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Non-separable bidimensional wavelet bases

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Abstract. We build orthonormal and biorthogonal wavelet bases of $L^2(\mathbb{R}^2)$ with dilation matrices of determinant 2. As for the one dimensional case, our construction uses a scaling function which solves a two-scale difference equation associated to a FIR filter. Our wavelets are generated from a single compactly supported mother function. However, the regularity of these functions cannot be derived by the same approach as in the one dimensional case. We review existing techniques to evaluate the regularity of wavelets, and we introduce new methods which allow to estimate the smoothness of non-separable wavelets and scaling functions in the most general situations. We illustrate these with several examples.

I. Introduction.

In the most general sense, wavelet bases are discrete families of functions obtained by dilations and translations of a finite number of well chosen mother functions. The most well known are certainly dyadic orthonormal bases of $L^2(\mathbb{R})$, of the type

$$(1.1) \qquad \psi_k^j(x) = 2^{-j/2}\psi(2^{-j}x - k), \quad j, k \in \mathbb{Z}.$$

These constructions have found many interesting applications, both in mathematics because they form Riesz bases for many functional spaces and in signal processing because wavelet expansions are more appropriate than Fourier series to represent the abrupt changes in non-stationary signals.

Several examples have been given by Meyer [Me1], Lemarié [Le] and Daubechies [Dau1], generalizing the classic Haar basis in which the mother wavelet $\psi = \chi_{[0,1/2]} - \chi_{[1/2,1]}$ suffers from a lack of regularity since it is not even continuous. All are based on the concept of multiscale analysis, *i.e.* a ladder of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ which approximates $L^2(\mathbb{R})$,

$$\{0\} \to \ldots V_1 \subset V_0 \subset V_{-1} \ldots \to L^2(\mathbb{R}) \ , \tag{1.2}$$

(note that in some papers and in Meyer's book, the converse convention is used, *i.e.* $V_j \subset V_{j+1}$) and satisfies the following properties,

$$f(x) \in V_j \iff f(2x) \in V_{j-1} \iff f(2^j x) \in V_0 \ , \tag{1.3}$$

(1.4) there exists a function $\varphi(x)$ in V_0 such that the set $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0 .

Since $V_0 \subset V_{-1}$, the scaling function $\varphi(x)$ has to be the solution of a two scale difference equation,

$$\varphi(x) \ = \ 2\sum_{n\in\mathbb{Z}} c_n \ \varphi(2x-n) \ . \tag{1.5}$$

The associated wavelet is then derived from the scaling function by the formula

$$\psi(x) \ = \ 2\sum_{n\in\mathbb{Z}}(-1)^n \ \bar{c}_{1-n} \ \varphi(2x-n) \ . \tag{1.6}$$

In the standard interpretation of a multiresolution analysis, the projections of a function f on the spaces V_j are viewed as successive approximations to f, with finer and finer resolution as j decreases. The wavelets can then be used to express the additional details needed to go from one resolution to the next finer level, since the $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ constitute an orthonormal basis for W_0, the orthogonal complement of V_0 in V_{-1}. The whole set $\{\psi_k^j(x)\}_{j,k\in\mathbb{Z}}$ forms then an orthonormal basis of $L^2(\mathbb{R})$.

We are here interested in similar constructions adapted to functions or signals of more than one variable.

The most commonly used method to build a multiresolution analysis and wavelet bases in $L^2(\mathbb{R}^n)$ is the tensor product of a multiresolution analyses of $L^2(\mathbb{R})$. In $L^2(\mathbb{R}^2)$ it leads to a ladder of spaces $\mathcal{V}_j = V_j \otimes V_j \subset \mathcal{V}_{j-1}$ generated by the families,

$$\Phi^j_{k\ell}(x,y) = 2^{-j}\varphi(2^{-j}x-k)\,\varphi(2^{-j}y-\ell), \quad k,\ell \in \mathbb{Z}\,. \tag{1.7}$$

Three wavelets are then necessary to construct the orthogonal complement of $\mathcal{V}_0$ in $\mathcal{V}_{-1}$, namely,

$$\Psi_a(x,y) = \varphi(x)\psi(y)\ , \tag{1.8}$$

$$\Psi_b(x,y) = \psi(x)\varphi(y)\ , \tag{1.9}$$

$$\Psi_c(x,y) = \psi(x)\psi(y)\ . \tag{1.10}$$

Actually, the theory of multiresolution analysis, as it was introduced by S. Mallat and Y. Meyer (see [Ma1] and [Me1]) was first motivated by the possibility of building these separable wavelets for the analysis of digital picture.

It is clear, however, that this choice is restrictive and that it gives a particular importance to the x and y directions, since Ψ_a and Ψ_b match respectively the horizontal and vertical details.

A more general way of extending multiresolution analysis to n dimensions consists in replacing the axioma (1.3) and (1.4) by

$$f(x) \in V_j \iff f(Dx) \in V_{j-1} \tag{1.11}$$

(1.12) There exists a function ϕ in V_0 such that the set $\{\phi(x-k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis for V_0 ,

where D is a $n \times n$ dilation matrix.

All the singular values $\lambda_1, \ldots, \lambda_n$ of D must satisfy

$$|\lambda_m| > 1\ , \tag{1.13}$$

to ensure that the approximation gets finer in every direction as j goes to $-\infty$. Furthermore, we require D to have integer entries. This condition means that the action of D on the translation grid $\mathbb{Z}^n$ leads to a sublattice $\Gamma \subset \mathbb{Z}^n$.

The number of basic wavelets required to characterize the orthogonal complement of V_0 in V_{-1} is in that case trivially given by the following heuristic argument. This complement should be generated by the action of $\mathbb{Z}^n$ on the basic wavelets, in the same way that V_0 is generated by the action of $\mathbb{Z}^n$ on ϕ, whereas V_{-1} is generated by the action of $D^{-1}\mathbb{Z}^n$. Consequently, each of the generating functions can be associated with an elementary coset of $D^{-1}\mathbb{Z}^n/\mathbb{Z}^n \sim \mathbb{Z}^n/D\mathbb{Z}^n$ except one which corresponds to the scaling function (see figure 1). Therefore, $d = |\det D| - 1$ different wavelets are needed. Note that it is not strictly necessary that the entries of D be integer to build wavelet bases using D as the elementary dilation.

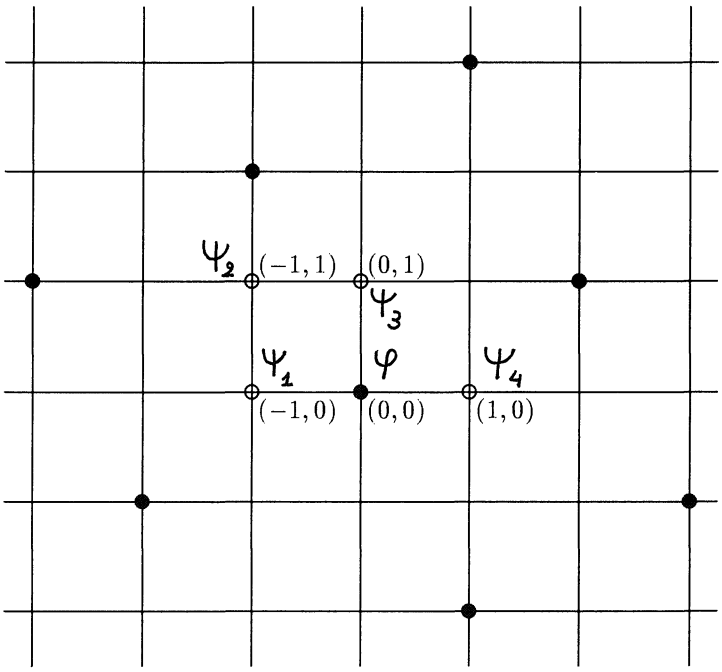


Figure 1

$\mathbb{Z}^2$ and $D\mathbb{Z}^2$ in the case where $D = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$.

