Revista Matemática Iberoamericana
Vol. 12, N.° 2, 1996

# A new technique to estimate the regularity of refinable functions

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**Abstract.** We study the regularity of refinable functions by analyzing the spectral properties of special operators associated to the refinement equation; in particular, we use the Fredholm determinant theory to derive numerical estimates for the spectral radius of these operators in certain spaces. This new technique is particularly useful for estimating the regularity in the cases where the refinement equation has an infinite number of nonzero coefficients and in the multidimensional cases.

## 1. Introduction.

Refinable functions are functions that satisfy a refinement equation, *i.e.*

$$\varphi(x) = 2 \sum_n c_n \, \varphi(2x - n) . \tag{1.1}$$

The coefficients $c_n$ are often, but not always, chosen finite in number. Such functions appear in different settings, most notably in subdivision schemes for computer aided design, where they are tools for the fast generation of smooth curves and surfaces (see Cavaretta, Dahmen and Micchelli (1991) and Dyn (1992) for reviews), and in the construction of wavelet bases and multiresolution analysis (see Mallat (1989), Daubechies (1988) and Meyer (1990)).

One of the earliest examples of refinable functions are the $B$-splines with equally spaced simple knots (see de Boor (1978) for a general review on splines), where

$$c_n = \binom{N}{n} 2^{-N+1},$$

and the corresponding $\varphi_N$ is a $C^{N-2}$ function, piecewise-polynomial of order $N-1$. Among refinable functions, the $B$-spline case is exceptional in that $\varphi(x)$ is given by an explicit analytical expression; in many other cases of interest, $\varphi$ is defined by fixing an appropriate choice for the $c_n$ in (1.1), and it is not immediately clear how smooth $\varphi$ is. Over the years, several techniques have been developed to determine the regularity of refinable functions. In this paper, we present a new technique for this purpose.

The regularity of a function $\varphi$ can be measured in different ways; we shall restrict ourselves to Hölder and Sobolev exponents. If $\varphi$ is in $C^n$ but not in $C^{n+1}$, then its Hölder exponent is given by $\mu = n + \nu$ with

$$\nu = \inf_x \left( \liminf_{|t|\to 0} \frac{\log |\varphi^{(n)}(x+t) - \varphi^{(n)}(x)|}{\log |t|} \right), \tag{1.2}$$

where $\varphi^{(n)}$ is the $n$-th derivative of $\varphi$.

The Sobolev exponent $s$ is defined by

$$s = \sup \left\{ \gamma : \int |\hat{\varphi}(\omega)|^2 (1 + |\omega|^2)^\gamma \, d\omega < +\infty \right\}, \tag{1.3}$$

where $\hat{\varphi}(\omega) = \int \varphi(x) e^{-i\omega x} dx$ is the Fourier transform of $\varphi$. One can generalize this to $L^p$-Sobolev exponents $s_p$ which, following Hervé (1995), we define by

$$s_p = \sup \left\{ \gamma : \int |\hat{\varphi}(\omega)|^p (1 + |\omega|^p)^\gamma \, d\omega < +\infty \right\}, \tag{1.4}$$

These different regularity indices are related to each other by $s = s_2$, $\mu \geq s_1$ and, by Hölder's inequality, $s_r + r^{-1} \leq s_p + p^{-1}$, for $0 \leq p \leq r$.

Most of the techniques developed to estimate the regularity of a refinable function concentrate on the case where only finitely many $h_n$ are non-zero, which was, until recently, the only case of interest for applications: in subdivision schemes it corresponds to finite masks; in wavelet constructions, to compactly supported scaling functions and wavelets. If there are only $N+1$ nonzero $c_n$, then Micchelli and Prautzsch (1989) and Daubechies and Lagarias (1991, 1992) showed how to find, at least in principle, the Hölder exponent of $\varphi$ by computing bounds on the norms of $N \times N$ matrices; in practice, this method becomes quickly impractical if $N$ is not small. Still for the case of finitely many nonzero $c_n$, a technique that can handle larger $N$ was proposed by Rioul (1992) and Dyn and Levin (1991); it is still the best available technique for finding the Hölder exponent when $N$ is finite. In general it is easier to compute the Sobolev exponents; these can moreover be used to find a bound on the Hölder exponent, since $\mu \geq s_1 \geq s_2 + 1/2$. In the case where $m(\omega) = \sum_n h_n e^{-in\omega}$ is a nonnegative trigonometric polynomial, one even has $\mu = s_1$, which was exploited in one of the first computations of the regularity of a refinable function in Dubuc (1986) for the special case $h_0 = 1$, $h_{\pm 1} = 9/16$, $h_{\pm 3} = -1/16$, all other $h_n = 0$, related to Lagrangian interpolation and later generalized in Dubuc and Deslauriers (1989). When $m(\omega)$ is not restricted to be nonnegative, most of the first developments were concentrated on the computation of $s_2 = s$; see Conze (1989), Eirola (1992) and the appendix in Daubechies (1988). Extensions to the computation of $s_p$ (including $p = 1, 2$) have appeared in Gripenberg (1992), Villemoes (1992) and Hervé (1992, 1995). With the exception of Hervé (1992, 1995), all the papers above apply only to $N$ finite. Most of them are also hard to generalize to the multidimensional case where (1.1) is replaced by

$$\varphi(x) = |D| \sum_n h_n \, \varphi(Dx - n)\,, \tag{1.5}$$

where $n \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$ and $D$ is a $d \times d$ matrix with integer entries and all eigenvalues strictly superior to 1 in absolute value; $|D|$ is the determinant of $D$. Examples of the type (1.5) occur in *e.g.* wavelet bases corresponding to quincunx subsampling in two dimensions proposed for image processing in Vetterli (1984), Feauveau (1989) and Kovačević and Vetterli (1990), with explicit orthonormal wavelet bases in Feauveau (1990), Kovačević and Vetterli (1992) and Cohen and Daubechies (1993). As illustrated by the trickiness of the estimates in Cohen and Daubechies (1993) and especially in Villemoes (1993), it is not easy to find the regularity in the multidimensional case by generalizing the approach referred to above, even when only finitely many $c_n$ are nonzero.

In this paper, we present a different technique for computing $s_p$ for a refinable function. This technique is independent of whether the $h_n$ are finite in number or not (like in Hervé (1992, 1995)) and it generalizes easily to the multidimensional case. Like most of the other approaches, our results hinge on the computation of the spectral radius of a particular operator (see sections 3 and 4 below). We introduce a different space on which this operator acts however, and we use a computation of the Fredholm determinant borrowed from Ruelle (1976) to compute its spectral radius.

This paper is organized as follows. In Section 2, we recall basic facts on trace-class operators and Fredholm determinants. In Section 3, we specialize these results to the particular case of transfer operators. In Section 4 we show how the spectral radius of such a transfer operator can be used to compute $s_p$. Although the connection between the spectral radius of a transfer operator and the Sobolev regularity index $s_p$ is not new (essentially all the computations of $s_2$ or $s_p$ to which we referred above, rely on such a connection), we present an essentially self-contained discussion in Section 4, for the sake of convenience, and also because we require some results tailored to our different framework. Next, we need to compute the spectral radius of the transfer operator, for which different techniques can be used. We propose three different procedures. The first two are similar (but not identical) to the formula in Hervé (1992, 1995), and have the same type of convergence: one can prove that the estimate $\rho_n$ at the $n$-th step approaches the true $\rho$ exponentially fast, $|\rho - \rho_n| \leq Ce^{-\alpha n}$, but $C$ and $\alpha$ are unknown. The third procedure, on which we concentrate almost exclusively, requires more computational work (*i.e.* a longer program), but at every step $n$ we have an explicit upper bound on $|\rho - \rho_n|$. In this procedure the spectral radius is found by determining the zero with the smallest absolute value of the corresponding Fredholm determinant. This determinant is an entire function which in practice needs to be truncated to its first order terms to derive numerical estimates. Section 5 shows how this is done, how the rest term can be controlled and how this translates into error estimates on our computation of $s_p$. In Section 6, we present many examples, in one as well as two dimensions, with finitely many as well as infinitely many nonzero $h_n$. In particular we apply our method to the easily implementable orthonormal wavelet filters recently introduced in Herley and Vetterli (1992) (which have an infinite number of nonvanishing $h_n$). Finally, in Section 7, we discuss whether and how the technique proposed here can be extended to a direct computation of the Hölder exponent $\mu$.

There is clearly some similarity of our results with those of Hervé (1992, 1995), the only other work in the literature so far that can deal with infinitely many nonvanishing $h_n$. Most of this work was carried out in the summer of 1992. At the time we were not aware yet of the results in Hervé (1992, 1995). Nevertheless, Hervé (1992, 1995) clearly has priority, since he proved his results earlier. Moreover, we initially developed our results only for $s_2$; after reading a preprint of Hervé (1995), we saw the interest of computing also $s_p$ for $p \neq 2$, and we extended our results to this case. If $p \neq 2$ then our method only works if we impose an extra technical condition which Hervé (1995) does not require; in this sense our results are weaker than Hervé's. On the other hand, our method has the new feature that it leads to explicit control over the error in the algorithm, as mentioned above, and as will be discussed in detail in Section 5.

## 2. Fredholm determinants of trace-class operators.

The most general treatment of Fredholm determinants is within the framework of Banach spaces, as in *e.g.* Grothendieck (1956). For the present paper, it suffices to work in an appropriate Hilbert space setting, where everything can be formulated and proved more simply. This section presents the results that will be needed in the sequel of the paper.

In this section we denote our (generic) Hilbert space by $\mathcal{H}$; we assume that $\mathcal{H}$ is separable. Recall that any bounded operator $A$ on $\mathcal{H}$ can be written as $A = U(A^*A)^{1/2} = U|A|$, where $U$ is a partial isometry with $\|Ux\| = \|x\|$ if $x \in \overline{\operatorname{Ran}|A|}$, $Ux = 0$ if $x \perp \operatorname{Ran}|A|$. If $A$ is a compact operator, then so is $|A|$; the spectrum of $|A|$ then consists of a decreasing sequence of nonnegative eigenvalues. The strictly positive eigenvalues $\lambda_m$ of $|A|$ are called the singular values of $A$. If $\varphi_m$ is a corresponding orthonormal system of eigenvectors of $|A|$, and we define another orthonormal system $\psi_m$ by $\psi_m = U\varphi_m$, then we have the following representation for $A$:

$$Ax = \sum_m \lambda_m \langle x, \varphi_m \rangle \psi_m . \tag{2.1}$$

Another useful representation of compact operators is obtained as follows. The spectrum of $A$ itself also consists of a sequence of discrete eigenvalues $\alpha_n$, which accumulate only at 0. For any $\alpha_n \neq 0$ we define its (algebraic) multiplicity $d_n$ by $d_n = \max_{k\geq 1} \dim(\mathrm{Ker}\,(A - \alpha_n \,\mathrm{Id})^k)$, which is necessarily finite because $A$ is compact. On $\mathrm{Ker}\,(A - \alpha_n \,\mathrm{Id})^k$ we can then choose a suitable basis in which $A$ is upper triangular (one can, for instance, reduce it to its Jordan normal form on this subspace). The union of all the different and independent bases constructed in this way for the distinct eigenvalues spans a closed subspace $\mathcal{H}_1$, which is invariant for $A$. By Gram-Schmidt orthogonalizing this basis we obtain an orthonormal basis $\{u_k\}$ for $\mathcal{H}_1$ in which $A|_{\mathcal{H}_1}$ is triangular:

$$\langle Au_k, u_\ell \rangle = 0\,, \qquad \text{if } \ell > k\,, \tag{2.2}$$

$$\langle Au_k, u_k \rangle = \alpha_k\,, \tag{2.3}$$

where the $\alpha_k$ now occur with their multiplicity. It is immediately clear that the same basis also gives a triangular representation for all the powers $A^m|_{\mathcal{H}_1}$, $m \geq 1$, with diagonal elements

$$\langle A^m u_k, u_k \rangle = \alpha_k^m\,. \tag{2.4}$$

Let us now assume that our compact operator is in fact trace class. This means that $A$ satisfies one of the following two equivalent statements:

$$\sum_m \lambda_m < \infty$$

or

$$\sum_n |\langle Ae_n, e_n \rangle| < \infty\,, \qquad \text{for all orthonormal systems } \{e_n\}\,.$$

If $A$ is trace-class, then the sum of the series $\sum_n \langle Ae_n, e_n \rangle$ is independent of the choice of the $\{e_n\}$, and is called the trace of $A$:

$$\mathrm{Tr}\,A = \sum_n \langle Ae_n, e_n \rangle\,.$$

In particular, using the representation (2.1) above, one has

$$\mathrm{Tr}\,|A| = \sum_m \langle |A|\varphi_m, \varphi_m \rangle = \sum_m \lambda_m\,.$$

In Lidskii (1959) it is proved that for any trace-class operator $A$ in a Hilbert space $\mathcal{H}$ one has

$$\operatorname{Tr} A = \sum_n \alpha_n \,, \tag{2.6}$$

where, as before, the $\alpha_n$ are the non-zero eigenvalues of $A$, taken with their algebraic multiplicity. This theorem, trivial in finite-dimensional spaces, is far less so in infinite dimensions. It is immediately clear from (2.2), (2.3) that the trace of $A|_{\mathcal{H}_1}$ is exactly $\sum_n \alpha_n$; the problem in proving (2.6) is to show that on the orthogonal complement $\mathcal{H}_0$ of $\mathcal{H}_1$, the operator $\operatorname{Proj}_{\mathcal{H}_0} A \operatorname{Proj}_{\mathcal{H}_0}$, which is of course trace-class too, and has spectrum consisting of only the point zero, has trace equal to zero. See *e.g.* Simon (1979) or Gohberg and Krein (1969) for other proofs of (2.6) than Lidskii's original proof.

The trace class operators form an ideal in the algebra of bounded operators: the product of a trace class operator and a bounded operator is again trace class. Consequently all the powers $A^n$, $n \geq 1$, of a trace class operator are trace class as well. Since $\operatorname{Proj}_{\mathcal{H}_0} A^n \operatorname{Proj}_{\mathcal{H}_0} = (\operatorname{Proj}_{\mathcal{H}_0} A \operatorname{Proj}_{\mathcal{H}_0})^n$ also has trace zero, we then see from (2.4) that

$$\operatorname{Tr} A^n = \sum_k \alpha_k^n \,. \tag{2.7}$$

(This can of course also be derived directly from Lidskii's theorem (2.6) and spectrum$(A^n) = \{\lambda^n : \lambda \in \text{spectrum } (A)\}$.) Finally, note that

$$\begin{aligned}
\sum_k |\alpha_k| &= \sum_k |\langle A u_k, u_k\rangle| \\
&\leq \sum_{k,m} \lambda_m \, |\langle u_k, \varphi_m\rangle| \, |\langle \psi_m, u_k\rangle| \\
&\leq \sum_m \lambda_m \Big(\sum_k |\langle u_k, \varphi_m\rangle|^2\Big)^{1/2} \Big(\sum_k |\langle \psi_m, u_k\rangle|^2\Big)^{1/2} \\
&\leq \sum_m \lambda_m \, \|\varphi_m\| \, \|\psi_m\| = \sum_m \lambda_m = \operatorname{Tr} |A| \,.
\end{aligned} \tag{2.8}$$

The Fredholm determinant of $A$ is defined as

$$D_A(z) = \det(I - zA) = \prod_{n=1}^{+\infty} (1 - z\,\alpha_n) \,, \tag{2.9}$$

where the $\alpha_n$ occur with their multiplicity. Because $\sum_n |\alpha_n| < +\infty$, $D_A(z)$ is an entire function with its zeros exactly at the $\alpha_n^{-1}$, with the same multiplicity. In particular, the spectral radius $\rho_A$ of $A$ is given by

$$\rho_A = (\min\{|z_0| : \; D_A(z_0) = 0\})^{-1} . \tag{2.10}$$

For sufficiently small values of $z$ (*e.g.* for $|z| < \rho_A^{-1}$), one can rewrite $D_A(z)$ as follows:

$$\begin{aligned} D_A(z) &= \exp\Big(\sum_{n=1}^{\infty} \log(1 - z\alpha_n)\Big) \\ &= \exp\Big(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\frac{1}{m}(z\alpha_n)^m\Big) \\ &= \exp\Big(-\sum_{m=1}^{\infty} \frac{1}{m} z^m \operatorname{Tr} A^m\Big) , \end{aligned} \tag{2.11}$$

a formula which already shows that $D_A$ is completely determined by the traces $\operatorname{Tr} A^m$. Expanding the infinite product in (2.9) leads to a different formula for $D_A(z)$,

$$D_A(z) = 1 + \sum_{k=1}^{+\infty} z^k \gamma_k , \tag{2.12}$$

with

$$\gamma_k = \sum_{l_1 < l_2 < \cdots < l_k} \alpha_{l_1} \cdots \alpha_{l_k} . \tag{2.13}$$

We then have

$$|\gamma_k| \le \frac{1}{k!}\Big(\sum_l |\alpha_l|\Big)^k \le \frac{1}{k!}(\operatorname{Tr}|A|)^k . \tag{2.13}$$

It follows that we can therefore always write

$$D_A(z) = D_A^N(z) + R_A^N(z) , \tag{2.14}$$

where $D_A^N(z)$ is the Taylor series for $D_A$ truncated after the term in $z^N$, and

$$|R_A^N(z)| \le \sum_{k=N+1}^{+\infty} \frac{1}{k!}(\operatorname{Tr}|A|)^k \, |z|^k . \tag{2.15}$$

We shall use this estimate to find the smallest zero of $D_A$: since $D_A^N$ is a polynomial, its smallest zero can be found by a host of different numerical methods, and the control we have via (2.15) on the rest term $R_A^N$ will tell us that the smallest zero of $D_A$ itself cannot be far from that of $D_A^N$ if $N$ is sufficiently large (see Section 5).

In order to identify the zeros of $D_A^N$, we need again a different representation; in particular, we are interested in a way of computing the Taylor coefficients of $D_A$ which does not require knowledge of the eigenvalues $\alpha_n$. To do this, let us start by restricting ourselves to the disk $B\left(0, \rho_A^{-1}\right)$. On this disk, we can write (using a trick going back to Newton)

$$
\begin{aligned}
\sum_{k=0}^{\infty}(k+1)\,\gamma_{k+1}\,z^k = D_A'(z) &= \sum_{n=1}^{\infty}(-\alpha_n)\prod_{\substack{m=1\\ m\neq n}}^{\infty}(1-\alpha_m z)\\
&= -D_A(z)\sum_{n=1}^{\infty}\frac{\alpha_n}{1-\alpha_n z}\\
&= -D_A(z)\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\alpha_n^{k+1}z^k\\
&= -D_A(z)\sum_{k=0}^{\infty}\operatorname{Tr}A^{k+1}\,z^k\\
&= -\sum_{r=0}^{\infty}z^r\sum_{m=0}^{r}\gamma_m\operatorname{Tr}A^{r-m+1}\,,
\end{aligned}
$$

where the reordering of the sums is allowed because the series converges absolutely for $|z| < (\rho_A)^{-1}$, and where we have introduced $\gamma_0 = 1$. It follows that

$$
\gamma_{k+1} = -\frac{1}{k+1}\sum_{m=0}^{k}\gamma_m\operatorname{Tr}A^{k+1-m}\,. \tag{2.16}
$$

(This derivation is in fact the standard way of relating the elementary symmetric functions $\sum_{i_1<i_2<\cdots<i_r}\alpha_{i_1}\cdots\alpha_{i_r}$ with the power sums $\sum_i \alpha_i^r$; see Macdonald (1979), p. 12-16.)

We now see the outline of a program that will be used below: given the $\operatorname{Tr}A^m$ for a trace-class operator $A$, we now know how to find, via the smallest zero of the Fredholm determinant, the spectral radius of $A$ and how to control the error. In the next section, we introduce specific operators $A$ in specific Hilbert spaces, we verify that they are trace-class, and we show how to compute $\mathrm{Tr}\, A^m$.

## 3. A special case: transfer operators.

The operators to which we shall apply the results in the previous section act on $2\pi$-periodic functions $f(\omega)$ and are defined by

$$(\mathcal{L}_w f)(\omega) = w\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) + w\left(\frac{\omega}{2}+\pi\right) f\left(\frac{\omega}{2}+\pi\right), \tag{3.1}$$

where $w$ is a $2\pi$-periodic weight function, for which several concrete choices will be proposed in Section 4. We shall say that $\mathcal{L}_w$ is the transfer operator associated with the function $w$; these operators are also called Perron-Frobenius operators or transition operators in the literature. We shall always assume that the Fourier coefficients of $w$ decay exponentially; that is,

$$w(\omega) = \sum_n w_n\, e^{-in\omega} \tag{3.2}$$

and

$$|w_n| \le C\, e^{-\gamma|n|}, \tag{3.3}$$

for some $C, \gamma > 0$. In terms of the Fourier coefficients $f_n$ of $f(\omega) = \sum_n f_n\, e^{-in\omega}$, (3.1) can also be rewritten as

$$(\mathcal{L}_w f)_n = \frac{1}{2\pi}\int_0^{2\pi} (\mathcal{L}_w f)(\omega)\, e^{in\omega}\, d\omega = 2\sum_k w_{2n-k}\, f_k\,. \tag{3.4}$$

When no confusion is possible, we shall often drop the subscript $w$ on $\mathcal{L}_w$.

