto

Two Recent Results on Wavelets: Wavelet Bases for the Interval, and Biorthogonal Wavelets Diagonalizing the Derivative Operator

Ingrid Daubechies

Abstract. The following two questions are often asked by researchers interested in applying wavelet bases to concrete numerical problems:

1) how does one adapt a wavelet basis on $\mathbb{R}$ to a wavelet basis on an interval without terrible edge effects?

2) how does the wavelet transform deal with the derivative operator?

This paper reviews several answers to each of these questions, including some recent constructions and observations.

§1 Introduction

The construction of orthonormal wavelet bases or of pairs of dual, biorthogonal wavelet bases for $L^2(\mathbb{R})$ is now well understood. For the construction of orthonormal bases of compactly supported wavelets for $L^2(\mathbb{R})$, in particular, one starts with a trigonometric polynomial $m_0(\xi) = \sum_n c_n e^{-in\xi}$, satisfying $m_0(0) = 1$ and $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$, as well as some mild technical conditions. The corresponding scaling function ϕ and wavelet ψ are defined by

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi)$$

and

$$\hat{\psi}(\xi) = e^{-i\xi/2}\, \overline{m_0(\xi/2 + \pi)}\, \hat{\phi}(\xi/2).$$

The functions $\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$, $j, k \in \mathbb{Z}$, then constitute an orthonormal basis for $L^2(\mathbb{R})$. For fixed $j \in \mathbb{Z}$, the $\phi_{j,k}(x) = 2^{-j/2}\ \phi(2^{-j}x-k)$

Recent Advances in Wavelet Analysis
Larry L. Schumaker and Glenn Webb (eds.) pp. 237–258.

ISBN 0-12-632370-4

are an orthonormal basis for a subspace $V_j \subset L^2(\mathbb{R})$; the spaces V_j constitute a *multiresolution analysis*, meaning in particular that

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots,$$

with

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}),$$

and

$$\mathrm{Proj}_{V_{j-1}} f = \mathrm{Proj}_{V_j} f + \sum_{k\in\mathbb{Z}} \langle f,\ \psi_{j,k}\rangle\ \psi_{j,k}.$$

See Mallat [16], Meyer [17] or Daubechies [9,11] for more details. Smoothness for ψ implies that m_0 has to have a zero at π of sufficiently high multiplicity. More precisely,

$$\psi \in C^k(\mathbb{R}) \Longrightarrow \int dx\ x^\ell \psi(x) = 0, \quad \ell = 0, \ldots, k$$

$$\Updownarrow$$

$$\frac{d^\ell}{d\xi^\ell} m_0 \bigg|_{\xi=\pi} = 0, \quad \ell = 0, \ldots, k\ .$$

This in turn implies that m_0 has at least $2k$ non-zero coefficients.

By far the oldest example of such an orthonormal basis of compactly supported wavelets is the Haar basis, with $m_0(\xi) = \frac{1}{2}(1+e^{-i\xi})$. Other examples, with arbitrarily high smoothness, were constructed in Daubechies [9]. They correspond to m_0 of the type $m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N Q_N(\xi)$, where $Q_N(\xi)$ is a polynomial of order $N-1$ in $e^{-i\xi}$. The resulting ϕ and ψ have support width $2N-1$; their degree of smoothness increases linearly with N.

These smoother wavelets provide not only orthonormal bases for $L^2(\mathbb{R})$, but also unconditional bases for function spaces consisting of more regular functions. In particular (Meyer [17]), if $\psi \in C^r(\mathbb{R})$, then the $\phi_{0,k}$, $k \in \mathbb{Z}$ and $\psi_{-j,k}$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$, provide an unconditional basis for the function spaces $C^s(\mathbb{R})$, for all $s < r$. The reason why wavelet bases (unlike Fourier series) can provide unconditional bases for C^r-spaces is essentially that the wavelets ψ have vanishing moments. Imposing such vanishing moments is equivalent to requiring that any polynomial of degree less than or equal to $N-1$ can be written as a linear combination of the $\phi(x-n)$.

Except for the Haar basis, the basic wavelet in an orthonormal basis of compactly supported wavelets cannot have a symmetry or antisymmetry axis. Symmetry can be recovered, without giving up the compact support, if the orthogonality requirement is relaxed. In that case one builds two different (but related) multiresolution hierarchies of spaces, $\ldots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots$ and $\ldots \tilde{V}_2 \subset \tilde{V}_1 \subset \tilde{V}_0 \subset \tilde{V}_{-1} \subset \tilde{V}_{-2} \subset \ldots$, corresponding to two scaling functions ϕ and $\tilde{\phi}$ and two wavelets ψ and $\tilde{\psi}$. They are defined by means of two trigonometric polynomials m_0 and $\tilde{m}_0$, satisfying

$$m_0(\xi)\ \overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\ \overline{\tilde{m}_0(\xi+\pi)} = 1.$$

We then have

$$\begin{aligned}\hat{\phi}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi),\\ \widehat{\tilde{\phi}}(\xi) &= (2\pi)^{-1/2} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi),\end{aligned}$$

and

$$\begin{aligned}\hat{\psi}(\xi) &= e^{-i\xi/2}\overline{\tilde{m}_0(\xi/2+\pi)}\hat{\phi}(\xi/2),\\ \widehat{\tilde{\psi}}(\xi) &= e^{-i\xi/2}\overline{m_0(\xi/2+\pi)}\widehat{\tilde{\phi}}(\xi/2).\end{aligned}$$

Under some extra technical conditions the $\psi_{j,k}$ and the $\tilde{\psi}_{j,k}$ constitute dual Riesz bases for $L^2(\mathbb{R})$, *i.e.*, $\langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = \delta_{j,j'}\delta_{k,k'}$. For proofs and examples, see Cohen, Daubechies and Feauveau [6]. There exist two possibilities leading to symmetry for ψ, $\tilde{\psi}$: 1) if $m_0, \tilde{m}_0$ have an even number of coefficients, then $\phi(x)$ is symmetric, and ψ is antisymmetric around $x = 1/2$, 2) if m_0 and $\tilde{m}_0$ have an odd number of coefficients, then ϕ and ψ are both symmetric, $\phi(x)$ around $x = 0$, $\psi(x)$ around $x = 1/2$. Smoothness for these "biorthogonal" wavelet bases again requires vanishing moments; we have now

$$\begin{aligned}\psi \in C^k(\mathbb{R}) \Longrightarrow \int dx\ x^\ell \tilde{\psi}(x) = 0, \quad \ell = 0, \ldots, k\\ \Updownarrow \qquad\qquad\\ \left.\frac{d^\ell}{d\xi^\ell} m_0\right|_{\xi=\pi} = 0, \quad \ell = 0, \ldots, k\,.\end{aligned}$$

Because wavelet bases have many mathematical properties and are associated with fast algorithms that are easy to implement, they are now being tried out for a host of applications. For many of these applications the constructions sketched above are not quite sufficient. In particular, one is often interested in problems confined to an interval rather than the whole line. Examples are numerical analysis (with boundary conditions at the edges of the interval), or image analysis (where the domain of interest is the cartesian product of two intervals). In this case, it is necessary to adapt the wavelet basis construction to "life on an interval". We review some of the "standard" ways of solving this problem, and point out some of their shortcomings. From the mathematical point of view, the construction by Y. Meyer [18] was the first satisfactory answer, in the sense that his construction led to interval-wavelets that were still unconditional bases for many function spaces, including the Hölder spaces. For the special case of semi-orthogonal spline wavelets, another construction was given in Chui and Quak [5]. This construction exploits the possibility of defining splines with multiple knots to take care of edge effects. A technique to incorporate boundary conditions into Meyer's construction was proposed in Auscher [1]. From a numerical point of view, Meyer's construction has some shortcomings; for this reason a different construction with the same nice mathematical properties was proposed independently but more or less simultaneously by B. Jawerth, by P.-G. Lemarié-Rieusset, and by A. Cohen, I. Daubechies and P. Vial. We explain this construction and its properties in the first part of the paper.