The scaling function and the four basic wavelets are indexed by an element of $\mathbb{Z}^2/D\mathbb{Z}^2$.

However, the condition seems to be necessary for the existence of a multiresolution analysis based on a single, real valued, compactly supported scaling function.

In this work we shall indeed focus on real valued, compactly supported scaling functions and wavelets. They have the advantage that the sequence $\{c_n\}_{n\in\mathbb{Z}}$ introduced in the two scale difference equation (1.5) is real and finite. These coefficients play an important part in the numerical applications because they are used directly in the Fast Wavelet Transform algorithm as decomposition and reconstruction filters. They constitute in that case an FIR (finite impulse response) filter which can be implemented very easily. Furthermore, this finite set of coefficients contains all the information about the multiresolution analysis since the functions φ and ψ can be constructed as solutions of (1.5) and (1.6). Our starting point to build wavelet bases will thus be a finite set of coefficients and the associate two-scale difference equation, rather than the approximation spaces V_j themselves.

The main difficulty in this approach is the design of the FIR filter $\{c_n\}_{n=0,\ldots,N}$ in such a way that φ and ψ are smooth and have orthonormal translates.

In the one dimensional case, it is shown in [Dau1] that orthonormal wavelets can be constructed by choosing a filter which corresponds to a particular case of exact reconstruction subband coding schemes, and which can be made arbitrarily regular by increasing the number of taps in a proper way. Several contributions have followed, giving supplementary information on the type of filter which has to be used (see [Me2], [DL], [Co1], [Dau2], [Co2], [Dau3]).

In the present bidimensional case, the design of filters associated to "nice wavelet bases" turns out to be more difficult because some of the one-dimensional techniques do not generalize trivially (or do not generalize at all!) to higher dimensions and new methods have to be introduced. This article concentrates on the situation where D is a 2×2 matrix with $|\det D| = 2$.

We deliberately restrict ourselves to this set of matrices for two reasons:

- These dilations have already been considered by electrical engineers and seem to have interesting applications in signal analysis and image processing. For example, since only one basic wavelet is required, one may hope for a more isotropic analysis than with the separable construction. Subband coding schemes with decimation on the quincunx sublattice have been studied in the works of J. C. Feauveau [Fea] and M. Vetterli and J. Kovacevic [KV]. Our work is complementary to their signal processing approach since we investigate here the mathematical properties, such as the Hölder regularity of the wavelet bases associated to these schemes. This regularity is important when one asks that the reconstruction of the signal from the coarse scales has a smooth aspect (see Section II.2).

- These dilations are simple and our study will be reduced to the case of two basic matrices. However, the difficulties which appear in the evaluation of the regularity of the corresponding wavelets are common to all the non-separable constructions, and the techniques that we develop to solve this problem can be used for other types of dilations. We believe that the set of integer matrices with $|\det D| = 2$ constitutes an interesting "laboratory case" in the general framework of multidimensional wavelets.

In the next section of this paper, we shall give an overview of different techniques which can be used in the construction of one dimensional compactly supported wavelets. Some new tools will be introduced specifically to be generalized and used in the multidimensional situation.

The third section examines the possible subband coding schemes with decimation on the quincunx sublattice and their general relations with non-separable wavelet bases.

In the fourth section, orthonormal bases of wavelets are constructed from such coding schemes. We show that for the same filters, different bases with widely differing regularity can be obtained, depending on the choice of the dilation matrix. Finally, we use a biorthogonal approach, in Section V, to construct more symmetrical wavelet bases corresponding to linear phase filters and allowing a more isotropic analysis. We show that arbitrarily high regularity can be attained and we give some asymptotical results.

II. The construction of compactly supported wavelets in one dimension: A complete toolbox.

The purpose of this section is to review, in the one dimensional case, many different techniques that can be used to build regular wavelets from subband coding schemes, theoretically and numerically. Some of these techniques, like the Littlewood-Paley estimation of smoothness, are not frequently used in the one dimensional case, but they turn out to be very useful for the non-separable bidimensional wavelets. For more details, the reader can also consult [Dau1], [Me1], [Ma1], [Ve1], [Dau2], [Me2], [Co2].

Wavelet bases and subband coding schemes.

II.1.a. The orthonormal case.

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis of $L^2(\mathbb{R})$. We can use the discrete Fourier transform of the finite sequence $\{c_n\}_{n=N_1}^{N_2}$, *i.e.* the transfer function

$$m_0(\omega) = \sum_{n\in\mathbb{Z}} c_n e^{-in\omega} = \sum_{n=N_1}^{N_2} c_n e^{-in\omega} , \tag{2.1}$$

to rewrite the two scale difference equation (1.5) that characterizes $\varphi(x)$. We suppose that the c_n are real. Taking the Fourier transform of (1.5) and (1.6) we obtain

$$\hat{\varphi}(2\omega) = m_0(\omega)\hat{\varphi}(\omega) \tag{2.2}$$

$$\hat{\psi}(2\omega) = e^{-i\omega}\,\overline{m_0(\omega+\pi)}\;\hat{\varphi}(\omega) = m_1(\omega)\hat{\varphi}(\omega) . \tag{2.3}$$

Two fundamental properties of $m_0(\omega)$ can be derived from the multiresolution analysis properties

- Since $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_0, the Fourier transform $\hat{\varphi}(\omega)$ satisfies a Poisson identity

$$\sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega+2n\pi)|^2 = 1 . \tag{2.4}$$

Combined with (2.2) this leads to

$$|m_0(\omega)|^2 + |m_0(\omega+\pi)|^2 = 1 \tag{2.5}$$

which may also be written as

$$2\sum_{n\in\mathbb{Z}} c_n c_{n+2k} = \delta_{k,0} \; (= 1 \text{ if } k=0,\; 0 \text{ otherwise}) . \tag{2.6}$$

- The denseness of $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ is equivalent to

$$\hat{\varphi}(0) = \int \varphi(x)dx = 1,$$

(see [Me1], [Ma1] or [Co1]).