Operators of the type (3.1) can be studied in many different function spaces. They have been linked with the study of refinable functions before; see *e.g.* Conze (1989), Eirola (1992), Villemoes (1992), Hervé (1995). They are special cases of the operators in Ruelle (1976, 1990). In this section we discuss their properties on some Hilbert spaces of analytic functions; in Section 7 we shall come back to their action on other, larger function spaces.

As candidates for the space $\mathcal{H}$ we define

$$E_\alpha = \Big\{ f \ 2\pi\text{-periodic} : f(\omega) = \sum_n f_n \, e^{-in\xi} \text{ and } \|f\|_\alpha^2 = \sum_n |f_n|^2 \, e^{2|n|\alpha} < \infty \Big\} . \tag{3.5}$$

(Note that these are different from the spaces $E^\alpha$ in Hervé (1995).) The $E_\alpha$ are Hilbert spaces of analytic functions ($f \in E_\alpha$ can be extended to complex $\omega = \omega_1 + i\omega_2$ and is then analytic for $|\omega_2| = |\mathrm{Im}\,\omega| < \alpha$); their inner product is given by

$$\langle f, g \rangle_\alpha = \sum_n f_n \, \overline{g}_n \, e^{2|n|\alpha} . \tag{3.6}$$

Note that for each $\alpha$, the constant function 1 is in $E_\alpha$; moreover, for $f \in E_\alpha$,

$$\langle f, 1 \rangle_\alpha = f_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \, d\omega . \tag{3.7}$$

The functions

$$e_{n,\alpha}(\omega) = e^{-|n|\alpha} \, e^{-in\omega} \tag{3.8}$$

constitute an orthonormal basis in $E_\alpha$.

In order to be able to apply Section 2 to $\mathcal{L}$ and $E_\alpha$, we need to verify: 1) that $\mathcal{L}$ is a bounded operator on $E_\alpha$, 2) that $\mathcal{L}$ is trace class on $E_\alpha$. We start by computing $\|\mathcal{L}f\|_\alpha$, using (3.4):

$$\begin{aligned}
\|\mathcal{L}f\|_\alpha^2 &= \sum_n |(\mathcal{L}f)_n|^2 \, e^{2|n|\alpha} \\
&= 4 \sum_n \Big| \sum_k w_{2n-k} \, f_k \Big|^2 e^{2|n|\alpha} \\
&\le 4 \, \|f\|_\alpha^2 \sum_{n,k} |w_{2n-k}|^2 \, e^{-2|k|\alpha} \, e^{2|n|\alpha} \\
&\le 4 \, C^2 \, \|f\|_\alpha^2 \sum_{n,k} e^{-2(\alpha-\gamma)|k|} \, e^{-2(2\gamma-\alpha)|n|} ,
\end{aligned}$$

where the last inequality used (3.3) and $|2n-k| \ge 2|n| - |k|$. It follows that $\mathcal{L}$ is a bounded operator on $E_\alpha$ if $\gamma < \alpha < 2\gamma$. We can us the same estimate to bound the matrix elements of $\mathcal{L}$ with respect to the orthonormal basis (3.8):

$$
\begin{aligned}
|\langle \mathcal{L}e_{k,\alpha}, e_{n,\alpha}\rangle_\alpha| &= |2\,w_{2n-k}\,e^{|n|\alpha}\,e^{-|k|\alpha}| \\
&\le 2\,C\,e^{-(\alpha-\gamma)|k|}\,e^{-(2\gamma-\alpha)|n|}\,.
\end{aligned} \tag{3.9}
$$

This implies that $\mathcal{L}$ is trace class on $E_\alpha$ if $\gamma < \alpha < 2\gamma$. By a simple application of Cauchy-Schwarz, we have indeed, for any orthonormal system $u_m$ in $E_\alpha$,

$$
\begin{aligned}
\sum_m |\langle \mathcal{L}u_m, u_m\rangle_\alpha| &\le \sum_{m,k,n} |\langle u_m, e_{k,\alpha}\rangle_\alpha|\,|\langle \mathcal{L}e_{k,\alpha}, e_{n,\alpha}\rangle_\alpha|\,|\langle e_{n,\alpha}, u_m\rangle_\alpha| \\
&\le \sum_{k,n} |\langle \mathcal{L}e_{k,\alpha}, e_{n,\alpha}\rangle_\alpha| < \infty\,.
\end{aligned} \tag{3.10}
$$

Since $\mathcal{L}$ is trace class, it has a representation of type (2.1) with $\sum_m \lambda_m < \infty$; in fact, by the same argument as in (3.10), we have

$$
\operatorname{Tr}|\mathcal{L}| = \sum_m \lambda_m = \sum_m \langle \mathcal{L}\varphi_m, \psi_m\rangle_\alpha \le \sum_{k,n} |\langle \mathcal{L}e_{k,\alpha}, e_{n,\alpha}\rangle_\alpha|\,.
$$

This leads to a bound on the sum of the absolute values $|\alpha_n|$ of the eigenvalues of $\mathcal{L}$:

$$
\sum_n |\alpha_n| \le \operatorname{Tr}|\mathcal{L}| \le 2\,C\Big(\sum_{k\in\mathbb{Z}} e^{-(\alpha-\gamma)|k|}\Big)\Big(\sum_{n\in\mathbb{Z}} e^{-(2\gamma-\alpha)|n|}\Big)\,. \tag{3.11}
$$

We shall need this bound in order to control the rest term when we try to locate the smallest zero of the Fredholm determinant after truncation (see (2.15) and Section 5 ).

We are therefore in a position to apply the results of Section 2 to $\mathcal{L}$ on $E_\alpha$; in the next section we shall see how this will then help to determine $s_p$. In order to be of practical use however, we need to be able to compute the Taylor coefficients of the Fredholm determinant $D_\mathcal{L}$ explicitly, and for this, we need the traces $\operatorname{Tr}\mathcal{L}^m$ (see (2.15)). Let us therefore now concentrate on their computation. As a warmup, let us compute $\operatorname{Tr}\mathcal{L}$ itself. We have

$$
\operatorname{Tr}\mathcal{L} = \sum_n \langle \mathcal{L}e_{n,\alpha}, e_{n,\alpha}\rangle_\alpha
$$

$$
\begin{aligned}
&= \sum_n (\mathcal{L}e_{n,\alpha})_n \, e^{|n|\alpha} \\
(3.12) \qquad &= \sum_n (\mathcal{L}e^{-in\,\cdot})_n \\
&= \frac{1}{2\pi} \sum_n \int_{-\pi}^{\pi} e^{in\omega} (\mathcal{L}e^{-in\,\cdot})(\omega) \, d\omega \, .
\end{aligned}
$$

To compute integrals of this type, we shall use the following standard lemma, which crops up in any study using these operators (see *e.g.* Cohen (1990), Eirola (1992), Villemoes (1992), Gripenberg (1992)). For the sake of completeness, we include its short proof.

**Lemma 3.1.** *Let $w$ be a $2\pi$-periodic function, and let $\mathcal{L}$ be defined as in (3.1). Then, for any $k > 0$ and any $f, g$ $2\pi$-periodic functions, we have*

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(\omega) \, (\mathcal{L}^k g)(\omega) \, d\omega &= \int_{-2^k\pi}^{2^k\pi} f(\omega) \Big( \prod_{\ell=1}^{k} w(2^{-\ell}\omega) \Big) \, g(2^{-k}\omega) \, d\omega \\
(3.13) \qquad &= 2^k \int_{-\pi}^{\pi} f(2^k\omega) \Big( \prod_{m=0}^{k-1} w(2^m\omega) \Big) \, g(\omega) \, d\omega \, .
\end{aligned}
$$

PROOF. By induction. For $k = 1$ we have

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(\omega) \, (\mathcal{L}g)(\omega) \, d\omega &= \int_{-\pi}^{\pi} f(\omega) \Big( w\big(\frac{\omega}{2}\big) \, g\big(\frac{\omega}{2}\big) + w\big(\frac{\omega}{2} + \pi\big) \, g\big(\frac{\omega}{2} + \pi\big) \Big) \, d\omega \\
&= 2 \int_{-\pi/2}^{\pi/2} f(2\omega) \, (w(\omega) \, g(\omega) + w(\omega + \pi) \, g(\omega + \pi)) \, d\omega \\
&= 2 \int_{-\pi/2}^{3\pi/2} f(2\omega) \, w(\omega) \, g(\omega) \, d\omega \\
&= 2 \int_{-\pi}^{\pi} f(2\omega) \, w(\omega) \, g(\omega) \, d\omega \\
&= \int_{-2\pi}^{2\pi} f(\omega) \, w\big(\frac{\omega}{2}\big) \, g\big(\frac{\omega}{2}\big) \, d\omega \, .
\end{aligned}
$$

If we now assume that (3.13) holds for $k$, then

$$
\int_{-\pi}^{\pi} f(\omega) \, (\mathcal{L}^{k+1} g)(\omega) \, d\omega
$$

$$= 2^k \int_{-\pi}^{\pi} f(2^k\omega) \Big( \prod_{m=0}^{k-1} w(2^m\omega) \Big) \Big( w(\frac{\omega}{2})\, g(\frac{\omega}{2}) + w(\frac{\omega}{2}+\pi)\, g(\frac{\omega}{2}+\pi) \Big)\, d\omega$$

$$= 2^{k+1} \int_{-\pi/2}^{\pi/2} f(2^{k+1}\omega) \Big( \prod_{m=1}^{k} w(2^m\omega) \Big) \Big( w(\omega)\, g(\omega) + w(\omega+\pi)\, g(\omega+\pi) \Big)\, d\omega$$

$$= 2^{k+1} \int_{-\pi}^{\pi} f(2^{k+1}\omega) \Big( \prod_{m=1}^{k} w(2^m\omega) \Big)\, g(\omega)\, d\omega\,.$$

Applying this to (3.12), we find

$$\operatorname{Tr} \mathcal{L} = \frac{1}{2\pi} \sum_n 2 \int_{-\pi}^{\pi} e^{2in\omega}\, w(\omega)\, e^{-in\omega}\, d\omega = 2 \sum_n w_n = 2\, w(0)\,.$$

Similarly, we can compute, for any $k \geq 1$,

$$(3.14) \qquad \begin{aligned} \operatorname{Tr} \mathcal{L}^k &= \sum_n (\mathcal{L}^k e^{-in\,\cdot})_n \\ &= \frac{1}{2\pi} \sum_n \int_{-\pi}^{\pi} e^{in\omega}\, (\mathcal{L}^k e^{-in\,\cdot})(\omega)\, d\omega \\ &= 2^k \sum_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(2^k-1)n\omega} \Big( \prod_{m=0}^{k-1} w(2^m\omega) \Big)\, d\omega \\ &= 2^k \sum_n (W_k)_{(2^k-1)n}\,, \end{aligned}$$

where the $2\pi$-periodic function $W_k$ is defined by

$$W_k(\omega) = \prod_{m=0}^{k-1} w(2^m\omega)\,,$$

and where its Fourier coefficients are denoted by $(W_k)_\ell$, as usual. Sums of the type (3.14) can be computed by means of the following lemma, which is essentially a version of the Poisson summation formula.

**Lemma 3.2.** *Let $f$ be a $2\pi$-periodic function, and $f_n$ its Fourier coefficients. Assume that $\sum_n |f_n| < \infty$. Then, for any $\ell \geq 1$,*

$$(3.15) \qquad \sum_{m=0}^{\ell-1} f\Big(\frac{2\pi m}{\ell}\Big) = \ell \sum_{n \in \mathbb{Z}} f_{\ell n}\,.$$

PROOF.

$$\sum_{m=0}^{\ell-1} f\left(\frac{2\pi m}{\ell}\right) = \sum_{m=0}^{\ell-1} \sum_{n\in\mathbb{Z}} f_n\, e^{-2\pi i m n/\ell}. \tag{3.16}$$

If $n \in \ell\mathbb{Z}$, then

$$\sum_{m=0}^{\ell-1} e^{-2\pi i m n/\ell} = \sum_{m=0}^{\ell-1} 1 = \ell. \tag{3.17}$$

If $n \in \ell\,\mathbb{Z} + k$, with $0 < k < \ell$, then

$$\sum_{m=0}^{\ell-1} e^{-2\pi i m n/\ell} = \sum_{m=0}^{\ell-1} e^{-2\pi i k m/\ell} = \frac{\left(e^{-2\pi i k/\ell}\right)^{\ell} - 1}{e^{-2\pi i k/\ell} - 1} = 0. \tag{3.18}$$

since (3.16) is absolutely summable, we may change the order of summations, and (3.15) then follows from (3.17) and (3.18).

This can then be used to give an explicit formula for $\operatorname{Tr}\mathcal{L}^k$. The following theorem summarizes the findings of this section so far.

**Theorem 3.3.** *Let $w(\omega)$ be a $2\pi$-periodic function with Fourier coefficients satisfying (3.3). Define $\mathcal{L}$ to be the corresponding transfer operator, as in (3.1), and let $E_\alpha$ be the Hilbert spaces defined by (3.5). If $\gamma < \alpha < 2\gamma$, then $\mathcal{L}$ is a trace class operator on $E_\alpha$. The spectrum of $\mathcal{L}$ does not depend on the choice of $\alpha$ in $]\gamma, 2\gamma[$, and for any $k \geq 1$, the traces $\operatorname{Tr}\mathcal{L}^k$ are given by the explicit formula*

$$\operatorname{Tr}\mathcal{L}^k = \frac{2^k}{2^k - 1} \sum_{m=0}^{2^{k-2}} \left( \prod_{\ell=0}^{k-1} w\left(2^\ell \frac{2\pi m}{2^k - 1}\right) \right). \tag{3.19}$$

PROOF. Most of the assertions were proved in our discussion above. Formula (3.19) is a direct consequence of applying Lemma 3.2 to (3.14). Since the Taylor coefficients of the Fredholm determinant $D_{\mathcal{L}}(z)$ are completely determined by the traces $\operatorname{Tr}\mathcal{L}^k$ (see Section 2), and the zeros of $D_{\mathcal{L}}$ are the inverses of the eigenvalues of $\mathcal{L}$, the fact that (3.19) does not depend on $\alpha$ immediately implies that the spectrum of $\mathcal{L}$ doesn't either.

REMARK. Another way of obtaining (3.19) is the following. The operator $\mathcal{L}_w$ can also be viewed as an integral operator,

$$(\mathcal{L}_w f)(w) = 2\int_{-\pi}^{\pi} w(\omega')\,\delta(\tau\omega' - \omega)\,f(\omega')\,d\omega'\,,$$

where $\tau$ on $]-\pi,\pi]$ is multiplication by 2, modulo $2\pi$. The trace of $\mathcal{L}_w$ can then be obtained by integrating the kernel $\mathcal{K}_w(\omega,\omega') = w(\omega')\,\delta(\tau\omega' - \omega)$ along the diagonal $\omega = \omega'$,

$$\operatorname{Tr}\mathcal{L}_w = 2\int_{-\pi}^{\pi} w(\omega)\,\delta(\tau\omega - \omega)\,d\omega = 2\,w(0)\,.$$

The integral kernel of $(\mathcal{L}_w)^k$ is given by

$$\int\cdots\int \mathcal{K}_w(\omega,\omega_1)\,\mathcal{K}_w(\omega_1,\omega_2)\cdots\mathcal{K}_w(\omega_{k-1},\omega')\,d\omega_1\cdots d\omega_{k-1}\;;$$

restricting this to $\omega = \omega'$ and integrating over $\omega$ leads to a delta-function $\delta(\tau^k\omega - \omega)$. This results in a sum of different contributions in the fixed points of $\tau^k$ (*i.e.* the points $2\pi m/(2^k-1)$), as in (3.19); the denominator $(2^k-1)$ multiplying these contributions results from the Jacobian of $\tau^k - \mathrm{Id}$.

In the next section, we shall see how, for a judicious choice of $w$, the operator $\mathcal{L}$ and in particular its spectral radius on $E_\alpha$ can be used to compute the Sobolev exponent $s_p$ of a refinable function. We shall be interested in multidimensional refinable functions as well. We will then need a slight generalization of the constructions above. Instead of (3.1), we have then, for $\omega \in [-\pi,\pi]^d$,

$$(\mathcal{L}f)(\omega) = \sum_{j=0}^{|\det D|-1} w(D^{-1}\omega + \xi_j)\,f(D^{-1}\omega + \xi_j)\,, \tag{3.20}$$

where $D$ is a $d\times d$ matrix with integer entries and all its eigenvalues strictly larger than 1 in absolute value, and where $f, w$ are functions in $d$ variables, $2\pi$-periodic in each (*i.e.* they are functions on the torus $\mathbb{T}^d$); the $\xi_j$ are defined by $\xi_j = D^{-1}\zeta_j$, where the $\zeta_j$ are distinct elements in $2\pi\mathbb{Z}^d/D\mathbb{Z}^d$, so that $D^{-1}\omega + \xi_j$ are exactly the $|\det D|$ distinct pre-images of $\omega$ under the map $D$. In the case where $d = 1$ and $D$ is multiplication by 2, (3.20) obviously reduces to (3.1). (3.20) can also be rewritten as

$$(3.21) \qquad (\mathcal{L}f)_n = |\det D| \sum_k w_{Dn-k}\, f_k \,,$$

where $f_k$ again denotes the $k$-th Fourier coefficient of $f$; the summation index $k$ now ranges over $\mathbb{Z}^d$. As before we shall assume

$$(3.22) \qquad |w_n| \leq C\, e^{-\gamma|n|}\,,$$

with $|n| = (n_1^2 + \cdots + n_d^2)^{1/2}$; the space $E_\alpha$ is then defined by

$$(3.23) \quad E_\alpha = \Big\{ f \text{ function on } [0, 2\pi]^d : \ \|f\|_\alpha^2 = \sum_{n \in \mathbb{Z}^d} |f_n|^2\, e^{2|n|\alpha} < \infty \Big\}.$$

Repeating the same arguments as before, one finds then the following generalization of Lemma 3.2:

**Theorem 3.4.** *Let $D$ be a $d \times d$ matrix with integer entries and with all its eigenvalues strictly larger than 1 in absolute value. Define the spaces $E_\alpha$ and the operator $\mathcal{L}$ as in (3.21)-(3.23). Then $\mathcal{L}$ is trace class on $E_\alpha$ if $\gamma < \alpha < r_D\gamma$, where $r_D = \min\{|\lambda| : \ \lambda$ is eigenvalue of $D\} > 1$, and the spectrum of $\mathcal{L}$ on $E_\alpha$ does not depend on $\alpha$.*

*Define now, for any $k \geq 1$, the set $F_k$ by*

$$(3.24) \qquad F_k = \{\eta \in\, ]-\pi, \pi[^d\colon\ D^k\eta - \eta \in 2\pi\mathbb{Z}^d\}\,;$$

*this set has exactly $|\det(D^k - \mathrm{Id})|$ elements, which are the fixed points in $\mathbb{T}^d$ of $D^k$. Then the traces $\mathrm{Tr}\,\mathcal{L}^k$ are given by the following explicit formula:*

$$(3.25) \qquad \mathrm{Tr}\,\mathcal{L}^k = \frac{|\det D|^k}{|\det(D^k - \mathrm{Id})|} \sum_{\eta \in F_k} \Big( \prod_{m=0}^{k-1} w(D^m\eta) \Big)\,.$$

The proof is exactly along the same lines as in the one-dimensional case, and we leave it to the reader to fill in the details. In order to obtain the explicit formula (3.25) one needs the following higher dimensional generalization of Lemma 3.2:

**Lemma 3.5.** *Let $L$ be an integer $d \times d$ matrix, and define the set $R$ by*

$$R = \{\zeta \in ]-\pi, \pi]^d : \; L\zeta \in 2\pi\mathbb{Z}^d\}.$$

*Then for any function $f$ on $\mathbb{T}^d$ such that $\sum_{n\in\mathbb{Z}^d} |f_n|$ is finite,*

$$\sum_{n\in\mathbb{Z}^d} f_{L^t n} = \frac{1}{|\det L|} \sum_{\zeta\in R} f(\zeta). \tag{3.26}$$

PROOF. As for Lemma 3.2, the proof hinges on the computation of $\sum_{\zeta\in R} e^{-in\cdot\zeta}$, where $n \in \mathbb{Z}^d$ and $n \cdot \zeta = n_1\zeta_1 + \cdots + n_d\zeta_d$. This can be done by a standard argument on character sums. The set $R$ is an additive group isomorphic with $\mathbb{Z}^d/L\mathbb{Z}^d$; it follows that

$$\Big(\sum_{\zeta\in R} e^{-in\cdot\zeta}\Big)^2 = \sum_{\zeta_1,\zeta_2\in R} e^{-in\cdot(\zeta_1-\zeta_2)} = (\#R)\sum_{\zeta\in R} e^{-in\cdot\zeta}.$$

Consequently we have either $\sum_{\zeta\in R} e^{-in\cdot\zeta} = \#R = |\det L|$ or $\sum_{\zeta\in R} e^{-in\cdot\zeta} = 0$. The former is possible only if each of the terms $e^{-in\cdot\zeta}$ equals 1, which is equivalent to the requirement $n = L^t k$ for some $k \in \mathbb{Z}^d$. Hence

$$\sum_{\zeta\in R} e^{-in\cdot\zeta} = \begin{cases} |\det L|, & \text{if } n \in L^t\mathbb{Z}^d, \\ 0, & \text{if } n \in \mathbb{Z}^d \backslash L^t\mathbb{Z}^d. \end{cases}$$

(3.26) then follows easily.