The second topic of this paper concerns how to deal with the derivative operator when working with wavelet bases. An expansion into wavelets, with their different scales, can often be used as an alternative to the Fourier expansion, with the added advantage of localization. A drawback is that the derivative operator, so easy (diagonal!) in the Fourier expansion, is less trivial when working with wavelets. Here too several approaches have been proposed; we review a straightforward approximation argument, as well as the observation by G. Beylkin that one can in fact use the "filter coefficients" associated with compactly supported orthonormal wavelet bases, to write the matrix for the derivative operator explicitly, even though no analytic expression is known for the wavelets themselves: If one is willing to give up orthonormality, then a recent observation by P.-G. Lemarié-Rieusset [14] gives an even simpler answer: there exist explicit biorthogonal wavelet bases in which the derivative is diagonal. The original observation appears in a paper in French where it is exploited to construct divergence-free wavelet bases for n-dimensional vector fields; it deserves to be known more widely, and this is the goal of the second part of this paper.

§2 Wavelets on the interval

Let us assume that the interval is $[0,1]$. It is very easy to restrict the Haar basis for $L^2(\mathbb{R})$ to a basis for $L^2([0,1])$; starting from the collection $\{\phi_{0,k};\ k \in \mathbb{Z}\} \cup \{\psi_{j,k};\ j \le 0, k \in \mathbb{Z}\}$, which is an orthonormal basis for $L^2(\mathbb{R})$, it suffices to take the restrictions of these functions to $[0,1]$; since every one of these functions has support either within $[0,1]$ or completely outside $]0,1[$, the resulting collection $\{\phi_{0,0}\} \cup \{\psi_{j,k};\ j \le 0,\ \ 0 \le k \le 2^{-j}-1\}$ is an orthonormal basis for $L^2([0,1])$. Things are not so trivial when one starts from smoother wavelet bases on the line. Assume that both ϕ and ψ have support width $2N-1$. In order to avoid having to deal with the two edges of $[0,1]$ at the same time, we can choose to start from the basis $\{\phi_{-j_0,k};\ k \in \mathbb{Z}\} \cup \{\psi_{j,k};\ j \le -j_0, k \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$, where $2^{j_0-1} \ge N$ so that none of the functions has support straddling both 0 and 1. Even so there will be $2N-2$ wavelets, at every resolution level and at every end of $[0,1]$, that straddle an endpoint, so that their support is neither completely in $[0,1]$ nor completely in $\mathbb{R}\setminus\ ]0,1[$. It is not a priori clear how to adapt them in such a way that the result is an orthonormal basis of $L^2([0,1])$.

Several solutions have been proposed for this problem. They all correspond to different choices of how to adapt the multiresolution hierarchy to the interval $[0,1]$.

Extending by zeros

This solution consists in not doing anything at all. A function f supported on $[0,1]$ can always be extended to the whole line by putting $f(x) = 0$ for $x \notin [0,1]$. This function can then be analyzed by means of the wavelets on the whole real line. There are two things wrong with this naive approach. First of all, this kind of extension typically introduces a discontinuity in f at $x = 0$ or 1 (except in those special cases where $f(x)$ already tends to 0 smoothly for $x \to 0$ and $x \to 1$), which will be reflected by "large" wavelet coefficients for fine scales (*i.e.*, wavelet coefficients which do not decay very fast) near the two edges, even if f itself is very smooth on $[0,1]$. The second "bad" aspect is that this approach uses "too many" wavelets. At scale $-j$, one finds $\langle f, \psi_{-j,k}\rangle \neq 0$ for typically $2^j + 2N - 1$ wavelets; intuitively one should have to use only 2^j wavelets, at scale $-j$, when looking at problems on $[0,1]$.

Periodizing

In this method, one expands a function f on $[0,1]$ into "periodized" wavelets defined by

$$\psi^{\text{per}}_{-j,k}(x) = \begin{cases} 2^{j/2} \sum_{\ell\in\mathbb{Z}} \psi(2^j x + 2^j \ell - k), & j \geq j_0 \geq 0 \\ 0, & j < 0, \end{cases}$$

for $0 \leq k \leq 2^j - 1$. These wavelets have to be supplemented by lowest resolution scaling functions $\phi^{\text{per}}_{-j_0,k}$, defined analogously; the result is an orthonormal basis of $L^2([0,1])$, associated with a multiresolution analysis in which V^{per}_{-j} is spanned by the $\phi^{\text{per}}_{-j,k}$. One now has exactly 2^j wavelets at scale $-j$, as well as 2^j scaling functions $\phi^{\text{per}}_{-j,k}$ in every V^{per}_{-j}. Since

$$\int_0^1 dx\ f(x)\ \psi^{\text{per}}_{-j,k}(x) = \int_{-\infty}^{\infty} dx \left[\sum_{\ell} f(x+\ell)\right]\ \psi_{-j,k}(x)\ ,$$

expanding a function on $[0,1]$ into periodized wavelets is equivalent to extending the original function into a periodic function with period 1 and analyzing this extension with the standard whole-line wavelets. Unless f was already periodic, this construction again introduces a discontinuity at $x = 0$, $x = 1$, which will show up as slow decay in the fine scale wavelet coefficients pertaining to the edges. Again, it will be impossible to characterize the one-sided regularity of f at 0 or 1 by looking at the decay of the $|\langle f, \psi^{\text{per}}_{j,k}\rangle|$ for $j \to -\infty$, unless f is periodic.

Reflecting at the edges

In this method, one extends the function f on $[0,1]$ by mirroring it at 0 and 1; beyond -1 and 2 one mirrors once more, and so on. The full extension is then defined by

$$f(x) = \begin{cases} f(2n - x), & 2n - 1 \leq x \leq 2n, \\ f(x - 2n), & 2n \leq x \leq 2n + 1. \end{cases}$$

If the original function on $[0,1]$ is continuous, then this extension will be continuous. Typically, however, the derivative of the extension has discontinuities at the integers. Expanding the "reflected" extension of a function on $[0,1]$ in a whole-line-basis of wavelets is equivalent to expanding the original function on $[0,1]$ with respect to "folded" wavelets $\psi_{j,k}^{\text{fold}}$ defined on $[0,1]$ by

$$\psi_{j,k}^{\text{fold}}(x) = \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(x-2\ell) + \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(2\ell - x) \ .$$

Starting from an orthonormal wavelet basis, this folding typically does not lead to an orthonormal wavelet basis on $[0,1]$. If ψ_{jk}, $\tilde{\psi}_{jk}$ are two biorthogonal wavelet bases, with ψ and $\tilde{\psi}$ both symmetric or antisymmetric around $1/2$, then their folded versions turn out to be still biorthogonal on $[0,1]$. The resulting biorthogonal multiresolution analysis hierarchies on $[0,1]$ have 2^j+1 (symmetric case) or 2^j (antisymmetric case) scaling functions and 2^j wavelets at resolution level j. Because the "reflected" extension typically has a discontinuous derivative, again we can not expect to characterize arbitrary regularity of f by means of the wavelet coefficients; decay of the $\langle f,\ \psi_{jk}^{\text{fold}}\rangle$ can characterize up to Lipschitz regularity (a gain over the two previous "solutions"), but not more, although one can do a little better by using two different pairs of biorthogonal bases. Explicitly, if the "original" (unfolded) ψ,ϕ are in C^r with $r>1$, one finds that a function f on $[0,1]$ is in $C^s([0,1])$, with $0<s<1$, if and only if

$$\sup_{\substack{j\geq 0 \\ 0\leq k\leq 2^j-1}} \quad 2^{j(s+1/2)} \left|\left\langle f, \tilde{\psi}_{-j,k}^{\text{fold}} \right\rangle\right| < \infty \ .$$