Consequently, we have

$$m_0(0) = 1 \quad\text{and}\quad m_0(\pi) = 0, \tag{2.7}$$

which may also be written as

$$\sum_{n=N_1}^{N_2} c_n = 1 \quad\text{and}\quad \sum_{n=N_1}^{N_2} (-1)^n c_n = 0. \tag{2.8}$$

The subband coding scheme associated to our multiresolution analysis appears clearly in the Fast Wavelet Transform Algorithm of S. Mallat [Ma2]. Let us recall how it works. The initial data are considered as the approximation of a continuous function at the scale $j = 0$,

$$S_k^0 = \langle f, \varphi(x-k)\rangle, \quad k \in \mathbb{Z}. \tag{2.9}$$

This allows the computation of the approximations and the details at coarser scales, *i.e.*

$$S_k^j = 2^{-j/2}\,\langle f, \varphi_k^j\rangle \quad\text{and}\quad D_k^j = 2^{-j/2}\,\langle f, \psi_k^j\rangle, \quad j > 0. \tag{2.10}$$

(The coefficients are normalized in such way that if $f \equiv 1$ locally, then $S_k^j = 1$ in that area). The sequence $\{S_k^j\}_{k\in\mathbb{Z}}$ (respectively $\{D_k^j\}_{k\in\mathbb{Z}}$) is then derived from $\{S_k^{j-1}\}_{k\in\mathbb{Z}}$ by a convolution with the filter $\overline{m_0(\omega)}$ (respectively, $\overline{m_1(\omega)}$) followed by a decimation of one sample out of two to keep the same total amount of information, *i.e.*

$$S_k^j = \sum_n c_{n-2k}\, S_n^{j-1}, \quad D_k^j = \sum_n (-1)^{n-1}\, c_{2k+1-n}\, S_n^{j-1}.$$

The algorithm then iterates on $\{S_k^j\}_{k\in\mathbb{Z}}$. Conversely, the sequence $\{S_k^{j-1}\}_{k\in\mathbb{Z}}$ can be recovered by applying the same filters $m_0(\omega)$ and $m_1(\omega)$ on $\{S_k^j\}_{k\in\mathbb{Z}}$ and $\{D_k^j\}_{k\in\mathbb{Z}}$ after inserting a zero between every pair of consecutive samples, and summing the two components (multiplied by two for normalization purposes), *i.e.*

$$S_n^{j-1} = 2\sum_k c_{n-2k}\, S_k^j + (-1)^{n-1}\, c_{2k+1-n}\, D_k^j \,.$$

All these operations, decomposition - decimation - interpolation - reconstruction, constitute a complete subband coding scheme as shown on figure 2. The property of exact reconstruction can now be derived in two ways. It is a natural consequence of the multiresolution approach, since $V_j = V_{j+1} \overset{\perp}{\oplus} W_{j+1}$ but it can also be viewed as a consequence of formula (2.5) for the filter m_0. This type of filter pair (m_0, m_1) is known as a pair of "conjugate quadrature filters" (CQF); they were first discovered by Smith and Barnwell in 1983, *cf.* [SB1]. The design of FIR pairs, with real coefficients and perfect reconstruction, has been generalized in [Dau1]. It also appears in [ASH], [SB2], [Ve1].

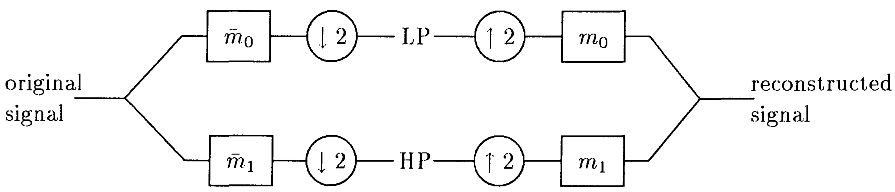


Figure 2

Subband coding scheme corresponding to the FWT algorithm.
The sign $2\downarrow$ stands for "decimation of one sample out of two" and $2\uparrow$ for the insertion of zeros at the intermediate values.

Since $m_0(\omega)$ is regular (it is a trigonometric polynomial) and since $m_0(0) = 1$, we can iterate (2.2) to obtain

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0(2^{-k}\omega) \,. \tag{2.11}$$

Given a conjugate quadrature filter $m_0(\omega)$ (*i.e.* a trigonometric polynomial satisfying (2.5) and (2.7)), it is thus possible to define the scaling function, either as a solution of the two scale difference equation (1.5), or explicitly with the above infinite product. However, this does not always lead to a multiresolution analysis: the function $\varphi(x) = \frac{1}{3}\chi_{[0,3]}$ generated by the CQF $m_0(\omega) = (1 + e^{3i\omega})/2$, for example, does not satisfy the orthonormality of the translates. Orthonormality of the $\varphi(x-k)$ turns out to be equivalent to the L^2 convergence of the truncated products $\hat{\varphi}_n(\omega) = \prod_{k=1}^{n} m_0(2^{-k}\omega)\chi_{[-2^n\pi, 2^n\pi]}(\omega)$ to $\hat{\varphi}(\omega)$ (because $\{\varphi_n(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal set as soon as (2.5) is satisfied).

More precisely, the following result characterizes the subclass of CQF filters leading to a multiresolution analysis and orthonormal basis of wavelets.

Theorem 2.1. *Let $m_0(\omega)$ be a Conjugate Quadrature Filter. Then, the infinite product* (2.11) *leads to a multiresolution analysis if and only if there exist a compact set $K \subset \mathbb{R}$ such that,*

i) *K contains a neighbourhood of the origin,*
ii) *$|K| = 2\pi$ and for all ω in $[-\pi, \pi]$, there exist $n \in \mathbb{Z}$ such that $\omega + 2n\pi \in K$,*
iii) *for all $n > 0$, $m_0(2^{-n}\omega)$ does not vanish on K.*

The set K is said to be "congruent to $[-\pi, \pi]$ modulo 2π" (figure 3). The proof of this result can be found in [Co1]. It exploits the continuity of m_0, the compactness of K and $m_0(0) = 1$ to show that (iii) is equivalent to $\hat{\varphi}(\omega) \geq c > 0$ on K. This is then sufficient to derive the L^2 convergence of the φ_n by Lebesgue's Theorem. We shall use a multidimensional generalization of Theorem 2.1 in the fourth section.

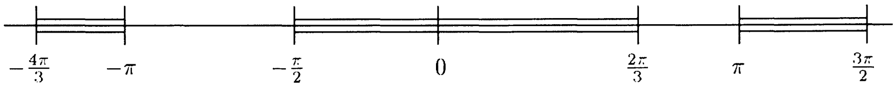


Figure 3
Example of compact set congruent to $[-\pi, \pi]$ modulo 2π.

II.1.b. The biorthogonal case.

The conjugate quadrature filters are a very particular case of subband coding scheme with perfect reconstruction, because identical filters (up to a complex conjugation) are used for both the decomposition and the reconstruction stages. If we do not impose this restriction, then the scheme uses four different filters: $\tilde{m}_0(\omega)$ and $\tilde{m}_1(\omega)$ for the decomposition, $m_0(\omega)$ and $m_1(\omega)$ for the reconstruction. Perfect reconstruction for any discrete signal is then ensured if,

$$\left\{\begin{array}{rcl} \overline{m_0(\omega)}\,\tilde{m}_0(\omega) + \overline{m_1(\omega)}\,\tilde{m}_1(\omega) & = & 1 \\ \tilde{m}_0(\omega+\pi)\,\overline{m_0(\omega)} + \tilde{m}_1(\omega+\pi)\,\overline{m_1(\omega)} & = & 0\,. \end{array}\right. \tag{2.12}$$