The following examples show how the explicit formulas for $\operatorname{Tr}\mathcal{L}^k$ can be used to determine the spectrum of $\mathcal{L}$ completely and explicitly in some simple cases.

EXAMPLE 3.6. $w(\omega) = ((1+e^{-i\omega})/2)^N$; this corresponds to the choice $w(\omega) = m(\omega)$ for the case where the refinable function is a $B$-spline (see Section 1). Then $w(2^\ell 2\pi m/(2^k-1)) = 1$, for all $\ell$, if $m = 0$, and for $m \neq 0$ we have

$$\prod_{\ell=0}^{k-1} w\Big(2^\ell \frac{2\pi m}{2^k-1}\Big) = \exp(i\pi N m)\left(\prod_{\ell=0}^{k-1} \frac{\sin\left(2^\ell 2\pi \frac{m}{2^k-1}\right)}{2\sin\left(2^\ell \pi \frac{m}{2^k-1}\right)}\right)^N = 2^{-Nk}.$$

Consequently

$$\begin{aligned}\operatorname{Tr}\mathcal{L}^k &= \frac{1}{1-2^{-k}} + \frac{2^k-2}{1-2^{-k}}\,2^{-Nk} \\ &= \sum_{\ell=0}^{N-2} 2^{-k\ell} + 2\,2^{-k(N-1)}\,.\end{aligned} \tag{3.27}$$

By (2.11) it then follows that

$$\begin{aligned}D_{\mathcal{L}}(z) &= \left(\prod_{\ell=0}^{N-2}\exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}(2^{-\ell}z)^m\Big)\right)\exp\Big(-2\sum_{m=1}^{\infty}\frac{1}{m}(2^{-N+1}z)^m\Big) \\ &= (1-z)\Big(1-\frac{z}{2}\Big)\cdots\Big(1-\frac{z}{2^{N-2}}\Big)\Big(1-\frac{z}{2^{N-1}}\Big)^2,\end{aligned}$$

implying that the nonzero eigenvalues of $\mathcal{L}$ are $1, 1/2, \ldots, 1/2^{N-2}$ (with multiplicity 1) and $1/2^{N-1}$ (with multiplicity 2); this can in fact also be read off from (3.27).

EXAMPLE 3.7. $w(\omega) = ((1+e^{-i\omega})/2)^N\, w_1(\omega)$; this corresponds to a factorization which we almost always impose (see Section 4); we assume $w_1(0) = 1$. Then the same computations as in the previous example give

$$\prod_{\ell=0}^{k-1} w\Big(2^\ell\,\frac{2\pi m}{2^k-1}\Big) = \begin{cases} 1, & \text{if } m = 0\,, \\ 2^{-kN}\displaystyle\prod_{\ell=0}^{k-1} w_1\Big(2^\ell\,\frac{2\pi m}{2^k-1}\Big), & \text{if } m \neq 0\,. \end{cases}$$

This leads to

$$\operatorname{Tr}\mathcal{L}^k = \frac{1-2^{-kN}}{1-2^{-k}} + 2^{-kN}\operatorname{Tr}\mathcal{L}_1^k\,,$$

where $\mathcal{L}_1$ is the operator (3.1) with $w$ replaced by $w_1$. By (2.11) we have therefore

$$D_{\mathcal{L}}(z) = (1-z)\Big(1-\frac{z}{2}\Big)\cdots\Big(1-\frac{z}{2^{N-1}}\Big)\,D_{\mathcal{L}_1}(2^{-N}z)\,.$$

The spectrum of $\mathcal{L}$ consists now of two parts: the eigenvalues $1, 1/2, \ldots, 1/2^{N-1}$, together with $2^{-N}\,\mathrm{spectrum}\,\mathcal{L}_1$.

The spectra for these simple $\mathcal{L}$ had been analyzed by other means before (see *e.g.* Daubechies and Lagarias (1992)), but their eigenvalues are recovered here in a particularly simple way. Let us consider a simple two-dimensional example next.

EXAMPLE 3.8. Take $d = 2$, $D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and

$$w(\omega) = \left(\frac{a(D\omega)}{2\,a(\omega)}\right)^N w_1(\omega),$$

where $a(\omega) \sim \omega_1^2 + \omega_2^2$ as $|\omega| \to 0$, $w_1(0) = 1$, and where we assume that $a$ and $w_1$ both satisfy (3.22), with moreover $\min\{|a(\omega)| : \omega \in [-\pi,\pi]^2\} > 0$. Then $w$ also satisfies a bound of the type (3.22). We have now, for $\eta \in F_k$ (defined by (3.24))

$$\prod_{m=0}^{k-1} w(D^k\eta) = \begin{cases} 1\,, & \text{if } \eta = 0\,, \\ 2^{-kN} \displaystyle\prod_{m=0}^{k-1} w_1(D^k\eta)\,, & \text{if } \eta \neq 0\,. \end{cases}$$

One easily checks that $|\det D| = 2$, $|\det(D^{2k} - \mathrm{Id})| = (2^k - 1)^2$ and $|\det(D^{2k+1} - \mathrm{Id})| = 2^{2k+1} - 1$. Consequently

$$\begin{aligned} D_{\mathcal{L}}(z) &= \exp\Big(-\sum_{m=1}^{\infty} \frac{z^{2m}}{2m} \frac{2^{2m}(1 - 2^{-2mN})}{(2^m - 1)^2}\Big) \\ &\quad \cdot \exp\Big(-\sum_{m=1}^{\infty} \frac{z^{2m+1}}{2m+1} \frac{2^{2m+1}(1 - e^{-(2m+1)N})}{2^{2m+1} - 1}\Big) D_{\mathcal{L}_1}(2^{-N}z) \\ &= \exp\Big(-\sum_{m=1}^{\infty}\sum_{\ell=0}^{N}\sum_{n=0}^{\infty} \frac{z^{2m}}{2m}(1 + 2^{-mN})\, 2^{-m(\ell+n)}\Big) \\ &\quad \cdot \exp\Big(-\sum_{m=1}^{\infty}\sum_{\ell=0}^{N} \frac{z^{2m+1}}{2m+1} 2^{-(2m+1)\ell}\Big) D_{\mathcal{L}_1}(2^{-N}z) \\ &= \Big(\prod_{\ell=0}^{N-1}(1 - 2^{-\ell}z)\Big)\Big(\prod_{\ell=0}^{2N-1}\prod_{\substack{n=0 \\ n\neq\ell}}^{\infty}(1 - 2^{-(\ell+n)}z^2)^{1/2}\Big) D_{\mathcal{L}_1}(2^{-N}z) \\ &= \Big(\prod_{\ell=0}^{N-1}(1 - 2^{-\ell}z)\Big)\Big(\prod_{k=0}^{\infty}(1 - 2^{-k}z^2)^{n(k)}\Big) D_{\mathcal{L}_1}(2^{-N}z)\,, \end{aligned}$$

where $n(k) = [(k+1)/2]$ if $k \le 2N - 1$, $n(k) = N$ if $k \ge 2N - 1$. This shows that in addition to the eigenvalues $1, 1/2, \ldots, 1/2^{N-1}$, which were also found in Cohen and Daubechies (1993), we have infinitely many eigenvalues $\pm 2^{-k/2}$, as well as of course the spectrum of $\mathcal{L}_1$ multiplied by $2^{-N}$.

## 4. Regularity estimates using the spectral radius of $\mathcal{L}$.

In this section, we discuss the relations between the Sobolev or Hölder (global) regularity of a refinable function and the spectral radius, in the previously described function spaces, of certain transfer operators that are associated to this function. More precisely, the study of these operators leads to an exact estimate of the $L^p$-Sobolev exponents $s_p$ defined in the introduction.

Let $\varphi(x)$ be an $L^1$ solution of a refinement equation

$$\varphi(x) = 2 \sum_{n \in \mathbb{Z}} h_n \, \varphi(2x - n) . \tag{4.1}$$

We assume that the coefficients $h_n$ are summable, and that $\varphi$ is normalized in the sense that $\int \varphi = 1$. By integrating on both sides of (4.1), we obtain

$$\sum_n h_n = 1 . \tag{4.2}$$

Define the continuous function $m(\omega) = \sum_n h_n \, e^{-in\omega}$. In all that follows, we shall assume that $m(\omega)$ can be put in the factorized form

$$m(\omega) = \cos^N(\omega/2) \, q(\omega) , \tag{4.3}$$

where $N$ is a strictly positive integer and $q(\omega)$ is a $2\pi$-periodic function whose Fourier coefficients $c_n$ satisfy a geometric decay estimate,

$$|c_n| \leq C \, e^{-\beta n} . \tag{4.4}$$

We shall often also impose that $q(\omega)$ does not vanish on $[0, 2\pi]$.

Consequently, $m(\omega)$ is a smooth $2\pi$-periodic function and (4.2) indicates that $m(0) = 1$. By applying the Fourier transform to (4.1), one obtains

$$\hat{\varphi}(\omega) = m(\omega/2) \, \hat{\varphi}(\omega/2) \tag{4.5}$$

and by iteration $\hat{\varphi}(\omega)$ can be written as the pointwise convergent infinite product

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m(2^{-k}\omega).$$

Before proceeding further, we would like to make some comments about the assumptions made on the functions $m(\omega)$ and $q(\omega)$ and their Fourier coefficients.

The infinite product formula (4.6) indicates that the function $\varphi$ is the limit of a stationary subdivision scheme, *i.e.* successive refinements of a Dirac sequence by means of interpolation: the sequence obtained at a given scale is filled with zeros at the mid-points and then convolved with the sequence $h_n$ and multiplied by 2. It is clear that a necessary condition for the existence of a non-trivial limit is $m(0) = 1$.

Formula (4.6) only indicates, however, that this scheme converges "weakly", *i.e.* in the sense of tempered distributions. The limit $\varphi$ itself may be a tempered distribution without any regularity. Clearly, for numerical applications, one is more interested in uniform (or strong) convergence of the subdivision scheme to a continuous or more regular limit function. It has been proved by Dyn and Levin (1990) that the limit function can be continuous only if $m(\pi)$=0. This justifies the factorization of $m(\omega)$ expressed in (4.3). In addition, strong convergence of the subdivision scheme follows if $\varphi \in C^0 \cap L^2$ and if there exists a compact set $K$ congruent to $[-\pi, \pi]$ modulo $2\pi$ (*i.e.* $|K| = 2\pi$, and for any $\omega \in [-\pi,\pi]$ there exists $\omega' \in K$ so that $\omega - \omega' \in 2\pi\mathbb{Z}$), containing a neighborhood of 0, such that $\inf_{j\geq 1, \omega\in K} |m(2^{-j}\omega)| > 0$. (See Cohen and Ryan (1995).) When this holds, we shall say that $m$ is of type $C$.

The next two lemmas give the decay of the Fourier coefficients of $|q(\omega)|^p$. Let us look first at the case $p = 2\ell$, $\ell \in \mathbb{N}$, $\ell > 0$.

**Lemma 4.1.** *The Fourier coefficients* $\{c_{n,2\ell}\}_{n\in\mathbb{Z}}$ *of* $|q(\omega)|^{2\ell}$, *with* $\ell \in \mathbb{N}$, *satisfy the estimate*

$$|c_{n,2\ell}| \leq C_{2\ell}\, e^{-\beta|n|}. \tag{4.7}$$

PROOF. We can write $q(\omega) = Q(e^{i\omega})$ where $Q(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ is analytic on the ring $R_\beta = \{e^{-\beta} < |z| < e^{\beta}\}$. We can define the function $Q_2(z) = Q(z)\,Q(z^{-1})$ that coincides with $|q(\omega)|^2$ on the unit circle $z = e^{i\omega}$. It is also analytic on $R_\beta$, as are all its integer powers, $(Q_2(z))^\ell$. The exponential decay (4.7) then follows immediately.

For $p \neq 2\ell$ we need an extra nonvanishing condition on $q(\omega)$.

**Lemma 4.2.** *Suppose that $q(\omega)$ does not vanish on $[0, 2\pi]$. Then there exists $\gamma$ in $]0, \beta]$ such that for all $p > 0$, the Fourier coefficients $\{c_{n,p}\}_{n\in\mathbb{Z}}$ of $|q(\omega)|^p$ satisfy the estimate*

$$|c_{n,p}| \leq C_p \, e^{-\gamma n} . \tag{4.8}$$

PROOF. Since $q(\omega)$ does not vanish, there exists $\gamma$ in $]0, \beta]$ such that $Q_2(z)$ does not vanish on $R_\gamma = \{e^{-\gamma} < |z| < e^{\gamma}\}$ (with the same notations as in the proof of Lemma 4.1 above). On this narrower ring, it is possible to define a set of analytic functions by

$$Q_p(z) = \exp\left(\frac{p}{2}\log Q_2(z)\right) = \sum_{n\in\mathbb{Z}} c_{n,p}\, z^n . \tag{4.9}$$

These functions are equal to $|q(\omega)|^p$ on the unit circle; their analyticity on $R_\gamma$ implies the estimate on the coefficients $c_{n,p}$.

From the results of the previous section, we thus know that the transfer operators associated to the functions $|q(\omega)|^p$ are trace class on $E_\alpha$, for any $\alpha \in \,]\gamma, 2\gamma[$. By Lemma 4.1, the transfer operators associated to the functions $|m(\omega)|^p$ will be trace class on $E_\alpha$ for $p \in 2\mathbb{N}$, but not for general $p$: because of the Hölder singularity at $\omega = \pi$ of $|m(\omega)|^p$ for $p \notin 2\mathbb{N}$, the Fourier coefficients of these functions do not decay exponentially. We have however the following result:

**Lemma 4.3.** *Let $\mathcal{L}_p$ (respectively $\mathcal{L}'_p$) be the transfer operators associated with $|q(\omega)|^p$ (respectively $|m(\omega)|^p$). For any $\alpha \in \,]\gamma, 2\gamma[$, $\mathcal{L}'_p$ acts as a trace class operator on the space $E'_\alpha$ composed of the functions $g(\omega) = |\sin(\omega/2)|^{Np} f(\omega)$ with $f \in E_\alpha$, the norm of $g$ in $E'_\alpha$ being identified to the norm of $f$ in $E_\alpha$. Moreover, if $f(\omega)$ is a continuous eigenfunction of $\mathcal{L}_p$ with eigenvalue $\lambda$, then $g(\omega) = |\sin(\omega/2)|^{Np} f(\omega)$ is a continuous eigenfunction of $\mathcal{L}'_p$ with eigenvalue $2^{-2Np}\lambda$.*

PROOF. It suffices to note that

$$\begin{aligned}\mathcal{L}'_p g(2\omega) &= |m(\omega)|^p\, g(\omega) + |m(\omega+\pi)|^p\, g(\omega+\pi) \\ &= |\sin(\omega/2)\cos(\omega/2)|^{Np}\left(|q(\omega)|^p\, f(\omega) + |q(\omega+\pi)|^p\, f(\omega+\pi)\right) \\ &= 2^{-2Np}\, |\sin(\omega)|^{Np}\, \mathcal{L}_p f(2\omega)\,.\end{aligned}$$

The operators $\mathcal{L}_p, \mathcal{L}'_p$ will be used to estimate the regularity of $\varphi$. In our proofs we shall use that $\mathcal{L}_p$ is a positive operator, in the sense that $(\mathcal{L}_p f)(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$ if $f(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$. Such operators have special spectral properties; see *e.g.* Schaefer (1966) or Schaefer (1974). To see how the general theorems on positive operators apply here, we first need to establish some facts about $E_\alpha$. Define $E_\alpha^+ = \{f \in E_\alpha : \ f(\omega) \geq 0 \text{ for } \omega \in [-\pi, \pi]\}$. This is a cone in $E_\alpha$, which contains in particular all the positive trigonometric polynomials. It follows that the closed linear span of $E_\alpha^+$ equals $E_\alpha$, or, in the terminology of Schaefer (1966), $E_\alpha$ is an ordered Banach space with total positive cone. It then already follows from the Krein-Rutman theorem (see *e.g.* Schaefer (1966), p. 265) that

**Lemma 4.4.** *The spectral radius $r_p$ of $\mathcal{L}_p$ in $E_\alpha$ is an eigenvalue for $\mathcal{L}_p$ and there exists a positive eigenfunction for this eigenvalue.*

(The statement of the Krein-Rutman theorem in Schaefer (1965) is for real ordered Banach spaces, but since $E_\alpha$ can easily be seen to be the complexification of $\{f \in E_\alpha : \ f(\omega) \in \mathbb{R} \text{ for all } \omega \in [-\pi, \pi]\}$, the theorem still applies.)

More restrictions on the spectrum of $\mathcal{L}_p$ can be derived if $q$ satisfies extra conditions. We shall need the following lemma:

**Lemma 4.5.** *Let $w(\omega)$ be a $2\pi$-periodic function satisfying* (3.3). *Assume furthermore that $w(\omega) \geq 0$ for all $\omega \in [-\pi, \pi]$, $w(0) = 1$, $w(\pi) \neq 0$, and that $w$ is of type $C$. Then, for all $f \in E_\alpha$ (with $\gamma < \alpha < 2\gamma$) with $f \geq 0$ and for all $\omega \in [-\pi, \pi]$, there exists $n \geq 1$ such that $(\mathcal{L}_w^n f)(\omega) > 0$.*

PROOF. 1) Since $w$ is of type $C$, we can find a compact set $K$, congruent with $[-\pi, \pi]$ mod $2\pi$, and a constant $C > 0$ so that, for all $\omega \in \mathbb{R}$, and all $j \geq 1$,

$$w(2^{-j}\omega) \geq C\, \chi_K(\omega)\,.$$

2) Assume now $\omega \neq 0$. Then

$$
\begin{aligned}
(\mathcal{L}_w^n f)(\omega) &= \sum_{m=-2^{n-1}+1}^{2^{n-1}} \Big(\prod_{j=1}^{n} w(2^{-j}(\omega + 2m\pi))\Big) f(2^{-n}(\omega + 2m\pi)) \\
&\geq C^n \sum_{m=-2^{n-1}+1}^{2^{n-1}} \chi_K(\omega + 2m\pi)\, f(2^{-n}(\omega + 2m\pi)).
\end{aligned}
$$

There exists $\tilde{\omega} \in K$, $m_1 \in \mathbb{Z}$, so that $\omega + 2m_1\pi = \tilde{\omega}$. Therefore, if $n$ is large enough, so that $2^{n-1} > |m_1|$, we have

$$
(\mathcal{L}_w^n f)(\omega) \geq C^n\, f(2^{-n}\tilde{\omega}).
$$

Since $\tilde{\omega} \neq 0$, and since $f \in E_\alpha$ is analytic, $f$ cannot vanish on all the $2^{-n}\tilde{\omega}$, implying that $(\mathcal{L}_w^n f)(\omega) > 0$ for some $n \geq 1$.

3) If $\omega = 0$, then

$$
\begin{aligned}
(\mathcal{L}_w^n f)(0) &= \sum_{m=-2^{n-1}+1}^{2^{n-1}} \Big(\prod_{j=1}^{n} w(2^{-j}2m\pi)\Big)\, f(2^{-n}2m\pi) \\
&\geq \sum_{\ell=-2^{n-2}+1}^{2^{n-2}-1} \Big(\prod_{j=1}^{n} w(2^{-j+1}(2\ell+1)\pi)\Big)\, f(2^{-n+1}(2\ell+1)\pi) \\
&= w(\pi) \sum_{\ell=-2^{n-2}+1}^{2^{n-2}-1} \Big(\prod_{j=1}^{n-1} w(2^{-j}(\pi + 2\ell\pi))\Big)\, f(2^{-n+1}(\pi + 2\ell\pi)).
\end{aligned}
$$

Again, there exists $\tilde{\omega} \in K$ and $\ell_1 \in \mathbb{Z}$ so that $\pi + 2\ell_1\pi = \tilde{\omega}$. If $2^{n-2} > |\ell_1|$, then it follows that

$$
(\mathcal{L}_w^n f)(0) \geq w(\pi)\, C^{n-1}\, f(2^{-n+1}\tilde{\omega}).
$$

Since $\tilde{\omega} \neq 0$, the conclusion then follows as in point 2) above.