(For $s=1$ a similar result holds, with C^1 replaced by a Zygmund-type space.) As usual, the "only if" part follows from

$$\int_0^1 dx\ \tilde{\psi}_{-j,k}^{\text{fold}}(x) = \int_{-\infty}^{\infty} dx\ \tilde{\psi}_{-j,k}(x) = 0,$$

while the "if" part follows from the smoothness of ψ. If one tries to see what goes wrong if $s>1$, say $1<s<2$, then the "only if" part would require $\int_0^1 dx\ x\tilde{\psi}_{-j,k}^{\text{fold}} = 0$, the "if" part $\psi \in C^r$ with $r>s$. The first requirement is equivalent to $\int_{-1}^1 dx\ (1-|x|)\tilde{\psi}(x-\ell) = 0$ for all $\ell\in\mathbb{Z}$, which is only possible if ϕ is the tent function

$$\phi(x) = \begin{cases} 1-|x|, & |x|\leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

But then $\psi \notin C^1$, and the "if" part fails. One can, however, characterize $f\in C^s$, $1<s<2$ if one uses two pairs of biorthogonal wavelet bases, one for the "if" part, one for the "only if" part. Values of $s>2$ cannot be attained. For more details, see Cohen, Daubechies and Vial [8].

The construction of Y. Meyer

A fourth solution was proposed in Meyer [18]. The starting point of this construction is any one of the compactly supported bases in Daubechies [9], with N vanishing moments, and support ψ = support $\phi = [-N+1, N]$. The basis on $[0,1]$ constructed by Y. Meyer is derived from a multiresolution analysis that "lives" on $[0,1]$. At sufficiently fine scales, the approximation spaces $V_{-j}^{[0,1]}$ consist of $2^j - 2N + 2$ "interior" functions, $2N-2$ "left edge" functions, and $2N-2$ "right edge" functions. The complement spaces $W_{-j}^{[0,1]}$ are generated by $2^j - 2N - 2$ "interior" wavelets, $N-1$ "left edge" wavelets, and $N-1$ "right edge" wavelets. The total number of wavelets at scale j is thus 2^j, but the total number of scaling functions is larger, 2^j+2N-2. The "interior" functions are simply those $\psi_{-j,k}$ or $\phi_{-j,k}$ (as they were defined on the whole line) which happen to have their support contained in $[0,1]$. The "edge" functions have to be constructed explicitly. In particular, the left edge functions $\phi_{-j,k}^{\text{left}}$ are obtained by orthonormalizing the $(2N-2)$ restrictions $\phi_{-j,k}|_{[0,1]}$ where k is chosen so that $0 \in$ interior support$(\phi_{-j,k})$. The right edge scaling functions are obtained similarly; the edge wavelets can then be computed from projections of those $\psi_{-j,k}|_{[0,1]}$ which straddle 0 or 1 and for which more than half the support is within $[0,1]$. For details, see Meyer [18]. The result of the construction is an orthonormal family of wavelets in $[0,1]$, with N vanishing moments, and the same regularity as the original ψ. Together with an orthonormal family of scaling functions on $[0,1]$ at the coarsest scale under consideration, these adapted wavelets constitute an orthonormal basis for $L^2([0,1])$. In addition, their regularity and vanishing moment properties ensure that they are unconditional wavelet bases for the Hölder spaces $C^s([0,1])$ for all $s < r$, where r is the regularity of the original wavelet basis, $\psi \in C^r$. In order to implement the scheme, all the orthonormalization and projection matrices have to be computed explicitly. This involves the computation of integrals of the type

$$\int_0^\infty dx\ \phi(x+k)\phi(x+\ell) \quad \text{with} \quad -N+1 < k, \ell < N \ .$$

Using the refinement equation for ϕ, these can be computed by solving an $(N-1)(2N-3)$ dimensional linear system. This system is however very badly conditioned, because, *e.g.*, $\int_0^\infty dx\ |\phi(x-N+2)|^2 >>> \int_0^\infty dx\ |\phi(x+N-1)|^2$.

The disparity among the $\int_0^\infty dx\ |\phi(x+k)|^2$ also expresses itself in other ways. One application of wavelet bases and multiresolution on the interval is the "natural" extension of functions living on the interval to functions on the whole line. Since the edge-wavelets and scaling functions can all be written as linear combinations of restrictions of whole-line functions, one can extend them trivially by "gluing on their tails again", *i.e.*, by replacing every $\phi_{-j,k}|_{[0,1]}$ by $\phi_{-j,k}$. If this is done for every edge term in the expansion of a function f on $[0,\infty)$, the result is a smooth function f^{ext} extending f to $\mathbb{R}$, with the appealing property that high frequency components in f spread out less to $(-\infty, 0]$ than low frequency components. At any scale j, the extension is limited to $[-2^{-j}(2N-2), \infty)$. This doesn't work so well in practice, however: the extension of those edge scaling functions that are obtained from restricting $\phi_{-j;k}$ to $[0,1]$ which have only a tiny piece of their support in $[0,1]$ can have a huge amplitude outside $[0,1]$. This is the reason why B. Jawerth, in an application involving such extension operators for surface design in collaboration with B. Dahlberg, decided to develop a construction different from Meyer's. Another instance where one can feel the imbalance among the $\phi_{0,k}^{\text{half}}$ is in the plots of the edge functions. Typically, $\phi_{0,-N+1}|_{[0,\infty)}$ has much faster high amplitude oscillations than ϕ itself (the same oscillations are of course present in the tail of ϕ, but with exceedingly small amplitude); because of the orthonormalization procedure, this oscillatory behavior spreads to several edge scaling functions. Figures 1 and 2 show the edge scaling functions for $N = 2$ and $N = 4$, at the left side of the interval $[0,1]$; they illustrate this oscillatory behavior.

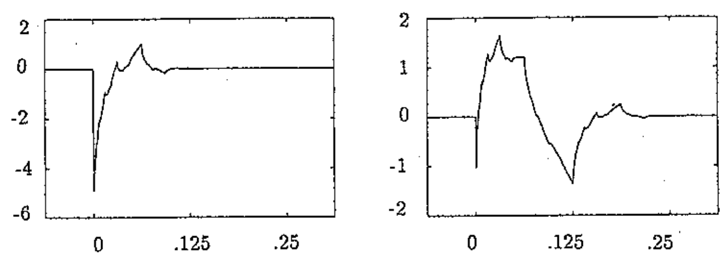

Figure 1. The adapted scaling functions in $V_{-3}^{[0,1]}$ at the left edge in the construction of Y. Meyer for $N = 2$.

One can check that Meyer's idea can be used also for biorthogonal wavelets. If one wants to use the interval wavelets to solve a differential equation (or in higher dimensions, a partial differential equation) with boundary conditions, then it is convenient to use a wavelet basis where all the wavelets themselves already satisfy the boundary conditions. Auscher [1] shows how Meyer's construction can be adapted to achieve this.