$\tilde{m}_0(\omega)$ and $\tilde{m}_1(\omega)$ may thus be regarded as the solutions of a linear system. However, to avoid the infinite impulse response solutions, we shall force the determinant of this system to be $\alpha e^{ik\omega}$, $\alpha \neq 0$, $k \in \mathbb{Z}$. For sake of convenience we take $\alpha = -1$ and $k = 1$ (a change of these values would only mean a shift and a scalar multiplication on the impulse response of our filters). This leads to

$$\overline{m_0(\omega)}\,\tilde{m}_0(\omega) + \overline{m_0(\omega+\pi)}\,\tilde{m}_0(\omega+\pi) = 1\,, \tag{2.13}$$

and

$$m_1(\omega) = e^{-i\omega}\,\overline{\tilde{m}_0(\omega+\pi)}\,, \quad \tilde{m}_1(\omega) = e^{-i\omega}\,\overline{m_0(\omega+\pi)}\,. \tag{2.14}$$

The formulas (2.13) and (2.14) are thus the most general setting for finite impulse response subband coders with exact reconstruction (in the two channel case). The functions $m_0(\omega)$ and $\tilde{m}_0(\omega)$ are called "dual filters". It is clear that the special case $m_0(\omega) = \tilde{m}_0(\omega)$ corresponds to the conjugate quadrature filters of II.1.a. However, dual filters are easier to design than CQF's. For example, if m_0 is fixed, $\tilde{m}_0$ can be found as the solution of a Bezout problem which is equivalent to a linear system. The coefficients of these filters can be very simple numerically (in particular they can have finite binary expansion which is very useful for practical implementation), furthermore they can be chosen symmetrical ("linear phase filter"), a property which is impossible to satisfy in the CQF case.

We can mimic, in this more general framework, the construction of orthonormal wavelets from CQF. Assuming that $m_0(0) = \tilde{m}_0(0) = 1$ and $m_0(\pi) = \tilde{m}_0(\pi) = 0$, we define

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0(2^{-k}\omega)\,, \tag{2.15}$$

$$\hat{\psi}(2\omega) = m_1(\omega)\hat{\varphi}(\omega) , \tag{2.16}$$

$$\hat{\tilde{\varphi}}(\omega) = \prod_{k=1}^{+\infty} \tilde{m}_0(2^{-k}\omega) , \tag{2.17}$$

$$\hat{\tilde{\psi}}(2\omega) = \tilde{m}_1(\omega)\hat{\tilde{\varphi}}(\omega) . \tag{2.18}$$

In [CDF], the following theorem was proved,

Theorem 2.2.

- *If* $\hat{\varphi}_n(\omega) = \prod_{k=1}^n m_0(2^{-k}\omega)\chi_{[-2^n\pi, 2^n\pi]}(\omega)$ *and* $\hat{\tilde{\varphi}}_n(\omega) = \prod_{k=1}^n \tilde{m}_0(2^{-k}\omega)\, \chi_{[-2^n\pi, 2^n\pi]}(\omega)$ *converge in* $L^2(\mathbb{R})$ *respectively to* $\hat{\varphi}(\omega)$ *and* $\hat{\tilde{\varphi}}(\omega)$*, then the following duality relations are satisfied*

$$\langle\varphi(x-k), \tilde{\varphi}(x-k')\rangle = \delta_{k,k'} \tag{2.19}$$

$$\langle\psi_k^j, \tilde{\psi}_{k'}^{j'}\rangle = \delta_{j,j'}\, \delta_{k,k'} \tag{2.20}$$

and for all f *in* $L^2(\mathbb{R})$ *one has the unique decomposition*

$$f = \lim_{J\to+\infty} \sum_{j=-J}^{J} \sum_{k\in\mathbb{Z}} \langle f, \psi_k^j\rangle\, \tilde{\psi}_k^j \tag{2.21}$$

(in the L^2 *sense).*

- *If* φ *and* $\tilde{\varphi}$ *satisfy* $|\hat{\varphi}(\omega)| + |\hat{\tilde{\varphi}}(\omega)| \le C(1+|\omega|)^{-1/2-\varepsilon}$ *for some* $\varepsilon > 0$*, then the families* $\{\psi_k^j\}_{j,k\in\mathbb{Z}}$ *and* $\{\tilde{\psi}_k^j\}_{j,k\in\mathbb{Z}}$ *are frames of* $L^2(\mathbb{R})$*.*
- *When these two properties hold, then* $\{\psi_k^j,\ \tilde{\psi}_k^j\}_{j,k\in\mathbb{Z}}$ *are biorthogonal (or dual) Riesz bases of* $L^2(\mathbb{R})$*.*

Many examples of these systems can be found in [CDF] and a sharper analysis of the frame conditions is developed in [CD]. We now recall a practical way of constructing φ and ψ numerically from a given subband coding scheme.

II.2. The cascade algorithm.

In the last section we saw that the scaling function $\varphi(x)$ could be approximated, at least in $L^2(\mathbb{R})$, by a sequence of band limited functions

$\{\varphi_n\}_{n>0}$ defined by

$$(2.22) \qquad \hat{\varphi}_n(\omega) = \prod_{j=1}^{n} m_0(2^{-j}\omega)\, \chi_{[-2^n\pi, 2^n\pi]}(\omega) .$$

These functions are characterized by their sampled values at the points $2^{-n}k$ $(k \in \mathbb{Z})$, *i.e.*

$$(2.23) \qquad s_k^n = \varphi_n(2^{-n}k) .$$

This sequence can also be considered as the impulse response of the transfer function

$$(2.24) \qquad S_n(\omega) = 2^n \prod_{j=1}^{n-1} m_0(2^j\omega) .$$

$S_n(\omega)$ can be obtained recursively by the formula

$$(2.25) \qquad S_{n+1}(\omega) = 2\, m_0(\omega)\, S_n(2\omega) .$$

In the time domain, (2.25) becomes an interpolation scheme; the sequence s_k^n is dilated by insertion of zeros $(S_n(\omega) \to S_n(2\omega))$ before being filtered (multiplication by $2\, m_0(\omega)$). We have thus,

$$(2.26) \qquad s_p^{n+1} = 2 \sum_{k\in\mathbb{Z}} c_{p-2k}\, s_k^n .$$

This iterative process, which computes the $\{s_k^n\}_{k\in\mathbb{Z}}$ sequences from an initial Dirac sequence $\delta_{0,k}$ is called the "cascade algorithm". We illustrate it on figure 4 (our sequences are represented by piecewise constant functions).

Note that it identifies exactly with the reconstruction stage in the FWT algorithm described in II.1.a. The scaling function is thus approached by the reconstructed signal from a single approximation coefficient at a coarse scale. Similarly, the wavelet will be obtained by starting the reconstruction from a detail coefficient at a coarse scale (and thus applying $m_1(\omega)$ at the first step of the cascade).

This explains why subband coding schemes associated with regular wavelets are particularly interesting: the smoothness of the wavelet determines the appearance of the coarse scale components of the reconstructed signal. A smooth appearance is important for many applications such as compression where a big part of the finer scale information is thrown away.