**Lemma 4.6.** *Let $m, q$ be as in (4.3), (4.4), with $q(0) = 1$, $q(\pi) \neq 0$. Assume moreover that one of the following two sets of conditions holds:*

1) *$p > 0$ and $q(\omega)$ does not vanish on $[-\pi, \pi]$, or*

2) *$p \in 2\mathbb{N}$, $p \geq 2$, and $m$ is of type $C$.*

*Then $r_p$, the spectral radius of $\mathcal{L}_p$ on $E_\alpha$, $\gamma < \alpha < 2\gamma$, is an eigenvalue of algebraic multiplicity* 1, *and the corresponding eigenfunction is strictly positive. Moreover $r_p > 1$.*

PROOF. If $q$ does not vanish on $[-\pi, \pi]$, then $|q|^p$ is obviously of type $C$. On the other hand, if $m$ is of type $C$, then so is $q$, hence $|q|^p$. In both cases we can therefore apply Lemma 4.5, and we find, for $\lambda > r_p$, and for any $f \geq 0$, any $\omega \in [-\pi, \pi]$,

$$\sum_{n=1}^{\infty} \lambda^{-n} (\mathcal{L}_p^n f)(\omega) > 0 .$$

In the terminology of Schaefer (1966), this means that $\mathcal{L}_p$ is irreducible. It then follows from Theorem 3.3 in the Appendix of Schaefer (1966) that the algebraic multiplicity of $r_p$ is 1 and that the associated eigenfunction is strictly positive; let us call this eigenfunction $F$. The inequality $r_p > 1$ follows from

$$\begin{aligned} r_p^n F(0) &= (\mathcal{L}_p^n F)(0) \\ &= F(0) + \sum_{\substack{m=-2^{n-1}+1 \\ m \neq 0}}^{2^{n-1}} \prod_{j=1}^{n} |q(2^{-j} 2m\pi)|^p \, F(2^{-n} 2m\pi) . \end{aligned}$$

The argument in point 3) of the proof of Lemma 4.5 shows that this second term must be strictly positive for some $n$, implying $r_p > 1$.

We prove one additional lemma, which we shall use in the next section, although we do not need it for Theorem 4.8 below. The argument in the proof is borrowed from Hervé (1995).

**Lemma 4.7.** *Let $m, q$ be as in Lemma 4.6. Then $r_p$ is the only eigenvalue of $\mathcal{L}_p$ in the peripheral spectrum, i.e. all the other eigenvalues $\lambda$ satisfy $|\lambda| < r_p$.*

PROOF. Let $F$ be the strictly positive eigenfunction of $\mathcal{L}_p$ in Lemma 4.6, and define the function

$$v(\omega) = \frac{1}{r_p \, F(2\omega)} \, F(\omega) \, |q(\omega)|^p .$$

Then $v$ is a continuous function, and it satisfies $v(\omega) + v(\omega + \pi) = 1$. It is then a consequence of results proved in Keane (1972) that, for any continuous $2\pi$-periodic function $g$, $(\mathcal{L}_v^n g)(\omega)$ converges uniformly to some constant $C_g$. Consequently, for any $f \in E_\alpha$,

$$r_p^{-n}\,(\mathcal{L}_p^n f)(\omega) = F(\omega)\,(\mathcal{L}_v^n(f/F))(\omega) \to F(\omega)\,C_{f/F}\,.$$

If now $f$ was an eigenvector of $\mathcal{L}_p$ with eigenvalue $\lambda$, with $\lambda = r_p\,\tilde{\lambda}$, $\tilde{\lambda} \neq 1$, $|\tilde{\lambda}| = 1$, then this would imply

$$\tilde{\lambda}^n f(\omega) \underset{n\to\infty}{\longrightarrow} C_{f/F}\,F(\omega)\,.$$

This is impossible (just take any $\omega$ such that $F(\omega) \neq 0 \neq f(\omega)$).

We are now ready to state the result that links the regularity of $\varphi$ with the spectral properties of transfer operators.

**Theorem 4.8.** 1) *Assume that $m(\omega)$, $q(\omega)$ satisfy the same conditions as in Lemma 4.6. Let $\mathcal{L}_p$ be the transfer operator associated to the function $|q(\omega)|^p$ and let $r_p$ be the spectral radius of this operator on $E_\alpha$, for any $\alpha \in ]\gamma, 2\gamma[$. Then the $L^p$-Sobolev exponent $s_p$ of $\varphi$ satisfies*

$$s_p = N - \frac{1}{p}\,\log_2(r_p)\,. \tag{4.10}$$

*Furthermore, one always has*

$$s_p < N\,. \tag{4.11}$$

2) *If $q(\omega)$ has some zeros in $[-\pi, \pi]$ and is not of type C, then we still have $s_p \geq N - \log_2(r_p)/p$, for $p \in 2\mathbb{N}$.*

PROOF. We start by proving that $s_p \geq N - \log_2(r_p)/p$ for general $q(\omega)$. Combining (4.3) and (4.6), we obtain

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} \cos^N(2^{-k-1}\omega) \prod_{k=1}^{+\infty} q(2^{-k}\omega) = \left(\frac{2\sin(\omega/2)}{\omega}\right)^N A(\omega)$$

with $A(\omega) = \prod_{k=1}^{+\infty} q(2^{-k}\omega)$.

To exploit Lemma 3.1, we remark that for all $\omega \in [-2^n\pi, 2^n\pi]$, we have

$$|A(\omega)|^p \le C_p \prod_{k=1}^{n} |q(\omega)|^p \tag{4.12}$$

with $C_p = \max_{\omega\in[-\pi,\pi]} |A(\omega)|^p$. By Lemma 3.1, we can write

$$\begin{aligned}\int_{-2^n\pi}^{2^n\pi} |A(\omega)|^p \, d\omega &\le C_p \int_{-2^n\pi}^{2^n\pi} \prod_{k=1}^{n} |q(2^{-k}\omega)|^p \, d\omega \\ &= C_p \int_{-\pi}^{\pi} (\mathcal{L}_p)^n 1(\omega) \, d\omega \\ &= 2\pi \, C_p \, \langle (\mathcal{L}_p)^n 1 | 1 \rangle_\alpha \,,\end{aligned}$$

where we have used (3.7).

It follows that for all $\varepsilon > 0$ and $p > 0$, there exists a constant $C_{p,\varepsilon}$ such that

$$\int_{-2^n\pi}^{2^n\pi} |A(\omega)|^p \, d\omega \le C_{p,\varepsilon} \, (r_p + \varepsilon)^n \,. \tag{4.13}$$

We now study the convergence of $\int |\hat{\varphi}(\omega)|^p (1+|\omega|^p)^s \, d\omega$ by a dyadic decomposition of the frequency domain:

$$\begin{aligned}\int_{\mathbb{R}} |\hat{\varphi}(\omega)|^p \, (1 + |\omega|^p)^s \, d\omega &= \int_{-\pi}^{\pi} |\hat{\varphi}(\omega)|^p \, (1 + |\omega|^p)^s \, d\omega \\ &\quad + \sum_{j=1}^{+\infty} \int_{|\omega|\in[2^{j-1}\pi, 2^j\pi]} |\hat{\varphi}(\omega)|^p \, (1 + |\omega|^p)^s \, d\omega \\ &\le C_1 + C_2 \sum_{j=1}^{+\infty} 2^{psj} \, 2^{-Npj} \int_{-2^n\pi}^{2^n\pi} |A(\omega)|^p \, d\omega \\ &\le C_1 + C_3 \sum_{j=1}^{+\infty} 2^{psj} \, 2^{-Npj} \, (r_p + \varepsilon)^j \,,\end{aligned}$$

where $C_1$, $C_2$ and $C_3$ only depend on the choice of $p$ and $\varepsilon$. Since $\varepsilon$ can be chosen arbitrarily small, it is clear that the integral will converge whenever $sp < Np - \log_2(r_p)$. This shows that we have

$$s_p \ge N - \frac{1}{p} \, \log_2(r_p) \,. \tag{4.14}$$

To sharpen this into an equality when $q(\omega)$ satisfies the extra conditions, we shall use the transfer operator $\mathcal{L}'_p$ associated to the function $|m(\omega)|^p$. According to Lemma 4.3, the spectral radius of $\mathcal{L}'_p$ on $E'_\alpha$ is $2^{-Np} r_p$. Moreover, since $F > 0$ is an eigenfunction of $\mathcal{L}_p$ for the eigenvalue $r_p$, it follows that $2^{-Np} r_p$ is an eigenvalue for $\mathcal{L}'_p$, with a corresponding positive eigenfunction $g(\omega) = |\sin(\omega/2)|^{Np} F(\omega)$.

Define now $S = \sup_\omega |F(\omega)|$, $G = \int_{-\pi}^{\pi} g(\omega)\,d\omega > 0$. Let $K$ be the compact set congruent with $[-\pi, \pi]$ for which $\inf_{j \geq 1,\, \omega \in K} |m(\omega)|^p > 0$. It then follows (see *e.g.* Cohen and Ryan (1995)) that

$$\rho = \inf_{\omega \in [-\pi,\pi]} |\hat{\varphi}(\omega)|^p > 0\,.$$

Define now

$$I_n = \int_{\omega \in 2^n K} |\omega|^{Np}\, |\hat{\varphi}(\omega)|^p\, d\omega\,. \tag{4.15}$$

Using again Lemma 3.1, we obtain, for all $n \geq 1$,

$$\begin{aligned}
I_n &\geq \rho^{-1} \int_{\omega \in 2^n K} |\omega|^{Np} \Big| \prod_{k=1}^{n} m(2^{-k}\omega) \Big|^p\, d\omega \\
&\geq \rho^{-1}\, 2^{Np(n+1)} \int_{\omega \in 2^n K} |\sin(2^{-n-1}\omega)|^{Np} \Big| \prod_{k=1}^{n} m(2^{-k}\omega) \Big|^p\, d\omega \\
&= \rho^{-1}\, 2^{Np(n+1)} \int_{-2^n\pi}^{2^n\pi} |\sin(2^{-n-1}\omega)|^{Np} \Big| \prod_{k=1}^{n} m(2^{-k}\omega) \Big|^p\, d\omega \\
&\geq (S\rho)^{-1}\, 2^{Np(n+1)} \int_{-2^n\pi}^{2^n\pi} g(2^{-n}\omega) \Big| \prod_{k=1}^{n} m(2^{-k}\omega) \Big|^p\, d\omega \\
&= (S\rho)^{-1}\, 2^{Np(n+1)} \int_{-\pi}^{\pi} (\mathcal{L}'_p)^n g(\omega)\, d\omega \\
&= G(S\rho)^{-1}\, 2^{Np}\, (r_p)^n = C\,(r_p)^n\,.
\end{aligned}$$

Since, for some $L < \infty$, $K \subset \{\omega : \ |\omega| \leq 2^L \pi\}$, it follows that

$$\begin{aligned}
\tilde{I}_n &= \int_{-2^n\pi}^{2^n\pi} |\omega|^{Np}\, |\hat{\varphi}(\omega)|^p\, d\omega \\
&\geq \int_{\omega \in 2^{n-L} K} |\omega|^{Np}\, |\hat{\varphi}(\omega)|^p\, d\omega \\
&= I_{n-L} \geq C'\,(r_p)^n\,.
\end{aligned} \tag{4.16}$$

If we now define

$$J_n = \tilde{I}_n - \tilde{I}_{n-1} = \int_{|\omega| \in [2^{n-1}\pi, 2^n\pi]} |\omega|^{Np} \, |\hat{\varphi}(\omega)|^p \, d\omega \,, \tag{4.17}$$

then (4.16) shows that for all $C, \varepsilon > 0$, there is an infinite number of $n \geq 1$ such that $J_n \geq C(r_p)^n \, 2^{-\varepsilon n}$ (here we use that, by Lemma 4.6, $r_p$ is strictly larger than 1), or equivalently

$$\int_{|\omega| \in [2^{n-1}\pi, 2^n\pi]} |\omega|^{Np - \log_2(r_p) + \varepsilon} \, |\hat{\varphi}(\omega)|^p \, d\omega \geq C \,. \tag{4.18}$$

This last inequality shows that $s_p$ is smaller than (and thus equal to) $N - \log_2(r_p)/p$. Finally, $s_p < N$ follows from $r_p > 1$.

## 5. Numerical precision.

Combining the results of the previous sections, we immediately obtain

**Theorem 5.1.** *For a function $\varphi$ as defined by (4.6), with $m(\omega)$, $q(\omega)$ satisfying the conditions in Lemma 4.6, the $L_p$-Sobolev exponent of $\varphi$ can be expressed as $s_p = N - \log_2(r_p)/p$, where $(r_p)^{-1} = x_p$ is the zero of smallest absolute value of the Fredholm determinant $d_p(z)$ of the operator $\mathcal{L}_p$ associated to the weight function $w(\omega) = |q(\omega)|^p$.*

Formulas (2.12), (2.16) and (3.19) give us an explicit expression for the Taylor series of the analytic function $d_p(z)$. In practice, to estimate $x_p$ numerically, we are obliged to truncate this series and work with the polynomials that are obtained from the first order terms. What is the effect of this truncation on the numerical precision of the estimate for $x_p$? More precisely, can we measure how well $x_p$ is approximated by the smallest zero of the truncated series at a given order?

We first discuss this problem in very general terms. Let $f(z)$ be an analytic function on $\mathbb{C}$ and suppose that we want to estimate the value of $z_0$, the zero of $f$ with the smallest absolute value. For a given $N$, let us denote by $P_N$ the polynomial corresponding to the $N$ first order terms in the Taylor development of $f$ around 0 and let $R_N(z) = f(z) - P_N(z)$ be the residual term. Then, for any fixed $A > 0$, and any $\lambda \in ]0, 1[$, one can find $C > 0$ so that $\sup_{|z| \leq A} \; |R_N(z)| \leq C\lambda^N$: the $R_N$ converge to zero faster than any geometric sequence, uniformly on the disk $|z| \leq A$. To exploit this estimate, we shall use a classic result of complex analysis (see, for example, Rudin (1967)) that we recall here:

**Rouché's Theorem.** *Let $g(z)$ and $h(z)$ be analytic functions on an open set $V$; let $D$ be an open set such that $\overline{D} \subset V$ and $\partial D$ is a Jordan curve. If $|h(z)| < |g(z)|$ for all $z \in \partial D$, then $g$ and $h+g$ have the same number of zeros inside $D$.*

This theorem leads to a systematic method for tracking the zeros of $f$:

- First, find $\varepsilon_0 > 0$, $A > 0$ and $N_0 \in \mathbb{N}$ such that $|P_{N_0}(z)| > \varepsilon_0$ on $S_A = \{|z| = A\}$, $|R_n(z)| < \varepsilon_0/4$ in $B_A = \{|z| \leq A\}$ for all $n \geq N_0$ and $P_{N_0}$ has at least one zero in $B_A$. This is always possible, if $f$ has at least one zero in $B_A$ and does not vanish on $S_A$. Moreover, Rouché's theorem implies that $f$ and $P_{N_0}$ have the same number $M$ of zeros in $B_A$.

- For a given zero $z_{0,j}$, $j \in \{1, \ldots, M\}$ of $P_{N_0}$ in $B_A$, consider then the parametrized curve $\Gamma_{0,j}(\theta) = z_{0,j} + u_{0,j}(\theta)\, e^{i\theta}$, $\theta \in [0, 2\pi]$, defined by

$$u_{0,j}(\theta) = \min\{u :\ P_{N_0}(z_{0,j} + u(\theta)\, e^{i\theta}) = \varepsilon_0\}\,. \tag{5.1}$$

  This is clearly a Jordan curve contained in $B_A$. By Rouché's theorem again, we know that the curve $\Gamma_{0,j}$ embraces the same number of zeros not only for $f$ and $P_{N_0}$ but also for all the $P_n$ for $n \geq N_0$ since, on $\Gamma_{0,j}$, we have $|P_n(z)| \geq |P_{N_0}(z)| + |R_{N_0}(z)| + |R_n(z)| \geq \varepsilon_0/2$.

- We now iterate this process by taking any sequence $\varepsilon_k$ starting with $\varepsilon_0$ and such that $0 < \varepsilon_k \leq \varepsilon_{k-1}/2$. After $k$ steps, we choose $N_k > N_{k-1}$ such that $|R_n(z)| < \varepsilon_k/4$ in $B_A = \{|z| \leq A\}$ for all $n \geq N_k$. The curves $\Gamma_{k,j}$ will be defined around the zeros $z_{k,j}$, $j \in \{1, \ldots, M\}$ of $P_{N_k}$ by $\Gamma_{k,j}\ (\theta) = z_{k,j} + u_{k,j}(\theta)\, e^{i\theta}$, $\theta \in [0, 2\pi]$, with

$$u_{k,j}(\theta) = \min\{u :\ P_{N_k}(z_{k,j} + u(\theta)\, e^{i\theta}) = \varepsilon_k\}\,. \tag{5.2}$$

  It is clear that $\Gamma_{k,j}$ lies completely within $\Gamma_{k-1,j}$ and that for fixed $k$, each $\Gamma_{k,j}$ contains at least one zero $z_j$ of $f$, $j \in \{1, \ldots, M\}$ (if it contains two zeros $z_i$ and $z_j$, then the curves $\Gamma_{k,i}$ and $\Gamma_{k,j}$ are necessarily identical).

The speed of convergence of this process will be measured by the decay of the diameter of the curves $\Gamma_{k,j}$ as $k$ goes to $+\infty$. We can estimate this diameter by remarking that $|f(z)| < 5\,\varepsilon_k/4$ within $\Gamma_{k,j}$. Consequently if $d$ is the maximal order of all the zeros $z_j$ of $f$ in $B_A$, $j \in \{1, \dots, M\}$, then there exists a constant $C$ such that

$$\begin{aligned}\max_j \operatorname{diam}(\Gamma_{k,j}) &\leq \max_{j,r}\{r : |z - z_j| < r \Rightarrow |f(z)| < 5\,\varepsilon_k/4\} \\ &< C(\varepsilon_k)^{1/d}.\end{aligned} \tag{5.3}$$

In particular, we see that since $\varepsilon_k$ has at least exponential decay, the speed of convergence will always be, at least, exponential. Note that for sufficiently large $k$, all the $\Gamma_{k,j}$ curves ($i = 1, \dots, M$) are disjoint from each other so that all the zeros are isolated and can be tracked separately.

In the particular case that we are interested in, some additional considerations can be made:

- We are looking for the zero with the smallest absolute value. By Lemma 4.6, this zero is unique and situated in $\mathbb{R}_+$. Furthermore, we know that it is contained in $[0, 1[$. Consequently, we may restrict our tracking process to this interval after the first step, using the fact that, for all $k$, the zero that we are looking for is necessarily situated between the two extremal intersections of a certain $\Gamma_{k,j}$ with the real axis.

- We have now a specific estimate for the rest $R_{N,p}(z)$ of $d_p(z)$: according to (2.11), for all $|z| \leq 1$,

$$|R_{N,p}(z)| \leq \sum_{k=N+1}^{+\infty} \frac{1}{k!} (\operatorname{Tr}|\mathcal{L}_p|)^k . \tag{5.4}$$

This estimate indicates that the estimation process that uses $P_k(z)$ at step $k$ should converge at least exponentially fast.

Before we proceed to the examples in the next section, we list a few comments on our procedure.

COMMENTS.