A different construction of interval wavelets

We now look at a fifth solution, also derived from compactly supported wavelet bases for $\mathbb{R}$. Like Meyer's construction, it uses "interior" and "edge" scaling functions at every resolution. We introduce fewer edge functions however, tailoring them so that the total number is exactly 2^j at resolution j. Moreover, as in Meyer's case, all the polynomials on $[0,1]$ of degree $\leq N-1$ can be

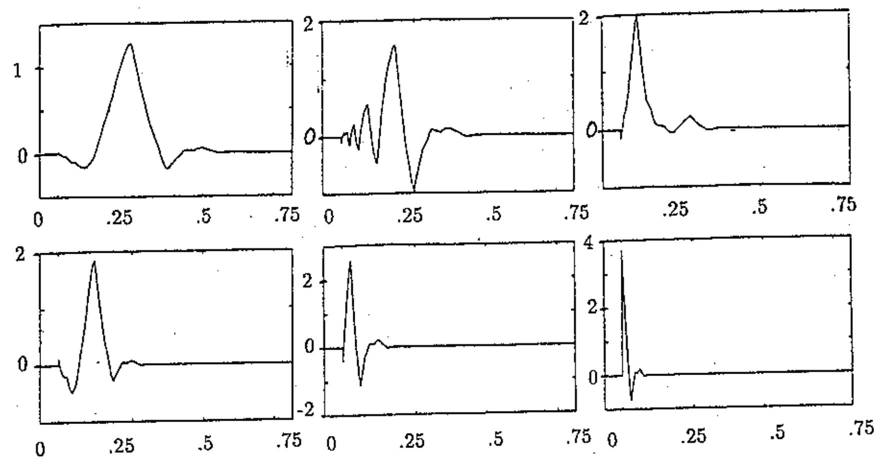

Figure 2. The adapted scaling functions in $V_{-3}^{[0,1]}$ at the left edge in the construction of Y. Meyer for $N = 4$.

written as linear combinations of the scaling functions at any fixed scale. It then follows that all the corresponding wavelets, at the edge as well as in the interior, have N vanishing moments, and this is sufficient to ensure that we have again unconditional bases for the $C^s([0,1])$-spaces, with $s < r$ if $\psi \in C^r$.

This construction was proposed independently and around the same time (1990-91) by B. Jawerth (to obtain better extension operators — see the paper by Andersson *et al.* in this volume for this and other applications), by A. Jouini and P.-G. Lemarié [13] (who also adapted the new construction to the treatment of boundary conditions), and by A. Cohen, I. Daubechies and P. Vial [8] (who wanted more symmetry between numbers of wavelets and numbers of scaling functions at every resolution, and who sought to avoid the oscillatory behavior of Meyer's edge functions). A joint announcement by two of these groups was made in Cohen, Daubechies, Jawerth and Vial [7]. A related but different construction, from the point of view of filter construction rather than with the goal of obtaining wavelets, is in Herley, Kovačević, Ramchandran and Vetterli [12].

Our starting point is again the N vanishing moment family of Daubechies [9], or a variant (see Daubechies [10,11]). We choose to translate them so that $\text{support}(\phi) = \text{support}(\psi) = [-N+1, N]$. Our goal is to retain the interior scaling functions, and to add adapted edge scaling functions in such a way that their union still generates all polynomials on $[0,1]$, up to a certain degree. Let us illustrate the principle of the construction by working on the half line $[0,\infty)$ instead of on $[0,1]$; we then only have to deal with the left edge, and it doesn't matter at which scale we work. The "interior" scaling functions at scale 0 are the $\phi_{0,k}$ with $k \geq N-1$; their support is contained in $[0,\infty)$. By themselves, the interior $\phi_{0,k}$ do not even generate the constants on $[0,\infty)$, as is clear from $\phi_{0,k}(0)=\phi(-k)=0$ for all $k \geq N-1$. Let us therefore add the constants "by hand". We define an edge function ϕ^0 by

$$\phi^0(x) = 1 - \sum_{k=N-1}^{\infty} \phi(x-k).$$

The interior $\phi_{0,k}$ and this edge function ϕ^0 together generate all the constants on $[0,\infty)$. Moreover, because $\sum_{k=-\infty}^{\infty} \phi(x-k) = 1$, we also have

$$\phi^0(x) = \sum_{k=-\infty}^{N-2} \phi(x-k) = \sum_{k=N+1}^{N-2} \phi(x-k),$$

for $0 \leq x < \infty$, showing that ϕ^0 has compact support. It also shows, incidentally, that ϕ^0 is orthogonal to all the interior $\phi_{0,k}$. The only thing that we have to check is that by adding functions in this ad hoc way we don't leave the framework of a multiresolution hierarchy. We have, however,

$$\phi(x-k) = \sqrt{2} \sum_{\ell=2k-N+1}^{N+2k} h_{\ell-2k}\, \phi(2x-\ell),$$

and

$$\begin{aligned}\phi^0 &= \phi^0(2x) + \sum_{\ell=N-1}^{\infty} \phi(2x-\ell) \left[1 - \sqrt{2} \sum_{k=N-1}^{\infty} h_{\ell-2k}\right] \\ &= \phi^0(2x) + \sum_{\ell=N-1}^{3N-4} \phi(2x-\ell) \left[1 - \sqrt{2} \sum_{k=\lceil(\ell-N)/2\rceil}^{\lfloor(\ell+N-1)/2\rfloor} h_{\ell-2k}\right],\end{aligned}$$

where we have used that $h_n = 0$ for $n < -N+1$ or $n > N$ and $\sum_n h_{2n} = \frac{1}{\sqrt{2}} = \sum_n h_{2n+1}$. It follows therefore that

$$\begin{aligned}V_0^{\text{left}} &= \overline{\text{Span}\,\{\phi^0,\ \phi_{0,k};\ k \geq N-1\}} \\ &\subset \overline{\text{Span}\,\{\phi^0(2\cdot),\ \phi_{-1,k};\ k \geq N-1\}} = V_{-1}^{\text{left}}.\end{aligned}$$

Similar inclusions hold immediately if we scale by other integer powers of 2, and we still have a hierarchy of nested spaces.

This is essentially all there is to the construction proposed here. If we want the set of edge plus interior scaling functions to generate more polynomials than only the constants, then we have to add, by hand, more edge functions (for the polynomials up to degree L, we add a total $L+1$ functions). If we work on the interval, then the same has to be done at the other edge as well. For many applications, it is desirable to have exactly 2^j scaling functions of scale j when working on $[0,1]$. Let us figure out how much room this leaves us for adding extra functions at the edges. If we start from a minimal support N-vanishing moment wavelet, then $\text{support}(\phi) = [-N+1, N]$, and for j sufficiently large we have exactly $2^j - 2N + 2$ interior scaling functions at scale j. This leaves room for adding $N-1$ ad hoc functions at each edge, so that the total family can generate polynomials of degree at most $N-2$. The unaltered whole-line scaling functions can generate all polynomials up to degree $N-1$, so that we seem to have "lost" one degree. In order to recover this one extra degree (and so be able to characterize the $C^s([0,1])$ spaces for the same range of s as we could on all of $\mathbb{R}$), we have to make room for one extra function at each edge of the interval. For this reason we abandon the two outermost interior scaling functions (one at each end of $[0,1]$), which corresponds to retaining only the $\phi_{0,k}$ with $k \geq N$ rather than $k \geq N-1$ on the half line. More precisely, we define the N edge functions $\tilde{\phi}^k$, $k = 0, \ldots, N-1$, on $[0,\infty)$ by

$$\tilde{\phi}^k(x) = \sum_{n=0}^{2N-2} \binom{n}{k} \phi(x+n-N+1) . \tag{1}$$

These are all compactly supported, and their supports are staggered, *i.e.*, $\text{support}(\tilde{\phi}^k) = [0, 2N-1-k]$; they are independent, and orthogonal to the $\phi_{0,m}$, $m \geq N$. Together with the $\phi_{0,m}$, $m \geq N$, they generate all the polynomials up to degree $N-1$ on $[0,\infty)$. Finally, there exist constants $a_{k,\ell}$, $b_{k,m}$ (which can be computed explicitly) so that

$$\tilde{\phi}^k(x) = \sum_{\ell=0}^{k} a_{k,\ell}\, \tilde{\phi}^\ell(2x) + \sum_{m=N}^{3N-2-2k} b_{k,m}\, \phi(2x-m) . \tag{2}$$

For proofs, see, *e.g.*, Cohen, Daubechies and Vial [8].