In the biorthogonal case, the analysis and the synthesis wavelets (ψ and $\tilde{\psi}$) need not have the same regularity. As just discussed, smoothness

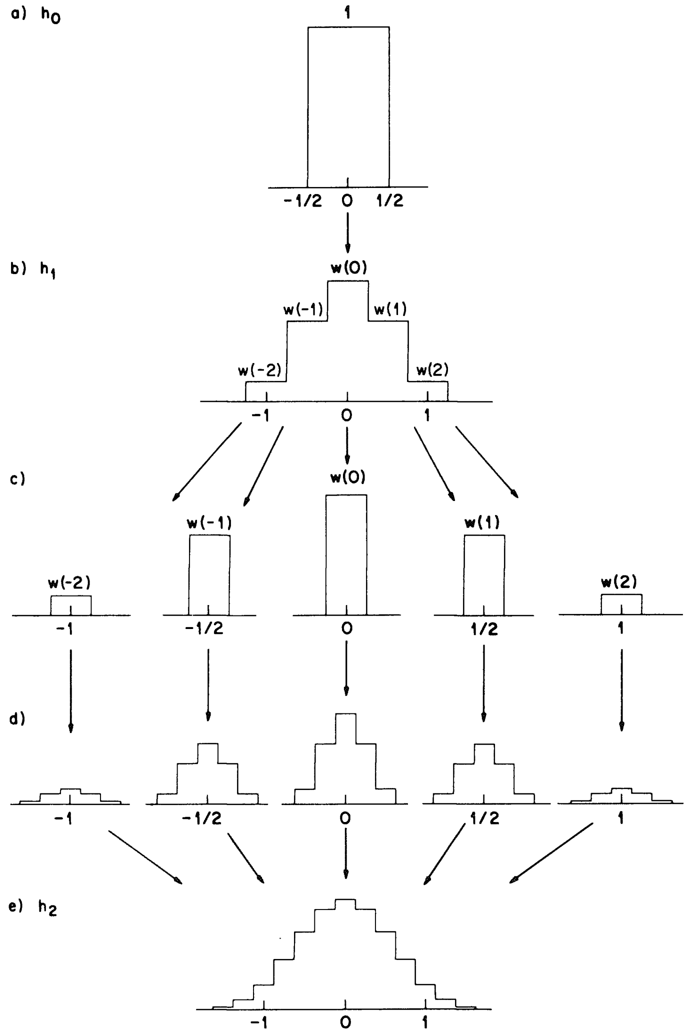


Figure 4
The cascade algorithm (from [Dau1]).

is important for the reconstructing function; the analyzing function needs only to be sufficiently regular to ensure that the wavelet bases are unconditional, so that the FWT algorithm is stable. Note that an important property on the analyzing wavelet is cancellation, *i.e.* vanishing moments, ensuring small high scale coefficients for smooth regions in the function or signal to be analyzed.

Let us finally mention that this type of "refinement method" is well known in approximation theory as "stationary subdivision" (*e.g.* [CDM], [DyL]). Most of these papers are motivated by interpolation problems, where smooth curves or surfaces need to be constructed, connecting (or close to) given sparse data points. Consequently, they are mainly concerned with what we call the reconstruction stage and they do not study the existence of an associated subband coding scheme. This also means that they do not care about an easy way of encoding or representing the extra "detail information" ($\longleftrightarrow W_j$) that can be added in going from one refinement level to the next one ($V_j \rightarrow V_{j-1}$). On the other hand, the subband coding literature seldom mentions the importance of the smoothness appearing in the cascade of the reconstruction from the low scales. Orthonormal and biorthogonal wavelet bases lead to an elegant combination of these two approaches.

We now present several different methods to estimate the regularity of the wavelets associated to a given subband coding scheme. We shall concentrate on the regularity of the scaling function which determines the regularity of the wavelet itself because $\psi(x)$ is a finite linear combination of translates of $\varphi(2x)$. Whatever the method used, if a global regularity of order r is achieved, then the cascade algorithm also converges uniformly up to this order (see [Dau1], [DL], [Co2]).

II.3. Regularity: the spectral approach.

II.3.a. A Fourier estimation of the Hölder exponent.

Let us denote by C^α the Hölder space defined as follows. For $\alpha = n+\beta$, $\beta \in [0,1[$, $f \in C^\alpha$ if and only if it is n times continuously differentiable and for all $x \neq y$,

$$\frac{|f^n(x) - f^n(y)|}{|x-y|^\beta} \leq C(f) .$$

Define also

$$\mathcal{F}_p^\alpha = \{f : \ (1+|\omega|)^\alpha \hat{f}(\omega) \in L^p\} \qquad (\alpha \geq 0, \ p \geq 1) . \tag{2.27}$$

It is well known (and easy to check) that $\mathcal{F}_\infty^{\alpha+1+\varepsilon} \subset \mathcal{F}_1^\alpha \subset C^\alpha$, for $\varepsilon > 0$. For compactly supported functions f, we also have

$$f \in C^\alpha \text{ implies } f \in \mathcal{F}_\infty^\alpha \tag{2.28}$$

so that the decay of the Fourier transform can be used to evaluate the global regularity. To estimate this decay in the case of the scaling function, it is possible to use the factorization of $m_0(\omega)$; due to its cancellation at $\omega = \pi$, we have indeed

$$m_0(\omega) = \left(\frac{1+e^{i\omega}}{2}\right)^N p(\omega) . \tag{2.29}$$

The infinite product (2.11) is thus divided in two parts. The first part, which comes from the factor $\left((1+e^{i\omega})/2\right)^N$ gives decay, since

$$\left|\prod_{k=1}^{+\infty}\left(\frac{1+e^{i2^{-k}\omega}}{2}\right)\right| = \left|\prod_{k=2}^{+\infty} \cos(2^{-k}\omega)\right| = \left|\frac{2}{\omega} \sin(\frac{\omega}{2})\right| . \tag{2.30}$$

The second part, which involves the factor $p(\omega)$, can be controlled by a polynomial expression. Indeed, since $p(0) = 1$ and p is a regular function, the infinite product generated by the second factor satisfies

$$\left|\prod_{k=1}^{+\infty} p(2^{-k}\omega)\right| \leq C \prod_{1\leq k<\log(1+|\omega|)/\log 2} |p(2^{-k}\omega)| . \tag{2.31}$$

Defining, for $j > 0$,

$$B_j = \sup_{\omega\in\mathbb{R}} \left|\prod_{k=0}^{j-1} p(2^k\omega)\right| \tag{2.32}$$

and

$$b_j = \frac{\log B_j}{j \log 2} , \tag{2.33}$$

we obtain

$$\left|\prod_{k=1}^{+\infty} p(2^{-k}\omega)\right| \leq C(B_j)^{\log(1+|\omega|)/\log 2} \leq C\,(1+|\omega|)^{b_j} \tag{2.34}$$

and

$$|\tilde{\varphi}(\omega)| \leq C\,(1+|\omega|)^{b_j - N} . \tag{2.35}$$

Consequently, φ is in $\mathcal{F}_1^{\alpha}$ and C^{α} if $\alpha < N - b_j - 1$ for some $j > 0$. We see here that N must be large to allow high regularity since b_j is always positive. In fact, one can prove that if the wavelet is r times continuously differentiable then it has at least $r+1$ vanishing moments (see [Me1], [Dau1]), *i.e.*

$$\left(\frac{d}{d\omega}\right)^n (\hat{\psi})(0) \;=\; \left(\frac{d}{d\omega}\right)^n (m_0)(\pi) \;=\; 0\,,$$

for $n = 0, \ldots, r+1$ and thus $N \geq r+1$. These cancellations are also known as the Fix-Strang conditions [FS]; they are equivalent to the property that the polynomials of order $N-1$ can be expressed as linear combinations of the $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$. However, these conditions are necessary but not sufficient to ensure the regularity of the scaling function since the effect of N may be killed by a large value of b_j. Fortunately, this can be avoided by a careful choice of the filter $m_0(\omega)$ (and, in the biorthogonal case, additionally $\tilde{m}_0(\omega)$).