1). According to Lemma 4.7, we can split the eigenvalues $\alpha_n$ of $\mathcal{L}_p$ into $\alpha_0 = r_p$, and all the other $\alpha_n$, $n \geq 1$, which satisfy $|\alpha_n| < r_p$. We can therefore write the following estimate for $\operatorname{Tr} \mathcal{L}_p^k$:

$$|\operatorname{Tr} \mathcal{L}_p^k - r_p^k| \leq r_p^k \sum_{\ell=1}^{\infty} p_\ell^k N_\ell , \tag{5.5}$$

where all the $p_\ell$ are $< 1$, where $N_\ell = \#\{\alpha_n : \ |\alpha_n| = p_\ell\}$, and $\sum_{\ell=1}^{\infty} p_\ell N_\ell = \sum_{n=1}^{\infty} |\alpha_n| < \infty$. This leads to two alternative formulas for the computation of $r_p$, starting from the traces $\operatorname{Tr} \mathcal{L}_p^k$:

$$r_p = \lim_{k\to\infty} |\operatorname{Tr} \mathcal{L}_p^k|^{1/k} , \tag{5.6}$$

and

$$r_p = \lim_{k\to\infty} \frac{|\operatorname{Tr} \mathcal{L}_p^{k+1}|}{|\operatorname{Tr} \mathcal{L}_p^k|} . \tag{5.7}$$

According to the estimate (5.5), we have indeed

$$|\operatorname{Tr} \mathcal{L}_p^k|^{1/k} = r_p + C_1(k)\, p_1^k \tag{5.8}$$

and

$$\frac{|\operatorname{Tr} \mathcal{L}_p^{k+1}|}{|\operatorname{Tr} \mathcal{L}_p^k|} = r_p + C_2(k)\, p_1^k , \tag{5.9}$$

where, for $k$ sufficiently large,

$$|C_1(k)| \leq (1+\epsilon)\, \frac{N_1}{k}\, r_p$$

and

$$|C_2(k)| \leq (1+\epsilon)\, (p_1 + 1)\, N_1\, r_p ,$$

which proves (5.6) and (5.7) (since $p_1 < 1$). Both formulas converge exponentially fast, with the same rate, but a slightly better multiplicative constant in (5.6) than in (5.7), according to (5.8), (5.9). Computing $r_p$ via either (5.6) or (5.7) is simpler than the Fredholm determinant method explained above, but although we have exponential convergence in (5.6), (5.7), we have no control over, or no good estimate for the rate of convergence. Since $\mathcal{L}_p$ is not selfadjoint, its eigenvalues can be complex, and it is conceivable that the largest $|\alpha_0|$ may be "masked" in the first sums. This is illustrated by the following example (which is admittedly ad hoc, and not computed as the spectrum of a true $\mathcal{L}_w$). Take

$$\begin{aligned} &\alpha_0 = 1\,, \qquad \alpha_n = (1-\epsilon)\, e^{2\pi i n/K}\,, \quad n = 1, \dots, K-1\,, \\ &\alpha_K = \gamma\,, \qquad \alpha_n = 0\,, \quad n > K\,. \end{aligned}$$

Then we have

$$\sum_{n=0}^{\infty} \alpha_n^k = \begin{cases} 1 + (K-1)(1-\epsilon)^k + \gamma^k\,, & \text{if } k = 0 \pmod{K}\,, \\ 1 - (1-\epsilon)^k + \gamma^k\,, & \text{if } k \neq 0 \pmod{K}\,. \end{cases}$$

Consequently the first $K-1$ sums $\sum_{n=0}^{\infty} \alpha_n^k$, $1 \le k \le K-1$, may lead to a very misleading picture. Figure 1a plots $\log\left(\sum_{n=0}^{\infty} \alpha_n^k\right)$ as a function of $k$ for $1 \le k \le 14$ for the choices $K = 15$, $\gamma = .9$, $\epsilon = .001$; the graph is virtually indistinguishable from the straight line $k \log \gamma$. The picture changes drastically when we reach $k = 15$, as shown in Figure 1b, which plots the behavior of $\sum_{n=0}^{\infty} \alpha_n^k$ for a much larger range of $k$. The point of this toy example is that the first $K-1$ sums $\sum_{n=0}^{\infty} \alpha_n^k$ contain no clue whatsoever indicating that the asymptotic regime is far from attained.

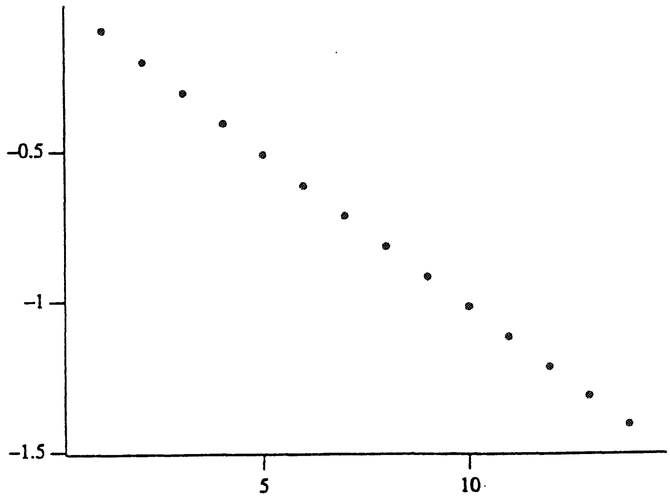


**Figure 1a.** Plot of $\log(\sum_{n=0}^{\infty} \alpha_n^k)$ for the values $k = 1, \dots, 14$ in the toy example.

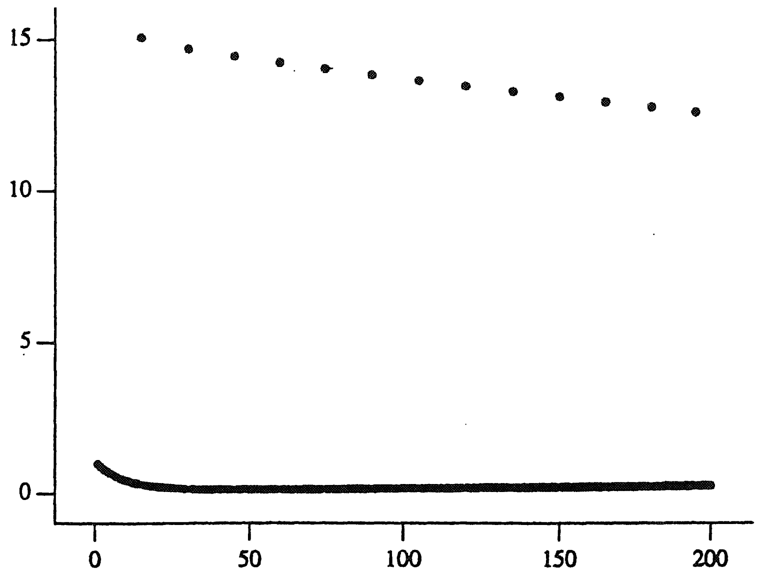


**Figure 1b.** Plot of $\sum_{n=0}^{\infty} \alpha_n^k$ for $k = 1, \ldots, 200$ in the toy example, showing the much slower exponential decay to the limit value 1.

The zero-tracking method above, in contrast, requires more work than (5.6) or (5.7), but it gives explicit error estimates at every step. An example of a true $\mathcal{L}_w$-spectrum that gives rise to a similar masking effect is given by $w(\omega) = |q(\omega)|^2$, with

$$q(\omega) = 100\,(1 - .9\,e^{3i\xi})\,(1 - .9\,e^{7i\xi})\,.$$

In this case $w$ is a trigonometric polynomial, so that the spectral radius $r_2$ of $\mathcal{L}_2 = \mathcal{L}_w$ is simply the largest eigenvalue of a $21 \times 21$ matrix, which can easily be computed explicitly; one finds $r_2 = 32\,642.525$. For small values of $k$, the traces of $\mathcal{L}_2^k$ are much smaller than $r_2^k$; this is due to the fact that $q$ is close to vanishing at the nontrivial fixed points of $\tau^2$ and $\tau^3$, where $\tau$ is the doubling operator, modulo $2\pi$, on $[0, 2\pi]$. Figure 2 shows the values of $|\mathrm{Tr}\ \mathcal{L}_2^k|^{1/k}$ for $k = 1$ to $25$.

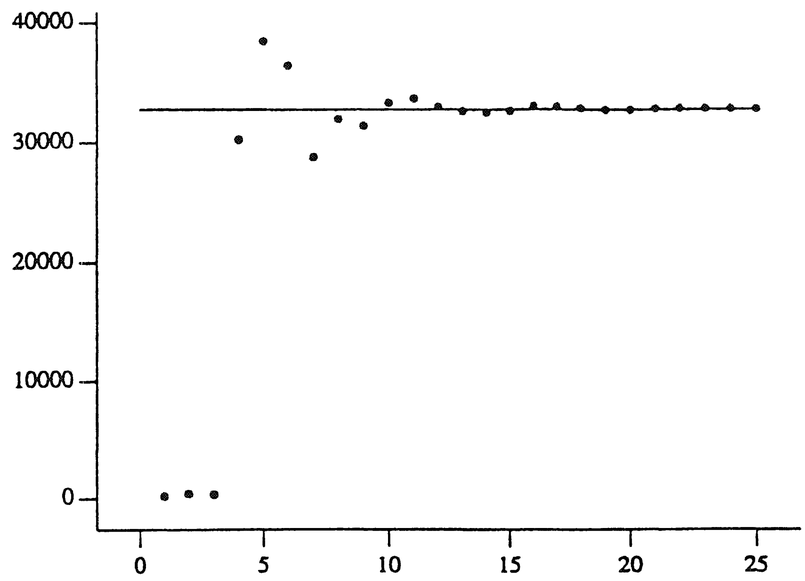


**Figure 2.** The values of $|\mathrm{Tr}\,\mathcal{L}_2^k|^{1/k}$ for $k = 1, \ldots, 25$. The first few values, for small $k$, give a misleading idea of what the limit value might be. The horizontal solid line indicates the true value of $r_2$, to which $|\mathrm{Tr}\,\mathcal{L}_2^k|^{1/k}$ is seen to converge.

The misleadingly small values (when compared with $r_2$) for $k = 2, 3$ can also be understood in terms of the eigenvalues of $\mathcal{L}_2$, illustrated in Figure 3, which do indeed fan out over different angles in the complex plane. The effect is not as pronounced here as in the ad hoc example above; it occurs only for $k = 2, 3$, and the first values of $\log|\mathrm{Tr}\,\mathcal{L}_2^k|$ do not line up along a line with misleading slope. In other, more complicated operators $\mathcal{L}_w$ the masking effect could well be more pronounced, and possibly be as strong as in the toy example. The Fredholm determinant method would of course not succeed any better in extracting $r_2$ from the values of $\mathrm{Tr}\,\mathcal{L}_2^k$ for small $k$; the "error bar estimate", computed from the Fourier coefficients of $w(\omega)$, would automatically tell us, however, that we have to look at larger $k$ in order to conclude something sensible.

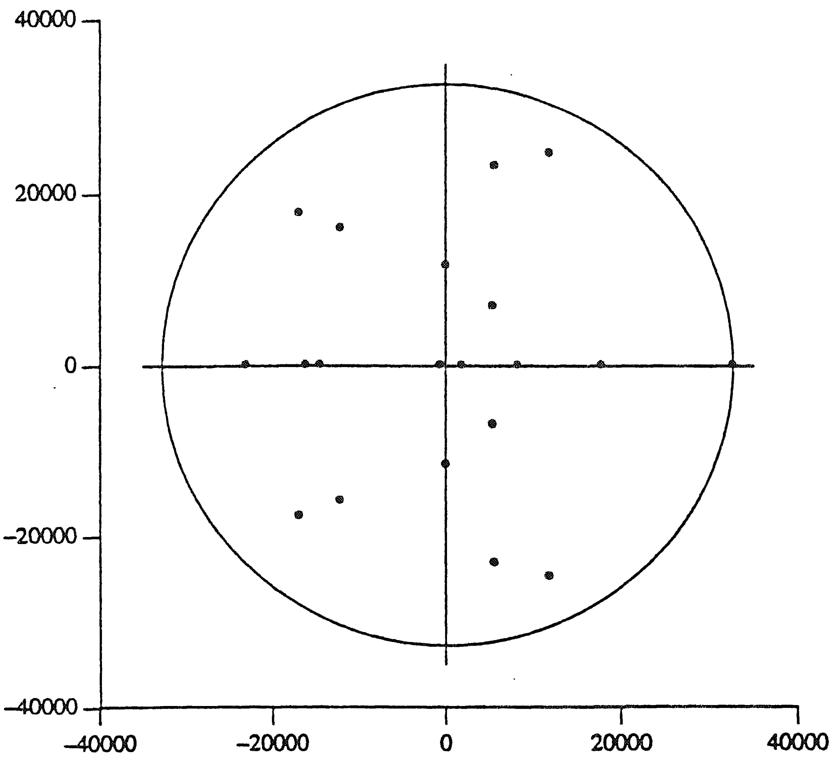


**Figure 3.** Position of the different eigenvalues of $\mathcal{L}_2$ for $q(\omega) = 100\,(1 - .9\,e^{3i\omega})\,(1 - .9\,e^{7i\omega})$.

2) If we replace $w$ in (3.19) by $|q|^k$, and substitute this into (5.7), then the resulting formula is very similar to formulas found in Hervé (1995) (see *e.g.* Theorem 6.2 in Hervé (1995) -beware of changes of notation). The only difference is that we have a denominator $2^k - 1$ in the arguments of $|q(\cdot)|^k$, where Hervé has $2^k$, because we sum over fixed points of $D^k$, where Hervé's approach sums over preimages of 0 under $D^k$ (where $D$ is multiplication by 2 on $[-\pi, \pi]$, $\bmod 2\pi$). The convergence of Hervé's formula is similar to that of (5.6), (5.7) above, and in principle, the same reservations as in point 1 above apply, although

we have not seen any cases where problems occurred in practice. In Hervé (1995) the operators $\mathcal{L}_p$ are studied on the much larger spaces $C^\gamma(-\pi,\pi)$ of Hölder continuous functions with exponent $\gamma$. On these spaces the $\mathcal{L}_p$ are not compact; they are quasi-compact, meaning that the radius of their essential spectrum is strictly smaller than the spectral radius itself; this corresponds to a spectrum where only discrete eigenvalues are possible in an outer annulus of the disk with spectral radius. Even though the operators are thus more complicated, Hervé's method has the advantage that he can treat also the case where $p \notin 2\mathbb{N}$ and $q$ has zeros in $[-\pi,\pi]$. It is interesting to note that the eigenvalues $\alpha_n$ in the annulus $\{\lambda;\ 2^{-\gamma}r < |\lambda| \leq r\}$ (where $\gamma$ is the Hölder regularity of $|q(\omega)|^p$) can still be tracked with the Fredholm determinant method (see Theorem 7.1 below), so that even in this case we can use our numerical approach and get absolute error estimates.

3) In practice, one can of course also use the reverse procedure: instead of setting first $\epsilon$ and then searching for the appropriate $N$, as sketched above, one can fix a (relatively large) value for $N$, find the corresponding smallest zero $z_{N,0}$ of $P_N$, and then identify $\epsilon_N$ so that $|R_N(z)| < \epsilon_N$ on the curve $\Gamma_{N,0}$ defined by (5.2).

## 6. Examples.

All our examples are motivated by wavelet constructions; we take the refinable function $\varphi$ to be either the orthonormal scaling function in a multiresolution analysis, or the autocorrelation function of a scaling function. We start by recalling some pertinent definitions.

The refinable function $\varphi(x)$ is said to be *cardinal interpolant* if it satisfies the condition

$$\varphi(k) = \delta_{0,k}\,, \qquad k \in \mathbb{Z}\,; \tag{6.1}$$

it is called *orthonormal* if

$$\int \varphi(x-k)\,\overline{\varphi(x-\ell)}\,dx = \delta_{k,\ell}\,, \qquad k,\ell \in \mathbb{Z}\,. \tag{6.2}$$

These properties correspond to special constraints on $m(\omega)$: (6.1) implies

$$m(\omega) + m(\omega+\pi) = 1\,, \tag{6.3}$$

whereas (6.2) can hold only if

$$|m(\omega)|^2 + |m(\omega + \pi)|^2 = 1\,. \tag{6.4}$$

The conditions (6.3) or (6.4) are necessary for (6.1) or (6.2) to hold, but not sufficient. Under additional technical conditions that ensure uniform convergence of the subdivision algorithm in the first case, or $L^2$-convergence in the second case, (6.3) implies (6.1) and (6.4) implies (6.2). (For a detailed discussion, see Chapter 6 in Daubechies (1992).)

It is clear that cardinal interpolation and orthonormality are linked: if $\phi$ is an orthonormal refinable function, then its autocorrelation function $\Phi(x) = \int \phi(y)\,\overline{\phi(x-y)}\,dy$ is interpolating; the corresponding functions $m_\phi$ and $m_\Phi$ are related by $m_\Phi = |m_\phi|^2$. In fact, compactly supported wavelets are usually constructed by first identifying a suitable positive $m_\Phi$ and then constructing $m_\phi$ so that $|m_\phi|^2 = m_\Phi$. It is then obvious that the $L^p$-Sobolev exponents of $\phi$ and $\Phi$ are related by

$$s_p(\Phi) = 2\,s_{2p}(\phi)\,. \tag{6.5}$$

In particular, using the definitions (1.2) and (1.3), we find

$$\mu(\Phi) = s_1(\Phi) = 2\,s_2(\phi) = 2\,s(\phi)\,, \tag{6.6}$$

where the first equality is a consequence of $\hat{\Phi}(\omega) \geq 0$ for all $\omega \in \mathbb{R}$.

In the first subsection below we concentrate on families of examples where $m_\Phi$ is a positive trigonometric polynomial of the form $P_\Phi(\cos\omega)$, so that $\Phi$ is real, symmetric and compactly supported. By Riesz' spectral factorization lemma, we can then find a trigonometric polynomial $m_\phi$, with real coefficients, so that $|m_\phi|^2 = m_\Phi$. The corresponding refinable functions $\phi$ are then compactly supported scaling functions from which compactly supported wavelets can be constructed; see Daubechies (1992). In the second subsection we consider examples where $P_\Phi(\cos\omega)$ is no longer a trigonometric polynomial, but the quotient of two such polynomials. The third subsection of examples looks at some two-dimensional examples $m(\omega_1, \omega_2)$ which cannot be written as products $m_1(\omega_1)\,m_2(\omega_2)$ of one-dimensional functions, with matrix dilations. Finally, in the fourth subsection we use the results of the previous subsections to deal with a problem on spline wavelet bases.

## 6.1. Interpolating and orthonormal scaling functions with compact support.

The minimal degree solution to (6.3) and the factorization requirement (4.3) is given by

$$m_N(\omega) = \left(\cos\frac{\omega}{2}\right)^{2N} \sum_{j=0}^{N-1} \binom{N-1+j}{j} \left(\sin\frac{\omega}{2}\right)^{2j}.$$

(We are interested in factoring out only even powers of $\cos(\omega/2)$ because we want $m_N(\omega) = P_N(\cos\omega)$.) In this case $q(\omega)$ is clearly strictly positive for all $\omega \in [0, 2\pi]$, so that we can apply our theorems for all values $p \geq 1$. The corresponding functions $\Phi_N$, $\phi_N$ have been studied extensively (see *e.g.* Daubechies (1992) for many references). We have computed the $L^p$-Sobolev exponents of these functions for different values of $N$. Table 1 and Figure 4 show the $s_p(\phi_N)$ for $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$.

| $N \setminus p$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|
| 1 | −0.322289 | 0.338856 | 0.669428 | 0.834714 |
| 2 | 0.521293 | 0.999820 | 1.220150 | 1.310014 |
| 3 | 0.979675 | 1.414947 | 1.587361 | 1.631686 |
| 4 | 1.391644 | 1.775305 | 1.896446 | 1.912144 |
| 5 | 1.767934 | 2.096541 | 2.171522 | 2.174682 |
| 6 | 2.116733 | 2.388060 | 2.430780 | 2.431755 |
| 7 | 2.441544 | 2.658569 | 2.680780 | 2.680307 |
| 8 | 2.746639 | 2.914556 | 2.926425 | 2.925926 |
| 9 | 3.035292 | 3.161380 | 3.166924 | 3.165533 |
| 10 | 3.309107 | 3.402546 | 3.405193 | 3.405141 |
| 11 | 3.572141 | 3.639569 | 3.641221 | 3.638529 |
| 12 | 3.825525 | 3.873991 | 3.874236 | 3.871917 |
| 13 | 4.071021 | 4.105802 | 4.105736 | 4.105305 |
| 14 | 4.311641 | 4.336042 | 4.336476 | 4.335502 |
| 15 | 4.547368 | 4.564708 | 4.564925 | 4.562449 |
| 16 | 4.780028 | 4.792323 | 4.792608 | 4.792645 |
| 17 | 5.010231 | 5.018884 | 5.018754 | 5.016283 |
| 18 | 5.238588 | 5.244390 | 5.244127 | 5.243230 |
| 19 | 5.464480 | 5.468841 | 5.468728 | 5.466868 |

**Table 1.** $L^p$-Sobolev exponents of $\varphi_N$, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$ (Polynomial).

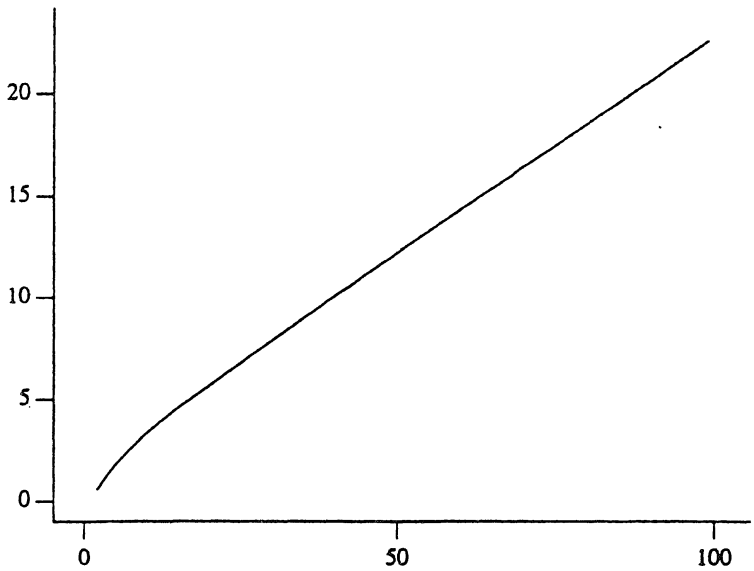


**Figure 4.** $s(\varphi_N)$ for $0 < N < 100$ (Polynomial).

An interesting observation is that $s_p(\varphi_N)$ becomes independent of $p$ as $N$ goes to $+\infty$. This reflects the fact that $\varphi_N$ has a lacunary structure in the Fourier domain and that this phenomenon grows with $n$. More precisely, if the Fourier transform of a function $\varphi$ decays uniformly at infinity in the sense that $C_1\,(1+|\omega|)^{-\alpha} \leq |\hat{\varphi}(\omega)| \leq C_2\,(1+|\omega|)^{-\alpha}$, then the exponents $s_p(\varphi) = \alpha - 1/p$ are related by $s_p - s_q = 1/q - 1/p$. This is true here only for $N = 1$ (which corresponds to the box function $\varphi_1(x) = \chi_{[0,1]}(x)$); $\hat{\varphi}_1(\omega) = (1 - e^{-i\omega})/(i\omega)$ decays uniformly in $|\omega|^{-1}$, up to the oscillation of the numerator. For larger $N$, the lacunary structure takes over. As shown in Volkmer (1992) and Cohen and Conze (1992), the worst decay occurs at the points $\omega_j = 2\pi\, 2^j/3$, $j > 0$. A possible explanation for our observation could be that the $L^p$ norm of $\hat{\varphi}_N$ concentrates at these points, as $N$ grows.