One can obtain an orthonormal basis for V_0^{left} by orthonormalizing the $\tilde{\phi}^k$, since they are already orthogonal to the orthonormal $\phi_{0,m}$; scaling them leads to an orthonormal basis for every V_j^{left}. If one orthonormalizes by a Gram-Schmidt procedure, starting with $\tilde{\phi}^{N-1}$, and working down to lower values of k, then the resulting orthonormal ϕ_k^{left}, $k = 0, \ldots, N-1$, still have staggered supports: $\text{support}(\phi_k^{\text{left}}) = [0, N+k]$. To carry out the Gram-Schmidt orthonormalization explicitly, we again need the overlap matrix $\langle \tilde{\phi}^k, \tilde{\phi}^\ell \rangle$. To compute this overlap matrix, we use the recurrence (2). For $k = 0$, for instance, we have

$$\|\tilde{\phi}^0\|^2 = a_{0,0}^2 \frac{1}{4} \|\tilde{\phi}^0\|^2 + \sum_{m=N}^{3N-2} b_{0,m}^2 \frac{1}{4} ,$$

from which we obtain $\|\tilde{\phi}^0\|^2$. It then follows that

$$\left\langle \tilde{\phi}^0, \tilde{\phi}^1 \right\rangle = a_{0,0}\, a_{1,0}\, \frac{1}{4} \|\tilde{\phi}^0\|^2 + a_{0,0}\, a_{1,1}\, \frac{1}{4} \left\langle \tilde{\phi}^0, \tilde{\phi}^1 \right\rangle + \frac{1}{4} \sum_{m=N}^{3N-4} b_{0,m}\, b_{1,m}$$

leading to an explicit formula for $\langle\tilde{\phi}^0, \tilde{\phi}^1\rangle$, since $\|\tilde{\phi}^0\|^2$ is known. One proceeds similarly for higher values of k; all this amounts to inverting a triangular matrix, and no ill-conditioning occurs.

The orthonormal $\phi_k^{\rm left}$, constructed with staggered supports along the lines indicated above, satisfy a recursion relation similar to (2) and inherited by all the scales j. Explicitly, there exist constants $H_{k,\ell}^{\rm left}$ and $h_{k,m}^{\rm left}$ (which can be computed explicitly from the $a_{k,\ell}$, $b_{k,\ell}$ in (2) and the orthonormalization procedure) such that

$$\phi_{-j,k}^{\rm left} = \sum_{\ell=0}^{N-1} H_{k,\ell}^{\rm left}\, \phi_{-j-1,\ell}^{\rm left} + \sum_{m=N}^{N+2k} h_{k,m}^{\rm left}\, \phi_{-j-1,m}\ . \qquad (3)$$

All this was on the half line. If we work on the interval $[0,1]$, and we start with a scale fine enough so that the two edges don't interact, *i.e.*, $2^j \geq 2N$, then there are $2^j - 2N$ interior scaling functions $\phi_{-j,N}, \ldots, \phi_{-j,2^j-N-1}$, and we add N functions at each end, following the principles outlined above. Together, these 2^j orthonormal functions span $V_{-j}^{[0,1]}$.

We now turn to the wavelets rather than the scaling functions. As usual, we define $W_{-j}^{[0,1]} = V_{-j-1}^{[0,1]} \cap \left(V_{-j}^{[0,1]}\right)^{\perp}$. From dimension counting, it immediately follows that $\dim W_{-j}^{[0,1]} = 2^j$. On the other hand it is easy to check that the $2^j - 2N$ functions $\psi_{-j,m}$, $m = N, \ldots, 2^j - N - 1$ are all in $W_{-j}^{[0,1]}$. Since they are all orthonormal, we therefore need to add an extra $2N$ wavelets (N at each edge) to provide an orthonormal basis for $W_{-j}^{[0,1]}$. How should they be constructed? To simplify notation, we return to the half line $[0,\infty)$. We define there $W_j^{\rm left} = V_{j-1}^{\rm half} \cap (V_j^{\rm half})^{\perp}$. The $\psi_{j,m}$, $m \geq N$ all belong to $W_j^{\rm left}$, and we are looking for N extra functions in $W_j^{\rm left}$, orthonormal to these $\psi_{j,m}$. Define

$$\tilde{\psi}^k = \phi_{-1,k}^{\rm left} - \sum_{m=0}^{N-1} \left\langle \phi_{-1,k}^{\rm left},\, \phi_{0,m}^{\rm left} \right\rangle \phi_{0,m}^{\rm left}\ . \qquad (4)$$

Then the $\tilde{\psi}^k$ are N independent functions in $W_0^{\rm left}$, orthogonal to the $\psi_{0,m}$, $m \geq N$. Because of the recursion relation (3), the $\tilde{\psi}^k$ can be written as a linear combination of $\phi_{-1,\ell}^{\rm left}$ and $\phi_{-1,m}$:

$$\tilde{\psi}^k = \sum_{\ell=0}^{k} c_{k,\ell}\, \phi_{-1,\ell}^{\rm left} + \sum_{m=N}^{3N-2} d_{k,m}\, \phi_{-1,m}\ . \qquad (5)$$

In a final step, these $\tilde{\psi}^k$ can be orthonormalized and we end up with an orthonormal family $\psi_k^{\rm left}$, $k = 0, \ldots, N-1$. It is possible to orthonormalize in such a way that the $\psi_k^{\rm left}$ have staggered supports, support$(\psi_k^{\rm left}) = [0, N+k]$. For any $j \in \mathbb{Z}$ we define again $\psi_{-j,k}^{\rm left}(x) = 2^{j/2}\, \psi_k^{\rm left}(2^j x)$; together with the $\psi_{-j,m}$, $m \geq N$, the $\psi_{-j,k}^{\rm left}$, $k = 0, \ldots, N-1$ provide an orthonormal basis for

W^{left}_{-j}; moreover, there exists constants $G^{\text{left}}_{k,\ell}$ and $g^{\text{left}}_{k,m}$ so that

$$\psi^{\text{left}}_{-j,k} = \sum_{\ell=0}^{N-1} G^{\text{left}}_{k,\ell}\, \phi^{\text{left}}_{-j-1,\ell} + \sum_{m=N}^{N+2k} g^{\text{left}}_{k,m}\, \phi_{-j-1,m} \,. \tag{6}$$

This completes our explicit construction, at least at a left end. The same has to be repeated at a right end. Combining the two leads to orthonormal bases for $W^{[0,1]}_{-j}$.

The result is an orthonormal basis for $L^2([0,1])$. If $\phi, \psi \in C^r$, this is also an unconditional basis for $C^s([0,1])$ for $s < r$. In particular, a bounded function f is in $C^s([0,1])$ if and only if

$$\left|\left\langle f,\ \psi^{\text{left}}_{-j,-k}\right\rangle\right|,\ \left|\left\langle f,\ \psi_{-j,m}\right\rangle\right|,\ \left|\left\langle f,\ \psi^{\text{right}}_{-j,2^j-N-k}\right\rangle\right| \le C 2^{-j(s+1/2)} \,,$$

where C is independent of j or m, k.

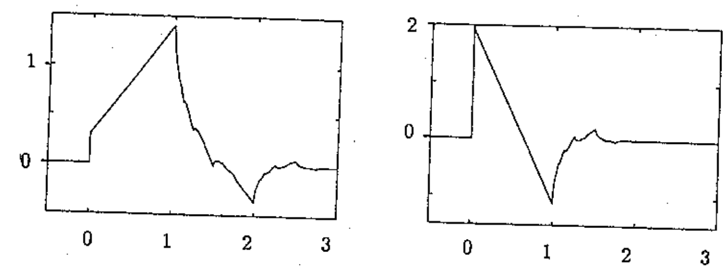

Figure 3. The adapted scaling functions in $V_0^{[0,\infty)}$ at the left edge in the new construction for $N = 2$.