In the CQF-orthonormal case, a particular family of FIR filters indexed by N has been constructed in [Dau1]. This construction uses the polynomial

$$P_N(y) \;=\; \sum_{j=0}^{N-1} \binom{N-1+j}{j}\; y^j \tag{2.36}$$

(with the shorthand notation $y = \sin^2(\omega/2)$), which is the lowest degree solution of the Bezout problem

$$P_N(y)(1-y)^N \;+\; y^N P_N(1-y) \;=\; 1\,. \tag{2.37}$$

The corresponding filters are defined by

$$m_0^N(\omega) \;=\; \left(\frac{1+e^{i\omega}}{2}\right)^N\; p_N(\omega) \tag{2.38}$$

with

$$|p_N(\omega)|^2 \;=\; P_N(y) \;=\; P_N\left(\frac{1-\cos\omega}{2}\right)\,. \tag{2.39}$$

The Fejer-Riesz lemma guarantees that there exists a FIR filter $p_N(\omega)$ which satisfies (2.39). It is clear that the CQF condition (2.5) is equivalent to (2.36) and the conditions in Theorem 2.1 are trivially satisfied with $K = [-\pi, \pi]$. For large values of N, the regularity $\alpha(N)$ of the associated scaling function is approximately $0.2\,N$ and the exact asymptotic ratio between $\alpha(N)$ and N can be determined. Intuitively speaking, this means that the contribution of $p_N(\omega)$ removes "eighty percent of the regularity" brought by the factor $\left((1+e^{i\omega})/2\right)^N$. For this estimation, we need to optimize the inequality (2.35), *i.e.* find the best possible exponent for the decay of $\hat{\varphi}(\omega)$.

II.3.b. Optimal and asymptotical Fourier estimation: The role of fixed points.

We start by defining "the critical exponent of $m_0(\omega)$":

$$b = \inf_{j>0} b_j = \inf_{j>0} \max_{\omega\in\mathbb{R}} \frac{1}{j\log 2} \log\left|\prod_{k=0}^{j-1} p(2^k\omega)\right| . \tag{2.40}$$

Then, it was proved in [Co2] that under the hypothesis $|p(\pi)| > |p(0)| = 1$ (satisfied in the present case (2.39)), $\hat{\varphi}(\omega)$ cannot have a better decay at infinity than $|\omega|^{b-N}$. If the infimum b is attained for some finite j, $b = b_j$, then this estimate is optimal.

How can we estimate the critical exponent? A first method consists in evaluating b_j for large values of j. Indeed, b is also the limit of the sequence b_j because the boundedness of p implies $b_J \leq b_j + O(j/J)$. This may however require heavy computations.

In several cases, it is possible to use a more powerful method based on the transformation $\tau : \omega \mapsto 2\omega$ mod-2π and the fixed points of its powers τ^n, $n > 0$. Indeed, let ω_0 be a fixed point of τ^n for $n > 0$ and define its orbit $\omega_j = \tau^j\omega_0$, for $j = 0, \ldots, n-1$. Since $p(\omega)$ has period 2π, we have

$$p(2^{nk}\omega_j) = p(\omega_j), \quad \text{for all } k > 0 \tag{2.41}$$

and consequently

$$b_{nk} \geq \frac{1}{n\log 2} \log\left|\prod_{j=0}^{n-1} p(\omega_j)\right| . \tag{2.42}$$

Letting k go to $+\infty$, this leads to

$$b \geq \frac{1}{n \log 2} \log \left| \prod_{j=0}^{n-1} p(\omega_j) \right| . \tag{2.43}$$

Fixed points of τ lead therefore to lower bounds for b and upper bounds for the regularity index. In fact they can do much better and provide optimal estimates for certain types of filters. Let us consider the smallest orbit of τ different from $\{0\}$, namely the pair $\{-2\pi/3,\ 2\pi/3\}$. Note that, because our filters have real coefficients, $|m_0(\omega)|$ and $|p(\omega)|$ are even functions so that $|p(2\pi/3)| = |p(-2\pi/3)|$. The following result associates the value $|p(2\pi/3)|$ and the critical exponent b.

Theorem 2.3. *Suppose that* $p(\omega)$ *satisfies*

$$|p(\omega)| \leq \left| p\left(\frac{2\pi}{3}\right) \right| \quad \text{if } |\omega| \leq \frac{2\pi}{3} , \tag{2.44}$$

$$|p(\omega) p(2\omega)| \leq \left| p\left(\frac{2\pi}{3}\right) \right|^2 \quad \text{if } \frac{2\pi}{3} \leq |\omega| \leq \pi . \tag{2.44'}$$

Then

$$b = \frac{1}{\log 2} \log \left| p\left(\frac{2\pi}{3}\right) \right| . \tag{2.45}$$

PROOF. We already know from (2.43) that $b \geq \log |p(2\pi/3)| / \log 2$. We now use the bounds on p to find an upper bound for b_j, $j > 0$. We can regroup the factors in (2.32) by packets of one or two elements in order to apply either (2.44) or (2.44$'$) on each block. Since only the last factor can miss one of these two inequalities, we obtain

$$\left| \prod_{k=0}^{j-1} p(2^k \omega) \right| \leq \left| p\left(\frac{2\pi}{3}\right) \right|^{j-1} \sup |p| , \tag{2.46}$$

and thus,

$$b_j \leq \frac{1}{\log 2} \left[\frac{j-1}{j} \log \left| p\left(\frac{2\pi}{3}\right) \right| + \frac{\sup\, [\log |p|]}{j} \right] , \tag{2.47}$$

which leads to

$$b \leq \frac{1}{\log 2} \log \left| p\left(\frac{2\pi}{3}\right) \right| \tag{2.48}$$

and to (2.45).

The equality (2.45) means that the worst decay of $\hat{\varphi}(\omega)$ occurs for the sequence $\omega_k = 2^n\pi/3$, $n > 0$. This is interesting, because (2.44) and (2.44′) turn out to be satisfied in many cases and in particular for the whole family of CQF defined by (2.38), (2.39). This is easy to check directly for small values of N, since the inequalities can be rewritten as

$$P_N(y) \leq P_N\left(\frac{3}{4}\right) \quad \text{if } y \leq \frac{3}{4}\,, \tag{2.49}$$

$$P_N(y)\, P_N(4y(1-y)) \leq \left(P_N\left(\frac{3}{4}\right)\right)^2 \quad \text{if } \frac{3}{4} \leq y \leq 1\,. \tag{2.49'}$$

The discussion for general N is more difficult and we refer to [CC] for a complete proof of (2.49), (2.49′). However, a similar result can be obtained in a simple way. To characterize the asymptotical behavior of the critical exponent when N goes to $+\infty$, one does not need the full force of (2.44), (2.44′), however. It can also be derived from a weaker, asymptotically valid inequality, as proved by H. Volkner in [V].