We thus conjecture that for all $p, q > 0$, $\lim_{N\to+\infty} |s_p(\varphi_N) - s_q(\varphi_N)| = 0$. If this is true, then we have in particular $\lim_{N\to+\infty} |s(\varphi_N) - \mu(\varphi_N)| = 0$ since $s_1(\varphi_N) \leq \mu(\varphi_N) \leq s(\varphi_N) = s_2(\varphi_N)$.

For large values of $N$, our method allows us to observe the asymptotic behavior of $\mu(\varphi_N)$.

It was proved by Volkmer (1992) that

$$\lim_{N\to+\infty}\frac{\mu(\varphi_N)}{N}=\lim_{N\to+\infty}\frac{s(\varphi_N)}{N}=1-\frac{\log_2 3}{2}\simeq 0.2075\,.$$

The graph of $s_2(\varphi_N)=s_2(N)$ presented in Table 1 shows in addition that $s_2(N)-(1-(\log_2 3)/2)N$ stays bounded by 3 for $N\le 100$.

### 6.2. Interpolating and orthonormal scaling functions with infinite support.

We now turn to the solutions of (6.3) that have the factorized form (4.3) but are not necessarily trigonometric polynomials.

We shall look for solutions of the type

$$m(\omega)=\cos^{2N}(\omega/2)\,R(\cos\omega)\,, \tag{6.7}$$

where $R(z)=P(z)/Q(z)$ is a rational function that is strictly positive on $[-1,1]$. Under this hypothesis we know that we can apply our method to estimate the $L^p$-Sobolev exponent $s_p$ of the associated scaling function since the Fourier coefficients of $|R(z)|^p$ have exponential decay. Note that the scaling function $\varphi$ is not compactly supported but typically still has exponential decay at infinity (some restrictions on $R$, always satisfied in practical examples, are needed to ensure this).

The choice of a rational function is still useful in the applications where one has to perform discrete convolutions with the Fourier coefficients of $m(\omega)$: although they are not finite in number, these convolutions can be implemented in a fast recursive way, the complexity being roughly $2S\times(\deg(P)+\deg(Q)+N)$ where $S$ is the size of the input data.

The simplest rational solution of (6.3) of the form (6.7) is given by the family

$$m_N(\omega)=\cos^{2N}(\omega/2)\,R_N(\cos\omega) \tag{6.8}$$

with $R_N(\cos\omega)=(\cos^{2N}(\omega/2)+\sin^{2N}(\omega/2))^{-1}$. These solutions are well known in signal processing as the transfer functions of the so-called "Butterworth filters" (see Oppenheim and Schafer (1975) for a detailed review).

As in the previous section, we give the estimate of $s_p$ for the orthonormal scaling functions $\varphi_N$, $1 \le N < 20$ and $p = 1, 2, 4, 8$. It is interesting to see that these exponents remain substantially different as $N$ grows: the lacunary behavior does not prevail as much as in the compactly supported case.

For large values of $N$, we have examined the evolution of $s(\varphi_N) = s(N)$ (see Table 2). It reveals a linear asymptotic behavior, similar to the compactly supported case.

| $N \setminus p$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|
| 1 | −0.322289 | 0.338856 | 0.669428 | 0.834714 |
| 2 | 0.677350 | 1.256211 | 1.495117 | 1.604344 |
| 3 | 1.561362 | 2.044109 | 2.269688 | 2.365870 |
| 4 | 2.370365 | 2.843768 | 3.059757 | 3.148599 |
| 5 | 3.183890 | 3.648646 | 3.857332 | 3.940563 |
| 6 | 3.999055 | 4.456118 | 4.658210 | 4.735925 |
| 7 | 4.815040 | 5.264533 | 5.460184 | 5.532265 |
| 8 | 5.630616 | 6.072947 | 6.262157 | 6.328326 |
| 9 | 6.446191 | 6.881125 | 7.063818 | 7.123827 |
| 10 | 7.260947 | 7.688598 | 7.864696 | 7.918627 |
| 11 | 8.075292 | 8.495600 | 8.664948 | 8.712863 |
| 12 | 8.888817 | 9.301894 | 9.464414 | 9.506534 |
| 13 | 9.701520 | 10.107480 | 10.263251 | 10.299921 |
| 14 | 10.513813 | 10.912358 | 11.061142 | 11.093166 |
| 15 | 11.325284 | 11.716526 | 11.858558 | 11.885986 |
| 16 | 12.135933 | 12.519984 | 12.655184 | 12.678805 |
| 17 | 12.946170 | 13.322968 | 13.451333 | 13.471625 |
| 18 | 13.755996 | 14.125241 | 14.247006 | 14.264159 |
| 19 | 14.564999 | 14.927039 | 15.042202 | 15.056836 |

**Table 2.** $L^p$-Sobolev exponents of $\varphi_N$, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$ (Butterworth).

Note that the limit ratio $s(N)/N \simeq .8$ seems to indicate that the worst decay of $\hat{\varphi}_N(\omega)$ occurs at the points $\omega_j = 2\pi\, 2^j/3$. Indeed, we have

$$|\hat{\varphi}_N(\omega_j)| = \left|\hat{\varphi}_N\Big(\frac{\pi}{3}\Big)\right| \left|m_N\Big(\frac{2\pi}{3}\Big)\right|^{j/2} = C\,|\omega_j|^{r_N} \tag{6.9}$$

with $r_N = \log_2(R_N(1/2))/2 - 1$. From the definition of $R_N$, we obtain

$$\lim_{N\to+\infty} \frac{r_N}{N} = -\frac{1}{2}\log_2 3 \simeq -0.7925, \tag{6.10}$$

which seems to coincide with the experimental asymptotic ratio.

The Butterworth functions $R_N(\cos\omega)$ correspond to a choice with $P(z) = 1$, $R(z) = 1/Q(z)$, which makes them in some sense opposites to the polynomial solutions of the previous subsection, for which $Q(z) = 1$, $R(z) = P(z)$. Recently, intermediate solutions that are equally balanced between the numerator and the denominator were proposed by Herley and Vetterli (1993). Such solutions can be built by the following procedure:

- fix $N > 0$ and $0 \leq k \leq N$ such that $N + k$ is odd.
- find a polynomial $P_k(z)$ such that

$$\left(\frac{1+z}{2}\right)^N P_k(z) + \left(\frac{1-z}{2}\right)^N P_k(-z)$$

has degree $N-k+1$. This can be done by solving $k$ linear equations.

- define

(6.11)

$$m_N^k(\omega) = \frac{\cos^{2N}(\omega/2)\, P_k(\cos\omega)}{\cos^{2N}(\omega/2)\, P_k(\cos\omega) + \sin^{2N}(\omega/2)\, P_k(-\cos\omega)} .$$

Note that the global complexity of the convolution by the discrete filter associated to $m_N^k(\omega)$ is given by

$$C \simeq 2S(k + (N - k + 1) + N) = 2S(2N + 1) \tag{6.12}$$

and is thus independent of $k$.

Here we have considered a family of intermediate solutions by taking $k$ close to $N/2$ so that the rational function $R_N^k(\cos\omega)$ has approximately the same number of poles as zeros. Unfortunately, and unlike the polynomial case (which can be viewed as a special case where $k = N - 1$), we do not have an explicit formula for $P_k(\cos\omega)$.

For the values $N = 4, 8, 12, 16, \ldots$, we have used $k(N) = N/2 + 1$ so that $N + k(N)$ is odd.

Figure 5 illustrates the evolution of $s(\varphi_N)$ for these particular intermediate solutions and compares it with the graphs obtained for polynomials and Butterworth solutions.

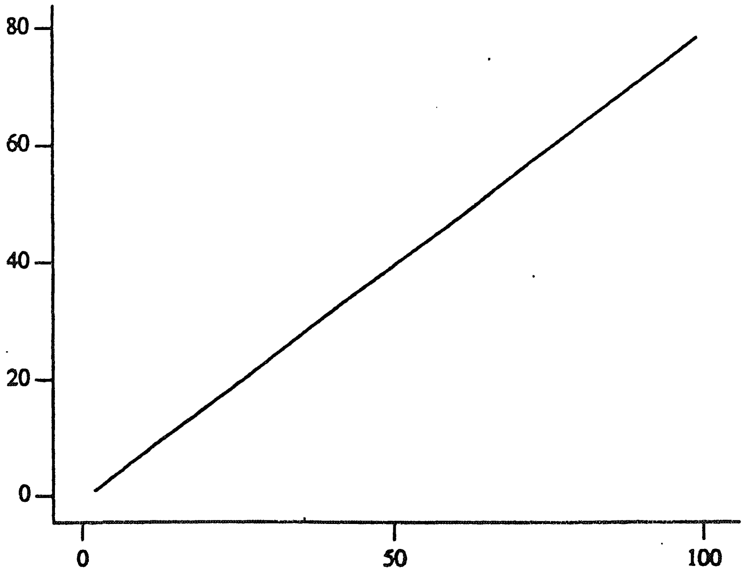


**Figure 5.** $s(\varphi_N)$ for $0 < N \leq 100$ (Butterworth).

A straightforward observation is that, although these intermediate solutions contain the same number of poles and zeros, the graph of $s(\varphi_N)$ is very close to the graph obtained for Butterworth scaling functions, making these particular discrete rational filters interesting for applications where regularity is desirable.

### 6.3. Nonseparable bidimensional scaling functions.

The simplest way to generate multivariate scaling functions is to use the tensor product, *i.e.* to define

$$\Phi(x_1, \ldots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n), \tag{6.113}$$

where $\varphi_1, \ldots, \varphi_n$ are univariate refinable functions. Note that if the univariate functions are cardinal interpolant or orthonormal, then the same property holds for $\Phi$. The analysis of the regularity of $\Phi$ then follows directly from the univariate analysis on the $\varphi_j$'s.

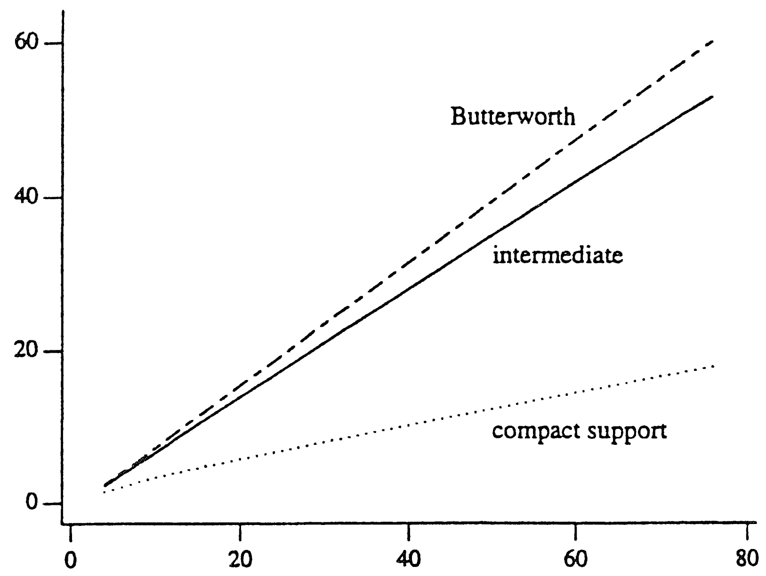


**Figure 6.** $s(\varphi_N)$, $N \leq 75$ for polynomial, Butterworth and intermediate solution.

One of the simplest -yet instructive- situations where non-separable scaling functions are unavoidable corresponds to the choice

$$D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{6.14}$$

for the dilation matrix, already introduced in Section 3 (Example 3.8). In that case the interpolatory condition has the following formulation:

$$M(\omega_1, \omega_2) + M(\omega_1 + \pi, \omega_2 + \pi) = 1 \,. \tag{6.15}$$

One can use the univariate functions $m_N(\omega)$, defined in the first subsection, to derive a solution of (6.15) as follows:

$$M_N(\omega_1, \omega_2) = \big(c(\omega_1, \omega_2)\big)^N \sum_{j=0}^{N-1} \binom{N-1+j}{j} \big(s(\omega_1, \omega_2)\big)^j \,, \tag{6.16}$$

where

$$c(\omega_1, \omega_2) = \frac{1}{2}\left(\cos^2\left(\frac{\omega_1}{2}\right) + \cos^2\left(\frac{\omega_2}{2}\right)\right)$$

and

$$s(\omega_1, \omega_2) = \frac{1}{2}\left(\sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right)\right).$$

We denote by $\Phi_N$ the nonseparable cardinal interpolant functions associated to $M_N$. Note however that the Riesz factorization lemma does not generalize in $nD$, $n > 1$, so that it is not possible to derive compactly supported orthonormal scaling functions from these $\Phi_N$. Using the preliminary results of Example 3.8, we can compute the Hölder exponents $\mu(\Phi_N) = s_1(\Phi_N)$ as well as $s_p(\Phi_N)$ for $p = 2, 4, 8$. We display their values, for $N = 1, \ldots, 19$ in Table 3. As in the univariate case, this table reveals the increasingly lacunary structure of the functions $\Phi_N$ in the Fourier domain.

| $N \setminus p$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|
| 1 | 0.611268 | 1.575915 | 1.939386 | 1.981617 |
| 2 | 2.285413 | 3.249338 | 3.684182 | 3.862247 |
| 3 | 3.881443 | 4.778977 | 5.146044 | 5.274342 |
| 4 | 5.395644 | 6.199651 | 6.487579 | 6.582461 |
| 5 | 6.841235 | 7.549708 | 7.780608 | 7.868502 |
| 6 | 8.233367 | 8.854675 | 9.054724 | 9.141341 |
| 7 | 9.584611 | 10.132740 | 10.318169 | 10.404515 |
| 8 | 10.904434 | 11.395230 | 11.573954 | 11.660232 |
| 9 | 12.200067 | 12.648141 | 12.823734 | 12.909992 |
| 10 | 13.477248 | 13.894538 | 14.068620 | 14.154872 |
| 11 | 14.740529 | 15.136095 | 15.309421 | 15.395672 |
| 12 | 15.993417 | 16.373816 | 16.546755 | 16.633005 |
| 13 | 17.238515 | 17.608370 | 17.781105 | 17.867355 |
| 14 | 18.477713 | 18.840232 | 19.012859 | 19.099109 |
| 15 | 19.712362 | 20.069763 | 20.242333 | 20.328582 |
| 16 | 20.943434 | 21.297249 | 21.469787 | 21.556036 |
| 17 | 22.171630 | 22.522919 | 22.695440 | 22.781690 |
| 18 | 23.397462 | 23.746966 | 23.919477 | 24.005727 |
| 19 | 24.621314 | 24.969550 | 25.142055 | 25.228305 |

**Table 3.** $L^p$-Sobolev exponent of $\Phi_n$, $p = 1, 2, 4, 8$, $N = 1, \ldots, 19$.

### 6.4. A problem on spline wavelet bases.

Spline wavelets are generated by a function $\psi$ that is piecewise polynomial (of a fixed degree $d$) on each integral $[k/2,(k+1)/2[$, $k \in \mathbb{Z}$. There exis' ` several types of spline wavelets:

a) Fully orthonormal wavelets: the family $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$ (Battle (1987), Lemarié (1988)). In that case, $\psi$ cannot be compactly supported except when $d = 0$, *i.e.* in the case of piecewise constant functions corresponding to the Haar system. (Note that in a generalized framework, where several scaling functions and wavelets are considered, even for the one dimensional case and dilation factor 2, compactly supported orthonormal wavelets are possible; see Donovan, Geronimo and Hardin (1994).)

b) Semi-orthonormal wavelets: the functions $\psi_{j,k}$ are orthogonal between levels $j \neq j'$ but not within one level $j$ for $k \neq k'$ (Chui-Wang 1990). In that case $\psi$ can be compactly supported but the dual function $\tilde{\psi}$ that generates the dual wavelet basis ($\langle\tilde{\psi}_{j,k},\psi_{j',k'}\rangle = \delta_{j,j'}\,\delta_{k,k'}$) is a noncompactly supported spline function.

c) Biorthogonal wavelet basis: $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis for $L^2(\mathbb{R})$ and there exists a dual system $\{\tilde{\psi}_{j,k}\}_{j,k\in\mathbb{Z}}$ as well as a dual scaling function $\tilde{\varphi}$ and multiresolution analysis $\tilde{V}_j$ (Cohen, Daubechies, and Feauveau (1992)). The functions $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$ may be simultaneously compactly supported, but $\tilde{\varphi}$ and $\tilde{\psi}$ are not spline functions in general.

Note that each construction is a particular case of the next one, and that in all cases the wavelet $\psi$ has the expression

$$\psi(x) = \sum_{n\in\mathbb{Z}} g_n\,\varphi(2n-n)\,, \tag{6.17}$$

where $\varphi = \chi_{[0,1]} * \chi_{[0,1]} \cdots * \chi_{[0,1]}$ ($d+1$ times) is the box spline of degree $d$, and $g_n$ is an oscillating $\ell^2$ sequence, *i.e.* $\sum_n g_n = 0$.

One can then address the following general problem: given an arbitrary oscillating sequence $g_n$, when does the corresponding combination (6.17) of box-splines generate a (Riesz) wavelet basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$? In particular, we have in mind very simple sequences such as $g_0 = 1$, $g_1 = -1$ or $g_0 = -1$, $g_1 = 2$, $g_2 = -1$, etc.

First, note that $\psi$ can be written in the Fourier domain as

$$\hat{\psi}(\omega) = m_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right), \tag{6.18}$$

where $m_1(\omega) = \sum_n g_n e^{-in\omega}$. Moreover, we have

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0(2^{-k}\omega), \tag{6.19}$$

where $m_0(\omega) = \left((1+e^{-i\omega})/2\right)^d$. In the case of biorthogonal wavelets (*i.e.* type c) above), $m_1(\omega)$ is equal to $e^{-i\omega}\,\overline{\tilde{m}_0(\omega+\pi)}$, where $\tilde{m}_0$ generates the dual scaling function $\tilde{\varphi}$ in the sense that

$$\hat{\tilde{\varphi}}(\omega) = \prod_{k=1}^{+\infty} \tilde{m}_0(2^{-k}\omega). \tag{6.20}$$

The biorthogonality constraint is expressed by the equation

$$\overline{m_0(\omega)}\,\tilde{m}_0(\omega) + \overline{m_0(\omega+\pi)}\,\tilde{m}_0(\omega+\pi) = 1, \tag{6.21}$$

or, equivalently,

$$e^{i\omega}\left(m_0(\omega)\,m_1(\omega+\pi) - m_0(\omega+\pi)\,m_1(\omega)\right) = 1. \tag{6.22}$$

It is clear that equation (6.22) is a strong restriction on $m_1$. Given a solution of (6.22), one can however construct other $m_1$ that still give rise to Riesz bases $\psi_{j,k}$. It suffices to take $m_1(\omega) = m(2\omega)M_1(\omega)$ where $M_1(\omega)$ satisfies equation (6.22) and $m(\omega)$ is a $2\pi$-periodic function such that

$$0 < c \le |m(\omega)| \le C < \infty, \tag{6.23}$$

almost everywhere with respect to $\omega \in \mathbb{R}$. This corresponds to the choice

$$\hat{\psi}(\omega) = m(\omega)\,M_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right). \tag{6.24}$$

We can then define $\Psi$ by $\hat{\Psi}(\omega) = M_1(\omega/2)\,\hat{\varphi}(\omega/2)$, and use the biorthogonal theory to study if $\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mathbb{R})$. If this is the case, then the same clearly holds for $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$. Note that

$m(\omega)$ and $M_1(\omega)$ are completely determined from $m_1(\omega)$ since we have $M_1(\omega) = m_1(\omega)/m(2\omega)$ and thus, by (6.22),

$$m(2\omega) = e^{i\omega}\left(m_0(\omega)\, m_1(\omega+\pi) - m_0(\omega+\pi)\, m_1(\omega)\right). \tag{6.25}$$

The system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ will thus constitute a Riesz basis if the two following conditions are satisfied:

i) the function $m(\omega)$ defined by (6.25) is bounded below and above by strictly positive constants.

ii) $\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis. A necessary and sufficient condition for this to hold was given in Cohen and Daubechies (1992). In our context this results in the following

**Theorem 6.1.** *Let*

$$\tilde{M}_0(\omega) = -e^{-i\omega}\,\overline{M_1(\omega+\pi)} \qquad \textit{and} \qquad M(\omega) = \frac{|\tilde{M}_0(\omega)|^2}{\cos^2(\omega/2)}\,.$$

*Assume that the Fourier coefficients of $M$ satisfy the decay condition* (3.3). *Define $d$ as the transition operator associated to $M(\omega)$ and denote by $\rho$ its spectral radius on a space $E_\alpha$ for any $\alpha\in]\gamma,2\gamma[$. Then $\{\Psi_{j,k}\}_{j,k\in\mathbb{Z}}$ constitutes a Riesz basis of $L^2(\mathbb{R})$ if and only if $\rho<4$.*

Note that, according to our results, this condition means that the $L^2$-Sobolev exponent of the scaling function $\tilde{\Phi}$ associated to $\tilde{M}_0$ is strictly positive.