Figures 3 and 4 plot the scaling functions for $N = 2$ and $N = 4$, at the left end of $[0,\infty)$. Note that, as on the whole line, we have no explicit analytic expression for the wavelets and scaling functions on the interval. For practical applications, all that is really needed are the filter coefficients. In addition to the h_m, $g_m = (-1)^m h_{2N+1-m}$, we now also have the $H^{\text{left}}_{-k,-\ell}$, $h^{\text{left}}_{-k,m}$, $G^{\text{left}}_{-k,-\ell}$, $g^{\text{left}}_{-k,m}$ from (3) and (6) (and the same on the right). Tables for these filter coefficients can be found in Cohen, Daubechies and Vial [8]. The adapted scaling functions in these plots are less oscillatory than those in Meyer's construction. On $[0,1]$ the N functions ϕ^{left}_k, $k = 0, \ldots, N-1$, are pure polynomials (of degree $N-1$). This is because all the scaling functions together on $[0,\infty)$ generate the polynomials up to degree $N-1$; since the interior scaling functions $\phi_{0,m}$, $m \ge N$, only start kicking in from $x \ge 1$ onward, the N adapted scaling functions have to be polynomials themselves.

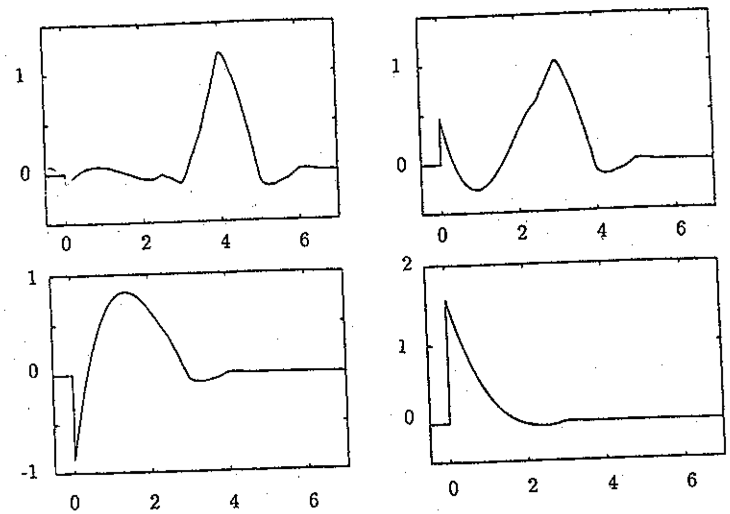

Figure 4. The adapted scaling functions in $V_0^{[0,\infty)}$ at the left edge in the new construction for $N = 4$.

Discussion of the new construction

Many variations are possible. One can, for instance, start from completely different families of whole-line wavelets, and adapt the number of additional edge scaling functions to their support width and to their number of vanishing moments.

We have assumed that we want the scaling functions to generate all possible polynomials up to a certain degree. If the interval wavelets are used to solve a differential equation, then it may be useful to adapt the construction so that all the scaling functions and wavelets involved satisfy certain prescribed boundary conditions. P. Auscher [1] adapted the original construction by Y. Meyer in this way; his scheme carries over entirely to the present construction (with more numerical stability). The construction by A. Jouini and P. G. Lemarié-Rieusset, which is essentially the same as ours, obtained independently, was carried out in view of this application.

The same ideas apply of course to biorthogonal wavelet bases. If one starts from a choice with (anti)symmetric wavelets and scaling functions, then the adapted scaling functions and wavelets at the right edge can be chosen to be the mirrors of their left edge equivalents. Since biorthogonality instead of orthonormality is wanted, there is more freedom in the choice of the edge functions, and one can optimize for extra criteria.

There is an important difference between wavelets on the line and wavelets on $[0,1]$, which results in the necessity, in at least some applications, to precondition the data (*e.g.*, an image) prior to their wavelet decomposition. Scaling functions on all of $\mathbb{R}$ have the property $\int dx\ \phi_{-j,k}(x) = 2^{-j/2}$, independently of k. A consequence of this is that the corresponding low pass filter preserves the sequence ...1111... . For the specially adapted scaling functions at the edge of $[0,1]$, we typically have $\int_0^1 dx\ \phi^{\text{edge}}_{-j,k}(x) \neq 2^{-j/2}$. The result is that the sequence invariant under low-pass filtering is not 111...111, but rather a sequence consisting of only 1's in the middle, but with different initial and final entries. Something similar happens for sequences corresponding to higher degree polynomials. In practical examples (*e.g.*, images) one still would like simple polynomial sequences like 1 1 1 1... or 1 2 3 4... to lead to a zero high-pass component, however. This can still be achieved if we perform a prefiltering on the data. The details of this scheme can be found in Cohen, Daubechies and Vial [8]. In the biorthogonal case, the extra freedom in the construction can be exploited so as to make this prefiltering unnecessary.

§3 Dealing with the derivative operator

The derivative operator is not diagonal in a wavelet basis. Nevertheless, compactly supported and reasonably smooth wavelets are well localized "on both sides of Fourier" (meaning that both $\psi(x)$ and $\hat{\psi}(\xi)$ are well localized), so that we expect the matrix of the derivative operator in such a wavelet basis to be sparse. More concretely, if $\hat{\psi}(\xi)$ is mainly localized in $\frac{\pi}{2} \leq |\xi| \leq 8\pi$, then we expect

$$\left\langle \frac{d}{dx}\ \psi_{j,k},\ \psi_{j',k'} \right\rangle \simeq 0$$

if $|j-j'| \geq 4$. Moreover, if $\text{support}(\psi) = [-N+1,\ N]$, then

$$\left\langle \frac{d}{dx}\ \psi_{j,k},\ \psi_{j',k'} \right\rangle = 0$$

for

$$-N(2^{j'-j}+1) + 2^{j'-j} \leq k - 2^{j'-j}k' \leq N(2^{j'-j}+1) - 1 \,.$$

Together, these two conditions mean that $\langle \frac{d}{dx}\ \psi_{j,k},\ \psi_{j',k'} \rangle$ is negligible unless (j,k) and (j',k') are "close".

This argument does not tell us how to compute the significant matrix elements, however. A first simple strategy for their computation can be found in Liandrat and Tchamitchian [15]: they worked with (truncated versions of) orthonormal cubic spline wavelets (from the Battle-Lemarié family), and approximated $\frac{d}{dx}\ \psi_{j,k}$ by

$$\frac{d}{dx}\ \psi_{j,k} = \sum_{\ell} \left\langle \psi'_{j,k},\ \phi_{j-2,\ell} \right\rangle \phi_{j-2,\ell}\ ,$$

which amounts to assuming that for $f \in W_j$, $\frac{d}{dx}f - \text{Proj}_{V_{j-2}} \frac{d}{dx} f \simeq 0$. Since $\langle \psi'_{j,k},\ \phi_{j-2,\ell} \rangle = 2^{-j} \langle \psi',\ \phi_{-2,\ell-2k} \rangle$, it suffices to compute the integrals $\int \psi'(x)\phi(4x-n)dx$; this was done numerically in Liandrat and Tchamitchian. For different wavelets it might be necessary to replace V_{j-2} by even finer scale spaces V_{j-L}, $L \geq 2$, in this approach.