Theorem 2.4. *Let $b(N)$ be the critical exponent associated to $m_0^N(\omega)$ and $\alpha(N)$ the Hölder exponent of the corresponding scaling function. Then*

$$\lim_{N\to+\infty} \frac{b(N)}{N} = \frac{\log 3}{2\log 2} \tag{2.50}$$

and

$$\lim_{n\to+\infty} \frac{\alpha(N)}{N} = \lim_{N\to+\infty} \frac{N-b(N)}{N} = 1 - \frac{\log 3}{2\log 2} \simeq 0.2075\,. \tag{2.50'}$$

PROOF. This result can be viewed as a consequence of Theorem 2.3, but it can also be proved directly by using some properties of $P_N(y)$. Let us write (2.36) in the following form:

$$P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} \left(\frac{1}{2}\right)^j (2y)^j\,. \tag{2.51}$$

From (2.36) we see that $P_N(1/2) = 2^{N-1}$; since P_N is an increasing function between 0 and 1, we have

$$P_N \leq (\max\{4y,\, 2\})^{N-1} = |g(y)|^{N-1} . \tag{2.52}$$

It is now trivial to check that (2.49) and (2.49′) are satisfied if we replace $P_N(y)$ by $g(y)$. The same argument used in the proof of Theorem 2.3 leads then to

$$b(N) \leq \frac{N-1}{2\log 2} \log\left|g\left(\frac{3}{4}\right)\right| = \frac{N-1}{2\log 2} \log 3 \tag{2.53}$$

but from (2.43) we get

$$\begin{aligned} b(N) &\geq \frac{1}{2\log 2} \log\left|P_N\left(\frac{3}{4}\right)\right| \\ &\geq \frac{1}{2\log 2} \log\left|\binom{2N-2}{N-1}\left(\frac{3}{4}\right)^{N-1}\right| \geq \frac{N-2}{2\log 2} \log 3 \,. \end{aligned} \tag{2.54}$$

This proves the limit (2.50), and consequently (2.50′) since the decay index of the Fourier transform is equivalent to the Hölder exponent when both tend to $+\infty$.

The use of fixed points for optimal estimations of the spectral decay is thus very efficient when one is looking for arbitrarily high regularity since a sharp asymptotical result is obtained. For small filters, this method does not give a good result because the error on the exact regularity may have the same order as the value of the Hölder exponent itself. For such filters, other methods, which take advantage of the small number of taps in the filter, can be used to derive more precise estimations. We now describe these methods; they are typically based on matrix computations.

II.4. Regularity: Matrix based sharper estimates.

II.4.a. The Littlewood-Paley approach.

We first recall some aspects of the Littlewood-Paley theory. Let $\gamma(x)$ be a real-valued, symmetrical function of the Schwartz class $\mathcal{S}(\mathbb{R})$, which satisfies

$$\begin{cases} \hat{\gamma}(\omega) = 0 & \text{if } |\omega| \leq 1/2 \text{ or } |\omega| \geq 5/2\,, \\ \hat{\gamma}(\omega) > 0 & \text{if } 1/2 < |\omega| < 5/2\,, \end{cases} \tag{2.55}$$

so that the frequency axis is covered by the dyadic dilations of γ. Indeed, we have

$$(2.56) \qquad 0 < C_1 \leq \sum_{j=-\infty}^{+\infty} \hat{\gamma}(2^j\omega) \leq C_2 \qquad \text{if } \omega \neq 0 \, .$$

Define for any f in $\mathcal{S}'(\mathbb{R})$ the dyadic blocks $\Delta_j(f)$ by

$$(2.57) \qquad \Delta_j(f) \;=\; 2^j\gamma(2^j\cdot) * f \iff \hat{\Delta}_j(f) \;=\; \hat{\gamma}(2^{-j}\cdot)\hat{f} \, .$$

The Littlewood-Paley theory tells us that several functional spaces can be characterized by examining only the L^p norm of these blocks. This is the case in particular for the Sobolev spaces $W^{p,s}$ and the Hölder spaces C^α, $\alpha > 0$. To do this, it is necessary to change slightly the definition of C^α when α is an integer; we shall say that a bounded function f is in C^n if and only if f^{n-1} belongs to the Zygmund class Λ, *i.e.* there exists a constant C such that, for all x and y, we have

$$(2.58) \qquad \left| f^{n-1}(x+y) + f^{n-1}(x-y) - 2f^{n-1}(x) \right| \leq C\,|y| \, .$$

With this convention, the Hölder space C^α is characterized by the following conditions,

$$(2.59) \qquad \|\Delta_j(f)\|_{L^\infty} \leq C\,2^{-\alpha j} \quad \text{when } j \geq 0 \, ,$$

$$(2.59') \qquad f \text{ is a bounded continuous function.}$$

Note that the choice (2.55) for γ is arbitrary and that more general functions could be chosen to divide the Fourier domain into dyadic blocks. To derive these types of estimates on the scaling function φ, we introduce a tool which will be very useful in the bidimensional case.

Definition 2.1. *Let $L^2[0,\, 2\pi]$ be the space of 2π-periodic, square integrable functions on $[0,\, 2\pi]$, and $C[0,\, 2\pi]$ the space of 2π-periodic continuous functions. Then, for any $m(\omega)$ in $C[0,\, 2\pi]$, we define the transition operator T_m associated to $m(\omega)$ by*

$$(2.60) \qquad \begin{cases} T_m : \; L^2[0,\, 2\pi] \longrightarrow L^2[0,\, 2\pi] \\ f \mapsto T_m f(\omega) \;=\; m\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) \\ \qquad\qquad\qquad + \, m\left(\frac{\omega}{2} + \pi\right) f\left(\frac{\omega}{2} + \pi\right) \, . \end{cases}$$

Note that when $m(\omega)$ is a trigonometric polynomial, the study of T_m can be made in a finite dimensional space. More precisely, if we define

$$(2.61)\quad E(N_1,\, N_2) \;=\; \left\{ \sum_{n=N_1}^{N_2} h_n e^{in\omega} : \quad (h_{N_1}, \ldots, h_{N_2}) \in \mathbb{C}^{N_2-N_1+1} \right\}$$

then we have clearly

$$(2.62)\qquad (f,\, m) \in [E(N_1,\, N_2)]^2 \text{ implies } T_m f \in E(N_1,\, N_2)\ .$$

This is due to the contraction $\omega \mapsto \omega/2$ which appears in the definition (2.60) of T_m. If c_n is the n-th Fourier coefficient of $m(\omega)$, then the matrix of T_m in the basis of the complex exponentials is given by

$$(2.63)\qquad T_{\ell,n} \;=\; (2\, c_{2\ell-n})\ .$$

The size of this matrix P in $E(N_1,\, N_2)$ is $L \times L$ with $L = N_2 - N_1 + 1$. This operator has been studied by J. P. Conze and A. Raugi and several ideas presented below are due to their work [CR], [Con]. We shall use it to derive Littlewood-Paley type of estimations for the Hölder continuity of the scaling function. For this, we need the following result.