Since we have

$$|\tilde{M}_0(\omega)|^2 = \frac{|m_1(\omega+\pi)|^2}{|m(2\omega)|^2}\,, \tag{6.26}$$

the function $M(\omega)$ is not a trigonometric polynomial in general, even when $\{g_n\}$ is a finite sequence. This made this application inaccessible to earlier methods that could only deal with finite masks.

An immediate application concerns the case of linear splines, *i.e.* $\varphi(x) = \sup\{0, 1-|x|\}$. In that case, we can propose three simple wavelets corresponding to different choices for the $g_n$ coefficients:

- $\psi_a(x) = \varphi(2x) - \varphi(2x-1)$,
- $\psi_b(x) = 2\,\varphi(2x) - \varphi(2x-1) - \varphi(2x+1)$,

- $\psi_c(x) = 2\varphi(2x-1) - \varphi(2x) - \varphi(2x-2)$.

One can easily check that for $\psi_b$, the associated function $m(\omega)$ vanishes at some point. It is also easy to check that this implies that Span $\{\psi_b(x-k)\}_{k\in\mathbb{Z}}$ cannot complement in a stable manner the space $V_0$ into $V_1$.

In the cases of $\psi_a$ and $\psi_c$, the associated function $m(\omega)$ does not vanish, so that we can further investigate the associated functions $M(\omega)$.

For $\psi_a$, we find

$$M(\omega) = \frac{16}{10 + 6\cos\omega}\,, \tag{6.27}$$

and our method shows that $\rho > 4$. For $\psi_c$ we find

$$M(\omega) = \cos^2\left(\frac{\omega}{2}\right)\left(\frac{4}{3+\cos\omega}\right)^2, \tag{6.28}$$

and in that case $\rho < 4$.

CONCLUSION: From the three functions above, only $\psi_c$ generates a Riesz wavelet basis.

## 7. Extension to the computation of the Hölder exponent.

The spaces $E_\alpha$ introduced in Section 3 have shown to be an excellent tool for the computation of $s_2$ in general, and of $s_p$ for $p \neq 2$ if the $h_n$ satisfy some additional conditions. Could a similar argument also be used for computing the Hölder exponent?

Let us restrict ourselves, for this discussion, to the one-dimensional case of equation (4.1). To start with, assume that only finitely many $c_n$ differ from zero, $c_n = 0$ for $n < 0$ or $n > K$. In this case Daubechies and Lagarias (1992) gave the following technique for computing the Hölder exponent $\mu$ of $\varphi$. First, we factor out all the zeros at $\omega = \pi$ of $m(\omega)$,

$$\sum_{n=0}^{K} c_n\, e^{-in\omega} = \left(\frac{1+e^{-i\omega}}{2}\right)^N \sum_{n=0}^{K-N} q_n\, e^{-in\omega}\,;$$

then we construct two $(K-N)\times(K-N)$ matrices $T_0$ and $T_1$ by taking

$$(T_j)_{k,\ell} = 2\,q_{2k-\ell+j}\,, \qquad 0 \le k,\ell \le K-N-1\,. \tag{7.1}$$

If there exist $\nu$, $C > 0$ so that, for all $k \in \mathbb{N}\backslash\{0\}$ and all $d_1, \ldots, d_k$ chosen in $\{0, 1\}$,

$$\|T_{d_1} \cdots T_{d_k}\| \le C\, 2^{\nu k}\,, \tag{7.2}$$

then $\mu \ge N - \nu$. (Note that we have changed notations slightly with respect to Daubechies and Lagarias (1992); we use the reduced representation that was there introduced in Section 5.) How can we distill from this a strategy that can be generalized to the case where infinitely many $c_n$ are nonzero? First of all, note that $T_0, T_1$ can be related simply to the operator $\mathcal{L}_w$ corresponding to the choice $w(\omega) = q(\omega)$. Comparing (7.1) with (3.4) we find indeed, for any $f(\omega) = \sum_{n=0}^{K-N} f_n\, e^{-in\omega}$ in the space $\mathcal{P}_{K-N}$ of one-sided trigonometric polynomials of degree $K - N$, that

$$\begin{aligned}(\mathcal{L}f)(\omega) &= \sum_{n=0}^{K-N} (T_0 f)_n\, e^{-in\omega}\,,\\ (\mathcal{L}Sf)(\omega) &= \sum_{n=0}^{K-N} (T_1 f)_n\, e^{-in\omega}\,,\end{aligned} \tag{7.3}$$

where we have introduced the shift operator $S$,

$$S\Big(\sum_\ell g_\ell\, e^{-i\ell\omega}\Big) = \sum_\ell g_{\ell+1}\, e^{-i\ell\omega} = e^{i\omega}\Big(\sum_\ell g_\ell\, e^{-i\ell\omega}\Big)\,. \tag{7.4}$$

From (3.1) one easily checks that $S\mathcal{L} = \mathcal{L}S^2$. The condition (7.2) can therefore be rewritten as: for all $k \in \mathbb{N}\backslash\{0\}$ and all $n$, $0 \le n \le 2^k - 1$,

$$\|\mathcal{L}^k S^n\, |_{\mathcal{P}_{K-N}}\| \le C\, 2^{\nu k}\,. \tag{7.5}$$

In this form it is easy to generalize (7.2): we could just drop the restriction to $\mathcal{P}_{K-N}$ in (7.5). There is one problem: in (7.5) it doesn't matter which norm we take, because for all $0 \le n \le 2^k - 1$ the operators $\mathcal{L}^k S^n$ map $\mathcal{P}_{K-N}$ to itself, so that we are dealing with a norm on matrices, and all matrix norms are equivalent. Once we look at the case of infinitely many nonvanishing $c_n$, and we drop the no longer relevant restriction to a finite-dimensional polynomial space, we need to specify which operator norm to use in (7.5). There is in fact a lot of freedom in the choice of this norm; in particular, if $E$ is a Banach space of $2\pi$-periodic functions such that

$$\sup_n |f_n| = \sup_n \left|\frac{1}{2\pi}\int_0^{2\pi} e^{in\omega}\, f(\omega)\, d\omega\right| \le C\, \|f\|_E\,, \tag{7.6}$$

then the operator norm

$$(7.7) \qquad |||A|||_E = \sup_{f\in E,\, \|f\|_E \neq 0} \frac{\|Af\|_E}{\|f\|_E}$$

will do. That is, if for all $k \in \mathbb{N}\backslash\{0\}$, all $0 \le n \le 2^k - 1$,

$$(7.8) \qquad |||\mathcal{L}^k S^n|||_E \le C\, 2^{\nu k}\,,$$

then it will follow that $\varphi$ has Hölder exponent at least $N - \nu$. The connection between (7.6), (7.8) and this Hölder continuity is explained in the Appendix. Note that (7.8) implicitly assumes that both $S$ and $\mathcal{L}$ map $E$ to itself.

Candidates for spaces $E$ that satisfy (7.6) abound. Examples are all the $E_\alpha$ of Section 3, as well as the $L^p(0, 2\pi)$-spaces, or the $C^\nu$-spaces of $2\pi$-periodic functions that have Hölder exponent $\nu$, and the $C^n$-spaces of $n$ times continuously differentiable periodic functions (including $C^0$). The $\ell^p$-spaces,

$$\ell^p = \left\{ f \ 2\pi\text{-periodic}:\ \|f\|_{\ell^p} = \Big(\sum_n |f_n|^p\Big)^{1/p} < \infty \right\},$$

also satisfy (7.6). For which of these spaces can we hope to verify (7.8) for some $\nu$?

The spaces $E_\alpha$, so convenient for the computation of the $s_p$, are completely useless here. Because $|||S^n|||_{E_\alpha} = |||S^{-n}|||_{E_\alpha} = e^{\alpha|n|}$, we have

$$|||\mathcal{L}^k S^n|||_{E_\alpha} = |||\mathcal{L}^k|||_{E_\alpha}\, e^{\alpha|n|}\,;$$

since $|n|$ can be as large as $2^k - 1$, the only $\mathcal{L}$ for which (7.8) can hold in $E_\alpha$ is the zero operator. In some sense, the $E_\alpha$-spaces are "too small" for our present purpose: their norm gets affected too much by $S$.

No such problem exists in the $L^p$, $\ell^p$ and $C^0$-spaces: they all share the property that $|||S|||_E = 1$. This reduces the estimate (7.8) to a spectral radius problem again: it suffices to prove that $\rho_E(\mathcal{L}) < 2^\nu$ in order to conclude that $\varphi$ has Hölder exponent at least $N - \nu$. If the space $E$ is chosen "too large", then we get a bad estimate for $\nu$, however. Take for example

$$m(\omega) = \left(\frac{1 + e^{-i\omega}}{2}\right)^2 \frac{1 + \gamma e^{i\omega}}{1 + \gamma} \qquad \text{with } \gamma > 1\,;$$

using the techniques of Daubechies and Lagarias (1992) one readily shows that the corresponding $\varphi$ has Hölder exponent exactly equal to $2-\log_2(2\gamma/(1+\gamma))$. On the other hand, it is easy to find eigenvectors for the corresponding $\mathcal{L}$ in the $\ell^p$-spaces. We have $q(\omega) = (1+\gamma e^{i\omega})/(1+\gamma)$, or

$$\mathcal{L}f = \mu f \quad \Longleftrightarrow \quad f_{2n} + \gamma f_{2n+1} = \frac{1}{2}\,\mu(1+\gamma)\, f_n\,. \tag{7.9}$$

Let now $\rho \in \mathbb{C}$ be arbitrary (to be fixed below), and define $f_n$ by

$$\begin{aligned} f_n &= 0\,, \qquad n \le 0\,,\\ f_1 &= 1\,,\\ f_{2n} &= \nu\Big(\frac{1+\gamma}{2} - \gamma\,\rho\Big)\, f_n\,, \qquad n \ge 1\,,\\ f_{2n+1} &= \rho\,\nu\, f_n\,, \qquad n \ge 1\,; \end{aligned} \tag{7.10}$$

then the $f_n$ obviously satisfy (7.9) with $\mu = \nu$. If, for some $\nu \in \mathbb{C}$, we can find $\rho \in \mathbb{C}$ so that the $f_n$ defined by (7.10) satisfy $\sum_n |f_n|^p < \infty$, then $\nu$ is an eigenvalue for $\mathcal{L}$ in $\ell^p$, and $\rho_{\ell^p}(\mathcal{L}) \ge |\nu|$. Let us check when this is true. First of all, note that

$$\begin{aligned} \sigma_N &= \sum_{n=2^N}^{2^{N+1}-1} |f_n|^p \\ &= \sum_{n=2^{N-1}}^{2^N-1} \big(|f_{2n}|^p + |f_{2n+1}|^p\big) \\ &= |\nu|^p\Big(\Big|\frac{1+\gamma}{2} - \gamma\,\rho\Big|^p + |\rho|^p\Big)\,\sigma_{N-1}\,, \end{aligned}$$

so that

$$\sum_{n=-\infty}^{\infty} |f_n|^p = \sum_{k=0}^{\infty} |\nu|^{pk}\Big(\Big|\frac{1+\gamma}{2} - \gamma\,\rho\Big|^p + |\rho|^p\Big)^k\,;$$

this is finite if and only if

$$|\nu|^p\,\Big(\Big|\frac{1+\gamma}{2} - \gamma\,\rho\Big|^p + |\rho|^p\Big) < 1\,.$$

Consequently

$$\rho_{\ell^p}(\mathcal{L}) \ge \max_{\rho\in\mathbb{C}}\Big(\Big|\frac{1+\gamma}{2} - \gamma\,\rho\Big|^p + |\rho|^p\Big)^{-1/p}\,.$$

In particular,

$$\rho_{\ell^2}(\mathcal{L}) \geq \frac{2\sqrt{1+\gamma^2}}{1+\gamma}\,, \qquad \rho_{\ell^1}(\mathcal{L}) \geq \frac{2\gamma}{1+\gamma}\,. \tag{7.11}$$

In fact, the lower bound for $\rho_{\ell^1}$ is exact, *i.e.* $\rho_{\ell^1}(\mathcal{L}) = 2\gamma/(1+\gamma)$. It is then clear from (7.11) that the spectral radius of $\mathcal{L}$ in the larger space $\ell^2$ is strictly larger, leading to a nonoptimal estimate of the Hölder exponent $\mu$ of $\varphi$. The same happens in the other $\ell^p$-spaces with $p > 1$.

This example teaches us that it is important to choose the space $E$ carefully. Note that the techniques in the literature for estimating $\mu$ can all be viewed in this way. The approaches of Rioul (1992) or Dyn and Levin (1991), Dyn (1991) correspond to estimates of the type

$$|||(\mathcal{L}^*)^k|||_{\ell^\infty} \leq C\,2^{\nu k}\,,$$

which is equivalent with

$$|||\mathcal{L}^k|||_{\ell^1} \leq C\,2^{\nu k}\,,$$

*i.e.* this corresponds to the choice $E = \ell^1$. In part of Hervé (1995), the choice $E = C^0$ is treated, a slightly larger space than $\ell^1$. One can show that the choices $E = \ell^1$ or $E = C^0$ lead to optimal values for $\mu$ (see Hervé (1995), Rioul (1992)). The example above also shows, however, that the operators $\mathcal{L}$ on $\ell^1$ or $C^0$ are far from their compact restrictions on the $E_\alpha$; in our example the entire disk $B(0, \rho_{\ell^1}) = \{z : |z| < \rho_{\ell^1}\}$ consists of eigenvalues of $\mathcal{L}$. This means that many iterative techniques, which usually rely on the fact that the largest eigenvalue is isolated, cannot be applied then.

So far we have seen that the $E_\alpha$ are "too small", the $\ell^p$ with $p > 1$ "too large" for our purposes; $\ell^1$ and $C^0$ are fine, but the spectrum of $\mathcal{L}$ on these spaces can consist of a disk of unisolated eigenvalues. It turns out that one can use slightly smaller spaces on which the largest eigenvalue of $\mathcal{L}$ becomes again isolated, and can be computed via the zeros of $\det(1 - z\mathcal{L})$. (These are, in fact, the spaces used for the computation of $s_p$ in Hervé (1995), who exploits the fact that the largest eigenvalue is isolated.) This is a consequence of a theorem in Ruelle (1990), of which the following statement is a special case, restricted to the situation under consideration here.

**Theorem 7.1.** *Let $q$ be a $2\pi$-periodic function satisfying* (4.4), *and let $\mathcal{L}$ be the associated transfer operator (obtained by replacing $w$ by $q$ in* *the definition* (3.1)). *Denote by $\mathcal{K}$ the operator obtained by replacing $w$ by $|q|$ in* (3.1). *Let $\rho$ be the spectral radius of $\mathcal{K}$ on $C^0(0,2\pi)$. Then, for any $\alpha > 0$, the spectral radius $\rho_\alpha$ of $\mathcal{K}$ on $C^\alpha(0,2\pi)$ equals $\rho$, $\rho_\alpha = \rho$; the spectral radius $\sigma_\alpha$ of $\mathcal{L}$ on $C^\alpha(0,2\pi)$ satisfies $\sigma_\alpha \leq \rho$. Moreover, the part of the spectrum of $\mathcal{L}$ on $C^\alpha(0,2\pi)$ that is contained in $\{\lambda : \ |\lambda| > 2^{-\alpha}\rho\}$ consists of only eigenvalues with finite multiplicities; these eigenvalues are exactly the inverses of the zeros of the Fredholm determinant $D(z)$,*

$$(7.12)\quad D(z) = \exp\Big(-\sum_{m=1}^{\infty}\frac{1}{m}z^m\frac{1}{1-2^{-m}}\sum_{k=0}^{2^m-1}\prod_{\ell=0}^{m-1}q\Big(2^\ell\frac{2\pi k}{2^m-1}\Big)\Big)$$

*in the region $\{z : \ |z| < 2^\alpha\rho^{-1}\}$, with the same multiplicities.*

If $q(\omega) \geq 0$ for $\omega \in [0,2\pi]$, then this theorem already implies that we can simply look for the smallest zero $z_0$ of (7.12), exactly like we did before. For any $\epsilon > 0$ it then follows that

$$|||\mathcal{L}^k|||_{C^0} \leq C\,(\rho+\epsilon)^k\,,$$

with $\rho = |z_0|^{-1}$, leading to the estimate $\mu \geq N - \log_2\rho$ for the Hölder exponent of $\varphi$. This is no surprise however: if $q(\omega) \geq 0$, then $|q(\omega)| = q(\omega)$, hence $\rho = r_1$, and $s_1 = N - \log_2 r_1 = N - \log_2\rho$. Since on the other hand

$$\hat{\varphi}(\omega) = \Big(\frac{1-e^{-i\omega}}{i\omega}\Big)^N\prod_{j=1}^{\infty}q(2^{-j}\omega)\,,$$

we have either

$$e^{i\omega L}\hat{\varphi}(\omega) \geq 0\,, \qquad \text{if } N = 2L \text{ is even,}$$

or

$$e^{iL\omega}\frac{1-e^{-i\omega}}{2}\,\hat{\varphi}(\omega) \geq 0\,, \qquad \text{if } N = 2L-1 \text{ is odd.}$$

It is well known from Littlewood-Paley theory that if $\hat{f}(\omega) \geq 0$ for all $\omega$, then the Hölder exponent of $f$ is exactly equal to the Sobolev exponent $s_1(f)$. It then follows easily, whether $N$ is even or odd, that the Hölder exponent $\mu$ of $\varphi$ is given by $\mu = N - \log_2\rho$.

If $q$ can also take negative values, then it follows from $\sigma_\alpha \leq \rho$ that for all $\epsilon > 0$,

$$|||\mathcal{L}^k|||_{C^\alpha} \leq C\,(\rho+\epsilon)^k\,.$$

Since $|||S^n|||_{C^\alpha} = |||S^{-n}|||_{C^\alpha} = |n|^\alpha$, we have therefore, for $0 \le n \le 2^k - 1$,

$$|||\mathcal{L}^k S^n|||_{C^\alpha} = |||\mathcal{L}^k|||_{C^\alpha}\ |n|^\alpha \le C\,(2^\alpha(\rho+\epsilon))^k\,.$$

Here $\alpha, \epsilon > 0$ can be chosen arbitrarily small, so that we have $\mu \ge N - \log_2 \rho$. This bound need not be sharp however: $\sigma_\alpha$ may well be smaller than $\rho$ for arbitrarily small $\alpha$. On the other hand, Theorem 7.1 also tells us that as we increase $\alpha$, the bothersome essential part of the spectrum of $\mathcal{L}$ on $C^\alpha$ shrinks, and at some point eigenvalues are uncovered which correspond to zeros of (7.12), which we can compute accurately. Let us imagine increasing $\alpha$ until Theorem 7.1 guarantees us that the spectral radius $\sigma_\alpha$ is exactly given by $|z_0|^{-1}$, with $z_0$ the smallest zero of (7.12). This happens when $2^{-\alpha}\rho \le |z_0|^{-1} < 2^{-\alpha}\rho + \delta$ for some small $\delta > 0$; we have then

$$\begin{aligned} |||\mathcal{L}^k S^n|||_{C^\alpha} &\le |||\mathcal{L}^k|||_{C^\alpha}\, 2^{k\alpha} \\ &\le C\, 2^{k\alpha}\,(|z_0|^{-1} + \epsilon)^k \\ &\le C\,(|2^{-\alpha} z_0|^{-1} + \epsilon')^k \\ &\le C\,(\rho + \epsilon'')^k\,, \end{aligned}$$

leading again to the same estimate $\mu \ge N - \log_2 \rho$.