Within the framework of orthonormal wavelets with compact support, G. Beylkin [2] makes the following interesting observation. Since

$$\phi(x) = \sqrt{2} \sum_{n=N_1}^{N_2} h_n \phi(2x-n) ,$$

it follows that the $a_k = \int \phi'(x)\, \phi(x-k)dx$, which differ from zero only if $|k| < N_2 - N_1$ (since support$(\phi) = [N_1, N_2]$), satisfy the equation

$$\begin{aligned} a_k &= 2 \sum_{n,m=N_1}^{N_2} h_n\, h_m\, a_{2k+m-n} \\ &= 2 \sum_{\ell=-(N_2-N_1)}^{N_2-N_1} \left(\sum_{n=N_1}^{N_2} h_n\, h_{\ell-2k+n} \right) a_\ell \; . \end{aligned}$$

The a_k can therefore be found by constructing the eigenvector with eigenvalue $\frac{1}{2}$ of the matrix A with entries

$$A_{k,\ell} = \sum_{n=N_1}^{N_2} h_n\, h_{\ell+n-2k},$$

for $|k|,\ |\ell| \le N_2 - N_1 - 1$. If $m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=N_1}^{N_2} h_n\, e^{-in\xi}$ has a zero at $\xi = \pi$ of multiplicity at least 2, then the matrix A does indeed have the eigenvalue 1/2, and it is nondegenerate. Moreover, Beylkin [2] also proves that

$$\sum_k k\, a_k = -1 \; .$$

(This is a direct consequence of $\sum_k \phi(x-k) = 1$ and $\sum_k (x-k)\phi(x-k) =$ constant.) This fixes the normalization of the a_k, so that they are uniquely determined, and can be computed directly from the h_n. Once the $\langle \frac{d}{dx}\, \phi_{0,0},\ \phi_{0,k} \rangle$ are known, all the $\langle \frac{d}{dx}\, \psi_{j,k},\ \psi_{j',k'} \rangle$ can be derived easily by the usual recurrences

$$\phi_{j,k} = \sum_n h_{n-2k}\, \phi_{j-1,k},$$

$$\psi_{j,k} = \sum_n g_{n-2k}\, \phi_{j-1,k} \; ,$$

with $g_n = (-1)^n\, h_{-n+1+2K}$, where K can be chosen so that the set of indices n where $h_n \neq 0$ coincides with the set where $g_n \neq 0$. Another way of viewing this is to start with a large array with entries $a_{k-\ell}$; successive high- and low-pass filterings on columns and rows (similar to the use of wavelets for image compression) then lead to the nonstandard form of the matrix for the derivative operator, *i.e.*, an array containing the entries

$$\left\langle \frac{d}{dx}\, \psi_{j,k},\ \phi_{j,k'} \right\rangle, \quad \left\langle \frac{d}{dx}\, \psi_{j,k},\ \psi_{j,k'} \right\rangle, \quad \left\langle \frac{d}{dx}\, \phi_{j,k},\ \psi_{j,k'} \right\rangle$$

for $j = 1, \dots J$, as well as the coarse scale $\langle \frac{d}{dx}\, \phi_{J,k}, \phi_{J,k'} \rangle$. Since

$$\left(\frac{d}{dx}\, \psi_{j,k}\right)(x) = 2^{-j} 2^{-j/2} \psi'(2^{-j}x - k),$$

we have $\langle \frac{d}{dx}\, \psi_{j,k}, \phi_{j,k'} \rangle = 2^{-j+1} \langle \frac{d}{dx}\, \psi_{1,k}, \phi_{1,k'} \rangle$ (and the same for other matrix elements), which means that we only need to compute one level. From the nonstandard form, one then obtains the standard form, *i.e.*, the array with entries

$$\left\langle \frac{d}{dx}\, \psi_{j,k},\ \psi_{j',k'} \right\rangle$$

by the usual additional filterings where needed (see Beylkin, Coifman and Rokhlin [3]). Note that this procedure leads to the *exact* values of the derivative operator matrix elements; the only approximation involved when this matrix is used consists in the choice of the finest scale space, implying an effective truncation of the derivative operator.

If one is willing to give up orthonormality, then an even more elegant representation of the derivative operator can be found, by using two special pairs of biorthogonal or dual bases of compactly supported wavelets. Such a pair of dual bases

$$\psi_{j,k}(x) = 2^{-j/2}\, \psi(2^{-j}x - k), \qquad \tilde{\psi}_{j,k}(x) = 2^{-j/2}\, \tilde{\psi}(2^{-j}x - k)\ ,$$

with corresponding scaling functions $\phi, \tilde{\phi}$, is defined by

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\xi)\ ,$$

$$\hat{\psi}(\xi) = e^{-i\xi/2}\, \overline{\tilde{m}_0\left(\frac{\xi}{2} + \pi\right)}\, \hat{\phi}\left(\frac{\xi}{2}\right)\ ,$$

with $\tilde{\phi}, \tilde{\psi}$ defined analogously (reverse the roles of m_0, $\tilde{m}_0$). Here m_0, $\tilde{m}_0$ are as described in §1. Many examples are constructed in Cohen, Daubechies and Feauveau [6]. In addition, there are also examples when m_0 and $\tilde{m}_0$ have the same (even) number of nonvanishing coefficients (C. Brislawn [4]). Duality is expressed by

$$\left\langle \psi_{j,k},\ \tilde{\psi}_{j',k'} \right\rangle = \delta_{j,j'} \delta_{k,k'}\ .$$

If ψ is in C^{L-1}, then $\tilde{\psi}$ must have L vanishing moments, *i.e.*,

$$\int dx\ \tilde{\psi}(x) x^{\ell} = 0, \qquad \ell = 0, \dots, L-1.$$

(This is a consequence of the duality — see, *e.g.*, Theorem 5.5.1 in Daubechies [11].) This implies that m_0 should be divisible by $(1 + e^{-i\xi})^L$; a similar statement holds of course if we reverse the roles of $\psi, \tilde{\psi}$ and $m_0, \tilde{m}_0$. In all interesting applications, m_0 and $\tilde{m}_0$ are symmetric, and can be written as

$$\begin{aligned} m_0(\xi) &= \left(\cos\frac{\xi}{2}\right)^L \; e^{-i\mathcal{K}\xi/2}\; P(\cos\xi), \\ \tilde{m}_0(\xi) &= \left(\cos\frac{\xi}{2}\right)^{\tilde{L}} \; e^{-i\mathcal{K}\xi/2}\; \tilde{P}(\cos\xi)\,, \end{aligned} \tag{7}$$

where

$$\mathcal{K} = \begin{cases} 1, & \text{if } L \text{ and } \tilde{L} \text{ are odd,} \\ 0, & \text{if } L \text{ and } \tilde{L} \text{ are even,} \end{cases}$$

(L and $\tilde{L}$ necessarily have the same parity). The polynomials P and $\tilde{P}$ satisfy the equation

$$(1+x)^K P(x)\tilde{P}(x) + (1-x)^K P(-x)\tilde{P}(-x) = 2^K\,, \tag{8}$$

where $2K = L + \tilde{L}$. The important thing to note here is that P and $\tilde{P}$ are determined solely by $L + \tilde{L}$; splitting up $2K$ into a different sum, $2K = L^\# + \tilde{L}^\#$, leads to different $m_0^\#$ and $\tilde{m}_0^\#$, but P and $\tilde{P}$ can be left untouched by this change. Substituting the formulas (7) into the infinite products defining $\hat{\phi}$ and $\hat{\tilde{\phi}}$ we find

$$\hat{\phi}(\xi) = (2\pi)^{-1/2} e^{-i\mathcal{K}\xi/2} \left[\frac{\sin\xi/2}{\xi/2}\right]^L \prod_{j=1}^{\infty} P(\cos 2^{-j}\xi)\,,$$

where we have used $\prod_{j=1}^{\infty} \cos(2^{-j}\alpha) = \frac{\sin\alpha}{\alpha}$. Consequently,

$$\hat{\psi}(\xi) = (2\pi)^{-1/2}\, i^{\mathcal{K}} \left(\sin\frac{\xi}{4}\right)^{\tilde{L}} \left[\frac{\sin\xi/4}{\xi/4}\right]^L \prod_{j=2}^{\infty} P(\cos 2^{-j}\xi)\,.$$

(Similar formulas hold for $\tilde{\psi}$, $\tilde{\phi}$.) The Fourier transform of the derivative ψ' of ψ is simply $i\xi\hat{\psi}(\xi)$, which can be written as

$$i\xi\; \hat{\psi}(\xi) = 4 i^{\mathcal{K}+1} (2\pi)^{-1/2} \left(\sin\frac{\xi}{4}\right)^{\tilde{L}+1} \left(\frac{\sin\xi/4}{\xi/4}\right)^{L-1} \prod_{j=2}^{\infty} P(\cos 2^{-j}\xi)\,.$$

Up to a multiplicative constant 4, this is exactly the Fourier transform of the wavelet $\psi^\#$ that would have corresponded to the same $P, \tilde{P}$ and K in (8), but with the choice $L^\# = L - 1$, $\tilde{L}^\# = L + 1$,

$$\begin{aligned} m_0^\#(\xi) &= \left(\cos\frac{\xi}{2}\right)^{L-1} \; e^{-i(1-\mathcal{K})\xi/2}\; P(\cos\xi), \\ \tilde{m}_0^\#(\xi) &= \left(\cos\frac{\xi}{2}\right)^{\tilde{L}+1} \; e^{-i(1-\mathcal{K})\xi/2}\; \tilde{P}(\cos\xi)\,. \end{aligned}$$

It follows that if we construct *two* pairs of biorthogonal wavelet bases, one using $\psi, \tilde{\psi}$, the other $\psi^{\#}, \tilde{\psi}^{\#}$, then

$$(\psi_{j,k})' = 2^{-j}\, 4\, \psi^{\#}_{j,k}\ ,$$

and hence

$$\left\langle \frac{d}{dx}\, \psi_{j,k},\ \tilde{\psi}^{\#}_{j',k'} \right\rangle = 4\ 2^{-j}\ \delta_{j,j'}\delta_{k,k'}\ .$$

We have indeed "diagonalized" the derivative operator! (Since we use two different bases, this is not a "true" diagonalization, however.) What this means in practice is that we can find the wavelet coefficients of $f'(x)$, *i.e.*,

$$\left\langle f',\ \psi_{j,k} \right\rangle = -\left\langle f, (\psi_{j,k})' \right\rangle$$

by simply computing the coefficients of f itself with respect to the $\psi^{\#}_{j,k}$:

$$\left\langle f', \psi_{j,k} \right\rangle = 4\ 2^{-j} \left\langle f,\ \psi^{\#}_{j,k} \right\rangle .$$

In a practical problem, we are usually given a discrete approximation of f to start with (*e.g.*, in the form of sampled values). From these, we have to determine the $\langle f, \phi_{0,k}\rangle$, which corresponds to a deconvolution, inverting the filters with coefficients $\phi(k)$:

$$f(n) = \sum_k \left\langle f, \phi_{0,k} \right\rangle\ \phi(k-n)\ .$$

We make the approximation here that $(\mathrm{Proj}_{V_0} f)(n) = f(n)$, which need not be correct, but without other a priori information on f we cannot hope to do better. With a priori information, we can adapt the deconvolution procedure. Similarly, we have to determine the $\langle f, \phi^{\#}_{0,k}\rangle$. Once these are obtained, we simply go through the usual high- and low-pass filtering plus decimation stages, with the filters m_0 and $\tilde{m}_0$ on one hand, $m_0^{\#}$ and $\tilde{m}_0^{\#}$ on the other hand, to obtain both the $\langle f, \psi_{j,k}\rangle$ and the $\langle f, \psi^{\#}_{j,k}\rangle = \frac{1}{4}\, 2^j \langle f', \psi_{j,k}\rangle$.

Another way of viewing the construction is to look at the link between ϕ and $\phi^{\#}$. We have

$$\hat{\phi}(\xi) = e^{i\xi/2} \left(\frac{\sin \xi/2}{\xi/2}\right) \hat{\phi}^{\#}(\xi) ,$$

or

$$i\xi\hat{\phi}(\xi) = (e^{i\xi} - 1)\hat{\phi}^{\#}(\xi) ,$$

implying

$$\phi'(x) = \phi^{\#}(x+1) - \phi^{\#}(x) .$$

The transition from the "ϕ-picture" to the "$\phi^{\#}$-picture" translates differentiation into a finite difference operation:

$$\begin{aligned} \mathrm{Proj}_{V_0} f' &= \sum_k \left\langle f', \phi_{0,k} \right\rangle \phi_{0,k} \\ &= -\sum_k \left\langle f, \phi'_{0,k} \right\rangle \phi_{0,k} \\ &= \sum_k \left\langle f, \phi^{\#}_{0,k} - \phi^{\#}_{0,k-1} \right\rangle \phi_{0,k} . \end{aligned}$$

This is closer to the presentation in Lemarié [14].

Note that Cohen, Daubechies and Feauveau [6] contains many examples of $\phi, \phi^{\#}, \psi, \psi^{\#}$ quadruplets, although the link with the differentiation operator was not noticed. Figure 5 plots one such quadruplet.

Acknowledgments. The first part of this paper was also presented at the International Conference on Wavelets in Toulouse, France, June 1992.

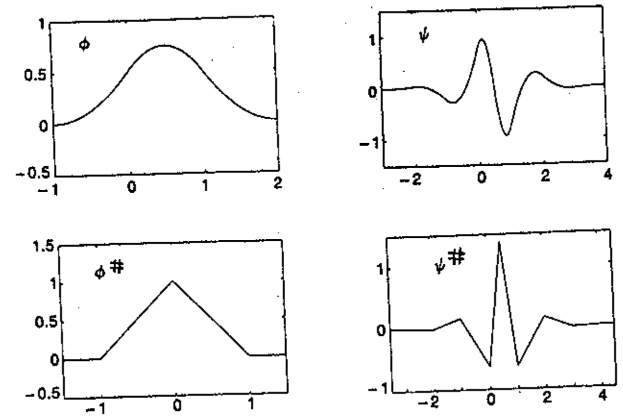

Figure 5. One example of a quadruplet ϕ, $\phi^{\#}$, ψ, $\psi^{\#}$, as found in Cohen, Daubechies and Feauveau [6]. Normalizations are slightly different from above; here $\psi^{\#} = -\frac{1}{4}\psi'$, and $\phi'(x) = \phi^{\#}(x) - \phi^{\#}(x-1)$.

References

1. Auscher, P., Ondelettes à support compact et conditions aux limites, J. Funct. Anal. **111** (1993), 29–43.
2. Beylkin, G., On the representation of operators in bases of compactly supported wavelets, SIAM J. Num. Anal. **29** (1992), 1716–1740.
3. Beylkin, G., R. Coifman, and V. Rokhlin, Fast wavelet transforms and numerical algorithms, Comm. Pure & Appl. Math. **44** (1991), 141–183.
4. Brislawn, C., Personal communication, 1991.
5. Chui, C. K. and E. Quak, Wavelets on a bounded interval, in *Numerical Methods in Approximation Theory*, D. Braess and L. L. Schumaker (eds.), Birkhauser, Boston, 1992, 53–75.
6. Cohen, A., I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, Comm. Pure & Appl. Math. **45** (1992), 485–560.
7. Cohen, A., I. Daubechies, B. Jawerth, and P. Vial, Multiresolution analysis, wavelets and fast algorithms on an interval, Comptes Rendus Acad. Sc. Paris **316** (série I), (1993) 417–421.
8. Cohen, A., I. Daubechies, and P. Vial, Wavelets and fast wavelet transforms on the interval, submitted to Applied and Computational Harmonic Analysis, 1992.
9. Daubechies, I., Orthonormal bases of compactly supported wavelets, Comm. Pure & Appl. Math. **41** (1988), 909–996.
10. Daubechies, I., Orthonormal bases of compactly supported wavelets. II. Variations on a theme, SIAM J. Math. Anal. **24** (1993), 499–519.