This discussion has given a unified picture of the techniques used to find Sobolev and Hölder regularity indices for refinable functions. It also shows that if $w$ is given, and we can prove, for some $\alpha > 0$, that $\sigma_\alpha < 2^{-\alpha}\rho$, then this would lead to a sharper estimate for the Hölder exponent than $s_1(|w|)$; since $\sigma_\alpha$ corresponds to the spectral radius on a smaller space than $C^0$ or $\ell^1$, it might be easier to tackle $\sigma_\alpha$ than $\sigma_0$. How to do this is an open question however.

**Acknowledgment.** This work was the direct result of conversations of one of us (I. Daubechies) with David Ruelle, during a visit to IHES in June 1992. We gratefully acknowledge his input. Combining the writing up of this paper with our busy schedules was more difficult than we expected, and it would have taken even longer without the hospitality of various institutes that made it possible for us to get together at various times. We would like to thank all of these institutes; IHES (ID, June '92), AT&T Bell Laboratories (AC, March '93 and July '94), Univ. Bielefeld (ID, August '93).

## Appendix. The link between estimates on $\mathcal{L}^k S^n$ and the Hölder exponent of $\varphi$.

Let $E$ be a Banach space of $2\pi$-periodic functions such that, for all $f$ in $E$,

$$\text{(A.1)} \qquad \sup_n |f_n| = \sup_n \left| \frac{1}{2\pi} \int_0^{2\pi} e^{in\omega} f(\omega)\, d\omega \right| \le C\, \|f\|_E\,,$$

where $C$ is independent of $f$. We assume that $E$ contains the function $f(\omega) \equiv 1$, and that $E$ is also invariant for the shift operator $S$, defined by

$$(Sf)(\omega) = e^{i\omega} f(\omega)\,,$$

or by

$$(Sf)_n = f_{n+1}\,.$$

For an operator $A$ from $E$ to itself, we denote by $\|A\|_E$ its standard operator norm, $\|A\|_E = \sup_{f\in E, f\neq 0} \|Af\|_E / \|f\|_E$. Let $\varphi$ be a refinable function,

$$\text{(A.2)} \qquad \varphi(x) = 2 \sum_n c_n\, \varphi(2x - n)$$

and assume that

$$\text{(A.3)} \qquad \sum_n c_n\, e^{-in\omega} = \Big(\frac{1 + e^{-i\omega}}{2}\Big)^N \sum_n w_n\, e^{-in\omega}\,,$$

where $\sum_n w_n = 1$, $\sum_n (-1)^n w_n \neq 0$. Assume that the associated transfer operator $\mathcal{L}_w$ (defined by (3.1)) maps $E$ to itself. We have one last technical requirement. Let us consider $N = 1$, for simplicity. If we compute $\mathcal{L}_c 1$ formally, where $\mathcal{L}_c$ is the transfer operator associated with $c(\omega) = \sum_n c_n\, e^{-in\omega}$, then

$$\begin{aligned}(\mathcal{L}_c 1)(\omega) &= \Big(\frac{1 + e^{-i\omega/2}}{2}\Big)\, w\big(\frac{\omega}{2}\big) + \Big(\frac{1 - e^{-i\omega/2}}{2}\Big)\, w\big(\frac{\omega}{2} + \pi\big) \\ &= \Big[\mathcal{L}_w \Big(\frac{1+S}{2}\Big) 1\Big](\omega)\,,\end{aligned}$$

and $(\mathcal{L}_c 1)(0) = w(0) = 1$. It follows that $(\mathcal{L}_c 1) - 1$ has a zero in $\omega = 0$; we shall require that

$$\text{(A.4)} \qquad (\mathcal{L}_c 1 - 1)(\omega) = (1 - e^{-i\omega})\, g(\omega)\,,$$

where $g \in E$, which can also be written as

$$\pounds_w\Big(\frac{1+S}{2}\Big)1 - 1 \in (1-S)E\,.$$

For general $N$, this requirement takes the form that for $1 \le k \le N$, we should have

$$\text{(A.5)}\quad \pounds_w(1+S)^k 1 \in \operatorname{Span}\{(1-S)^k E,\ (1-S)^m 1, \text{ with } 0 \le m \le k-1\}\,.$$

Then we have the following

**Proposition.** *Let $E$, $w$ satisfy all the conditions above. If there exist $c > 0$ and $0 \le \nu < N$ so that, for all $k > 0$ and all $n$ between $0$ and $2^k$,*

$$\text{(A.6)}\qquad \|\pounds_w^k\, S^n\|_E \le C\, 2^{\nu k}\,,$$

*then $\varphi$ has Hölder exponent $\mu \ge N - \nu$.*

PROOF. The proof is essentially a generalization of the arguments in Daubechies and Lagarias (1992), adapted to the case with infinitely many coefficients. This means that alternatives have to be found for some matrix arguments in Daubechies and Lagarias (1992). We shall restrict ourselves here to the case $N = 1$, and discuss the proof in detail for this case; for general $N$, similar but slightly longer generalizations of Section 3 in Daubechies and Lagarias (1992) do the trick.

1). We start by defining a space $\tilde{E}$ by

$$\tilde{E} = \{f:\ f(\omega) = (1 - e^{-i\omega})\, g(\omega) + C \text{ with } c \in \mathbb{C} \text{ and } g \in E\}\,;$$

the norm on $\tilde{E}$ is simply

$$\|c + (1 - e^{-i\omega})\, g\|_{\tilde{E}} = |c| + \|g\|_E\,.$$

$\tilde{E}$ is clearly a Banach space which contains all trigonometric polynomials (since $1 \in E$ and $E$ is invariant under $S$). In $\tilde{E}$ we consider the subspace $\tilde{E}_1$ defined by

$$\tilde{E}_1 = \{f \in \tilde{E}:\ \sum_n f_n = f(0) = 0\}\,;$$

$\tilde{E}_1$ can be identified with the original space $E$, since

$$\tilde{E}_1 = (1 - e^{-i\omega})E\,.$$

2). Let $\mathcal{L}_c$ be the transfer operator associated with the weight function $c(\omega) = \sum_n c_n\, e^{-in\omega}$. Because of the factorization (A.3) we have, for $f \in \tilde{E}_1$,

$$\begin{aligned}(\mathcal{L}_c f)(\omega) &= \Big(\frac{1+e^{-i\omega/2}}{2}\Big)\, w\big(\frac{\omega}{2}\big)\,(1-e^{-i\omega/2})\, g\big(\frac{\omega}{2}\big) \\ &\quad + \Big(\frac{1-e^{-i\omega/2}}{2}\Big)\, w\big(\frac{\omega}{2}+\pi\big)\,(1+e^{-i\omega/2})\, g\big(\frac{\omega}{2}+\pi\big) \\ &= \frac{1}{2}(1-e^{-i\omega})\,(\mathcal{L}_w g)(\omega)\,.\end{aligned}$$

$\tilde{E}_1$ is thus invariant for $\mathcal{L}_c$, and the action of $\mathcal{L}_c$ on $\tilde{E}_1$ is equivalent to the action of $\mathcal{L}_w$ on $E$. We have moreover for $f \in \tilde{E}_1$ and $k \geq 1$,

$$\|\mathcal{L}_c^k f\|_{\tilde{E}} = 2^{-k}\, \|\mathcal{L}_w^k g\|_E \leq 2^{-(1-\nu)k}\, \|f\|_{\tilde{E}}\,. \tag{A.7}$$

3). On the other component of $\tilde{E}$, the action of $\mathcal{L}_c$ is completely determined by $\mathcal{L}_c 1$. Because of (A.4), we have $\mathcal{L}_c 1 = 1 + r$, with $r \in \tilde{E}_1$. It follows then from (A.7) that

$$\mathcal{L}_c^k 1 = 1 + \sum_{m=1}^{k-1} \mathcal{L}_c^m r$$

converges, for $k \to \infty$, to $1 + R = a$, with $R \in \tilde{E}_1$; $a$ is an eigenvector of $\mathcal{L}_c$ with eigenvalue 1.

4). Next we rewrite the refinement equation (A.2). For $x \in [0,1]$, we define the sequence-valued function $v(x)$ by

$$[v(x)]_n = \varphi(x+n)\,, \qquad n \in \mathbb{Z}\,.$$

Then (A.2) implies that

$$v(x) = \begin{cases} \mathcal{L}_c v(2x)\,, & \text{if } 0 \leq x \leq 1/2\,, \\ \mathcal{L}_c(Sv)(2x-1)\,, & \text{if } 1/2 \leq x \leq 1\,. \end{cases} \tag{A.8}$$

If, as in Daubechies and Lagarias (1992), we write $d_k(x)$ for the $k$-th digit in the binary expansion of $x$, then this becomes

$$v(x) = \mathcal{L}_c(S^{d_1(x)} v)(\sigma x)\,, \tag{A.9}$$

where $\sigma x = 2x$ if $x < 1/2$, $\sigma x = 2x - 1$ if $x \geq 1/2$. Smoothness for $\varphi$ on $\mathbb{R}$ implies smoothness for $v$; conversely, smoothness for $v$ on $[0,1]$ together with consistency conditions at the edges (of the style $[v(0)]_{n+1} = [v(1)]_n$) implies smoothness for $\varphi$. Solving (A.2) and proving smoothness for $\varphi$ therefore amounts to finding a fixed point $v(x)$ for the equation (A.9) and proving smoothness for $v(x)$.

5). Define now $v_0(x)$ by

$$[v_0(x)]_n = a_n\,(1-x) + a_{n+1}\,x\,,$$

where $a_n$ is the $n$-th component of the eigenvector $a$ of $\mathcal{L}_c$ obtained in point 3). This clearly satisfies the consistency condition at $x = 0$ and $x = 1$; moreover $v_0(0)$ is an eigenvector of $\mathcal{L}_c$ with eigenvalue 1, as it should be, according to (A.8). We also define, for $j \geq 1$, and $x \in [0,1]$

$$\begin{aligned} v_j(x) &= \mathcal{L}_c S^{d_1(x)} v_{j-1}(\sigma x) \\ &= \mathcal{L}_c S^{d_1(x)} \mathcal{L}_c S^{d_2(x)} \cdots \mathcal{L}_c S^{d_j(x)} v_0(\sigma^j x)\,. \end{aligned} \tag{A.10}$$

Every component of $v_j(x)$ is a piecewise linear spline with nodes at the dyadic rationals $2^{-j}k$ in $[0,1]$. Since $S\mathcal{L}_c = \mathcal{L}_c S^2$, we can also rewrite (A.10) as

$$v_j(x) = \mathcal{L}_c^j S^{D_j(x)} v_0(\sigma^j x)\,, \tag{A.11}$$

where $D_j(x) = \sum_{\ell=1}^{j} 2^{j-\ell} d_\ell(x)$.

6). Now note that, for all $x \in [0,1]$,

$$\sum_n [v_0(x)]_n = \sum_n a_n = 1\,,$$

since $a = 1 + R$ with $R \in \tilde{E}_1$. Since this will be preserved by both $\mathcal{L}_c$ and $S$, this implies, for all $j \geq 0$ and all $x \in [0,1]$,

$$\sum_n [v_j(x)]_n = 1\,.$$

It follows that $v_{j+1}(x) - v_j(x) \in \tilde{E}_1$ for all $j \geq 0$ and all $x \in [0,1]$. Because of (A.6), (A.7) and (A.11) this implies

$$\|v_{j+1}(x) - v_j(x)\|_{\tilde{E}} \leq C\,2^{-(1-\nu)j} \sup_{y\in[0,1]} \|v_1(y) - v_0(y)\|\,.$$

We can use this first to show that the $v_k(x)$ are uniformly bounded in $\tilde{E}$,

$$\begin{aligned}\|v_k(x)\|_{\tilde{E}} &\leq \|v_0(x)\|_{\tilde{E}} + \sum_{j=1}^{k-1} \|v_{j+1}(x) - v_j(x)\|_{\tilde{E}} \\ &\leq \|v_0(x)\|_{\tilde{E}} + \frac{C}{1 - 2^{-(1-\nu)}} \|v_1(x) - v_0(x)\|_{\tilde{E}} \ ;\end{aligned}$$

$\|v_0(x)\|_{\tilde{E}}$ is obviously bounded uniformly in $x$, and one easily checks that $\|v_1(x)\|_{\tilde{E}} = \|\mathcal{L}_c S^{d_1(x)} v_0(\sigma x)\|_{\tilde{E}}$ is as well. Next, we use the estimate again to prove that the $v_k(x)$ constitute a Cauchy sequence in $\tilde{E}$, uniformly in $x$,

$$\|v_{k+m}(x) - v_k(x)\| \leq C'' \, 2^{-(1-\nu)m} \, .$$

7). It follows that the $v_j(x)$ tend to a limit $v(x)$ in $\tilde{E}$, uniformly in $x$. If we "unfold" the $v_j(x)$ and $v(x)$ to define functions $\varphi_j$, $\varphi$ by

$$\varphi_j(x) = (v_j(x - \lfloor x \rfloor))_{\lfloor x \rfloor}$$

(similarly for $\varphi$), then $\varphi_j$ is piecewise linear with nodes at the $2^{-j}k$, $k \in \mathbb{Z}$, and, for any $x \in \mathbb{R}$, $x = n + y$ with $y \in [0, 1[$,

$$\begin{aligned}|\varphi_j(x) - \varphi(x)| &= |[v_j(y) - v(y)]_n| \\ &\leq C \, \|v_j(y) - v(y)\| \\ &\leq C' \, 2^{-(1-\nu)j} \, .\end{aligned}$$

where the second inequality follows because of property (A.1), which $\tilde{E}$ inherits from $E$. It then follows from standard results on approximation by splines that $\varphi$ is Hölder continuous with exponent $1 - \nu$.

For larger values of $N$, the proof runs along the same lines; the space $\tilde{E}$ is defined by adding the $N$ elements $1$, $(1 - e^{-i\omega}), \ldots, (1 - e^{-i\omega})^{N-1}$ to $(1 - e^{-i\omega})^N E$, which then lead to eigenvectors for $\mathcal{L}_c$ with eigenvalues $1, 1/2, \ldots, 2^{-N+1}$. The corresponding eigenvectors are used to define a spline starting point $v_0(x)$ which is piecewise polynomial of degree $N$, and one ends up with an estimate of type $\|v_j(x) - v(x)\|_{\tilde{E}} \leq C \, 2^{-(N-\nu)j}$, leading to the desired result.

## References.


Battle, G., A block spin construction of ondelettes, Part I: Lemarié functions. *Comm. Math. Phys.* **110** (1987). 601-615.

Cavaretta, A. S., Dahmen, W., Micchelli, C., Stationary subdivision. *Mem. Amer. Math. Soc.* **93** (1991), 1-186.

Chui, C. K., Wang, J. Z., 1991. A cardinal spline approach to wavelets. *Proc. Amer. Math. Soc.* **113** (1991), 785-793.

Cohen, A., Conze, J. P., Régularité des bases d'ondelettes et mesures ergodiques. *Revista Mat. Iberoamericana.* **8** (1992), 351-366.

Cohen, A., Daubechies, I., Non-separable bidimensional wavelet bases. *Revista Mat. Iberoamericana.* **9** (1993), 51-137.

Cohen, A., Daubechies, I., Feauveau, J. C., Biorthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **45** (1992), 485-560.

Cohen, A., Ryan, R., *Wavelets and Multiscale Signal Processing.* Chapman and Hall, 1995.

Conze, J. P., Sur le calcul de la norme de Sobolev des fonctions d'échelles. Preprint, Dept. of Math., Université de Rennes, France, 1989.

Daubechies, I., Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.* **41** (1988), 909-996.

Daubechies, I., *Ten Lectures on Wavelets.* CBMS-NSF Series in Applied Mathematics **61**, SIAM, 1992.

Daubechies, I., Lagarias, J. C., Two-scale difference equations, I. Global regularity of solutions. *SIAM J. Math. Anal.* **22** (1991), 1388-1410.

Daubechies, I., Lagarias, J. C., Two-scale difference equations II. Infinite matrix products, local regularity bounds and fractals. *SIAM J. Math. Anal.* **23** (1992), 1031-1079.

de Boor, C., *A practical guide to splines.* Applied Math. Science Series **27**, Springer, 1978.

Donovan, G., Geronimo, J., Hardin, D., Intertwining multiresolution analyses and the construction of piecewise polynomial wavelets. Preprint, Dept. of Math., Georgia Tech, Atlanta, 1994.

Dyn, N., Subdivision schemes in computer-aided design. In *Advances in Numerical Analysis II, Wavelets, Subdivision Algorithms and Radial Functions*, 36-104. Light, W. A. (ed), Oxford Univ. Press, 1992.

Dyn, N., Levin, D., Interpolating subdivision schemes for the generation of curves and surfaces. In *Multivariate Interpolation and Approximation*, 257-268. Haussmann, W., Jeller, K. (eds), Birkhauser, 1990.

Eirola, T., Sobolev characterization of solutions of dilation equations. *SIAM J. Math. Anal.* **23** (1992), 1015-1030.

Feauveau, J. C., Presentation at the 1989 International Conference on Wavelets and Applications. Marseille, 1989.

Feauveau, J. C., Analyse multirésolution par ondelettes non orthogonals et bancs de filtres numériques. Ph.D. thesis, Université de Paris-Sud, France, 1990.

Gohberg, I., Krein, M., *Introduction to the theory of linear nonselfadjoint operators.* Amer. Math. Soc. (Translation of 1965 Russian original), 1969.

Gripenberg, G., Unconditional bases of wavelets for Sobolev spaces. *SIAM J. Math. Anal.* **24** (1993), 1030-1042.

Grothendieck, A., 1956. La théorie de Fredholm. *Bull. Soc. Math. France,* **84** (1956), 319-384.

Herley, C., Vetterli, M., Wavelets and recursive filter banks. *IEEE Trans. Sign. Proc.*, **41** (1992), 2536-2556.

Hervé, L., Régularité et conditions de bases de Riesz pour les fonctions d'échelle. *C. R. Acad. Sci. Paris, Série I* **315** (1992), 1029-1032.

Hervé, L., Construction et régularité des fonctions d'échelle. *SIAM J. Math. Anal.* **26** (1995), 1361-1385.

Keane, M., Strongly mixing $g$-measures. *Invent. Math.* **16** (1972), 309-324.

Kovačević, J., Vetterli, M., Perfect reconstruction filter banks for HDTV representation and coding. *Image Comm.* **2** (1990), 349-364.

Kovačević, J., Vetterli, M., Nonseparable multidimensional perfect reconstruction filter banks and wavelet bases for $\mathbb{R}^n$. *IEEE Trans. Inf. Theory* **38** (1992), 533-555. (Special issue on wavelets).

Lemarié, P. G., Une nouvelle base d'ondelettes de $L^2(\mathbb{R}^n)$. *J. Math. Pures et Appl.* **67** (1988), 227-236.

Lidskii, V. B., Non-selfadjoint operators with a trace. *Dokl. Akad. Nauk SSSR*, **125** (1959), 485-487. English translation, *Amer. Math. Soc. Transl.* **47** (1965), 43-46.

Macdonald, I., *Symmetric functions and Hall polynomials.* Oxford Math. Monographs, Oxford Univ. Press, 1979.

Mallat, S., Multiresolution approximation and wavelet orthonormal bases of $L^2$. *Trans. Amer. Math. Soc.* **315** (1989), 69-88.

Meyer, Y., *Ondelettes ét opérateurs, I: Ondelettes, II: Opérateurs de Calderón-Zygmund, III: Opérateurs multilinéaires.* Hermann, 1990. An English translation is being prepared by the Cambridge Univ. Press.

Micchelli, C. A., Prautzsch, H., Uniform refinement of curves. *Linear Algebra Appl.* **114/115** (1989), 841-870.

Oppenheim, A., Schafer, R., *Digital signal processing.* Prentice Hall, 1975.

Rioul, O., Simple regularity criteria for subdivision schemes. *SIAM J. Math. Anal.* **23** (1992), 1544-1576.

Rudin, W., *Real and complex analysis.* McGraw-Hill, 1974.

Ruelle, D., Zeta-functions for expanding maps and Anosov flows. *Invent. Math.* **34** (1976), 231-242.

Ruelle, D., An extension of the theory of Fredholm determinants. *Inst. Hautes Etudes Sci. Publ.* **72** (1990), 175-193.

Schaefer, H., *Topological vector spaces.* MacMillan, 1966.

Schaefer, H., *Banach lattices and positive operators.* Springer, 1974.

Simon, B., *Trace ideal methods.* London Math. Soc. Lecture Notes **35**, Cambridge Univ. Press, 1979.

Vetterli, M., Multidimensional subband coding: some theory and algorithms. *Signal Processing* **6** (1984), 97-112.

Villemoes, L., Energy moments in time and frequency for 2-scale difference equation solutions and wavelets. *SIAM J. Math. Anal.* **23** (1992), 1519-1543.

Villemoes, L., Continuity of nonseparable quincunx wavelets. *ACHA* **1** (1994), 180-187.

Volkmer, H., On the regularity of wavelets. *IEEE Trans. Inf. Theory* **38** (1992), 872-876. (Special issue on wavelets).

*Recibido:* 25 de septiembre de 1.995

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