Lemma 2.5. *For all* $n > 0$,

$$(2.64)\qquad \int_{-\pi}^{\pi} (T_m)^n f(\omega) d\omega \;=\; \int_{-2^n\pi}^{2^n\pi} f(2^{-n}\omega) \prod_{k=1}^{n} m(2^{-k}\omega) d\omega\ .$$

PROOF. We prove it by induction. It is clear for $n = 1$ since

$$\begin{aligned}\int_{-\pi}^{\pi} T_m f(\omega) d\omega &= \int_{-\pi}^{\pi} \left[m\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right) \;+\; m\left(\frac{\omega}{2}+\pi\right) f\left(\frac{\omega}{2}+\pi\right)\right]\, d\omega \\ &= 2\int_{-\pi/2}^{\pi/2} [m(\omega)f(\omega) + m(\omega+\pi)f(\omega+\pi)]\, d\omega \\ &= 2\int_{-\pi}^{\pi} m(\omega)f(\omega)d\omega \;=\; \int_{-2\pi}^{2\pi} m\left(\frac{\omega}{2}\right) f\left(\frac{\omega}{2}\right)\, d\omega\ .\end{aligned}$$

Assuming (2.64) for n, we obtain at the next step,

$$\int_{-\pi}^{\pi} (T_m)^{n+1} f(\omega)\, d\omega = \int_{-\pi}^{\pi} (T_m)^n T_m f(\omega)\, d\omega$$

$$= \int_{-2^n\pi}^{2^n\pi} \left[\prod_{k=1}^{n} m(2^{-k}\omega)\right] \left[m(2^{-n-1}\omega)f(2^{-n-1}\omega) + m(2^{-n-1}\omega+\pi)f(2^{-n-1}\omega+\pi)\right] d\omega$$

$$= 2^{n+1} \int_{-\pi/2}^{\pi/2} \left[\prod_{k=1}^{n} m(2^{k}\omega)\right] \left[m(\omega)f(\omega) + m(\omega+\pi)f(\omega+\pi)\right] d\omega$$

$$= \int_{-2^{n+1}\pi}^{2^{n+1}\pi} \left[\prod_{k=1}^{n+1} m(2^{-k}\omega)\right] f(2^{-n-1}\omega)\, d\omega \; .$$

This concludes the proof.

We now suppose that $m(\omega)$ is a positive trigonometric polynomial in $E_M = E(-M, M)$ and that $m(0) = 1$ and $m(\pi) = 0$. Then m can be factorized as

$$m(\omega) = \cos^{2N}\left(\frac{\omega}{2}\right) p(\omega) \tag{2.65}$$

where $p(\omega)$ is a trigonometric polynomial that does not vanish for $\omega = \pi$. Note that necessarily $N \leq M$. From this cancellation property, we can derive,

Lemma 2.6. *$\{1, 1/2, \ldots, 2^{-2N+1}\}$ are eigenvalues of T_m. The row vectors $p_j = (n^j)_{n=-M,\ldots,M}$, for $0 \leq j \leq 2N-1$ generate a subspace which is left invariant by T_m and contains one eigenvector for each of these $2N$ eigenvalues.*

Consequently, the orthogonal subspace defined by

$$F_N = \left\{ \sum_{n=-M}^{M} h_n e^{-in\omega} : \sum_{n=-M}^{M} n^j h_n = 0,\ j = 0, \ldots, 2N-1 \right\} \tag{2.66}$$

is right invariant by T_m .

PROOF. The factorization in (2.65) is equivalent to the cancellation rules

$$\sum_{n=-M}^{M} (-1)^n\, n^j c_n = 0 \quad \text{for } j = 0, \ldots, 2N-1 \; . \tag{2.67}$$

In particular, for $j = 0$, we have

$$(2.68) \qquad \sum_n c_{2n} = \sum_n c_{2n+1} = \frac{1}{2} \quad (\text{because } m(0) = 1) \; .$$

This means that the sum of each column in the matrix of T (2.63) is equal to 1 and that $p_0 = (1, \ldots, 1)$ is a left eigenvector for the eigenvalue 1. For $0 < j \leq 2N - 1$ we define $q_j = p_j P = \left(q_j^{-M}, \ldots, q_j^M\right)$; we have,

$$(2.69) \qquad q_j^\ell = \sum_n n^j c_{2n-\ell} \; .$$

Thus, if ℓ is even

$$(2.70) \qquad q_j^\ell = \sum_n \left(n + \frac{\ell}{2}\right)^j c_{2n}$$

and if ℓ is odd

$$(2.70') \qquad q_j^\ell = \sum_n \left(n + \frac{1}{2} + \frac{\ell}{2}\right)^j c_{2n+1} \; .$$

Using the binomial formula and the cancellation rules (2.67), we see that q_j is a linear combination of p_k for $k = 0, \ldots, j$. The coefficient of p_j is given by the last term of the binomial and is thus equal to 2^{-j}. Consequently $\{p_j\}_{j=0,\ldots,2N-1}$ is a triangular basis for the left action of T_m and the eigenvalues are $\{2^{-j}\}_{j=0,\ldots,2N-1}$.

We now come back to the scaling function φ, given by the infinite product

$$(2.71) \qquad \hat{\varphi}(\omega) = \prod_{k=0}^{+\infty} m(2^{-k}\omega) \; .$$

Theorem 2.7. *Let F_N be the invariant subspace of T_m defined by* (2.66). *If λ is the eigenvalue of T_m restricted to F_N with largest modulus, and if $|\lambda| < 1$, then, we have, with $\alpha = -\log|\lambda| / \log 2$ (> 0),*

- *φ is in $C^{\alpha-\varepsilon}$ for all $\varepsilon > 0$,*

- *φ is in C^α if the restriction of T_m to the invariant subspace F_λ of eigenvalue λ is purely diagonal (i.e. $= \lambda I$).*

These two estimates are optimal if $\hat{\varphi}(\omega)$ does not vanish on $[-\pi, \pi]$.

PROOF. Consider the trigonometric polynomial

$$C_N(\omega) = (1 - \cos\omega)^N \ . \tag{2.72}$$

It clearly belongs to F_N.

Consequently, for all $n > 0$,

$$\begin{aligned} &\int_{-\pi}^{\pi} (T_m)^n C_N(\omega) d\omega \\ &\leq (2\pi)^{1/2} \left(\int_{-\pi}^{\pi} |(T_m)^n C_N(\omega)|^2 d\omega \right)^{1/2} \\ &\leq C(|\lambda| + \varepsilon)^n \quad \text{or } C\,|\lambda|^n \text{ if } T_m|_{F_\lambda} = \lambda I\,. \end{aligned} \tag{2.73}$$

We now use Lemma 2.5 combined with the inequality

$$C_N(\omega) \geq 1 \quad \text{when } \frac{\pi}{2} \leq |\omega| \leq \pi \ . \tag{2.74}$$

This leads us to

$$\begin{aligned} \int_{2^{n-1}\pi \leq |\omega| \leq 2^n \pi} \hat{\varphi}(\omega)\, d\omega &\leq C \int_{2^{n-1}\pi \leq |\omega| \leq 2^n \pi} \prod_{k=1}^{n} m(2^{-k}\omega)\ d\omega \\ &\leq C \int_{-2^n\pi}^{2^n \pi} C_N(2^{-n}\omega) \prod_{k=1}^{n} m(2^{-k}\omega)\ d\omega \\ &= C \int_{-\pi}^{\pi} (T_m)^n\ C_N(\omega)\, d\omega \ . \end{aligned}$$

Consequently the Littlewood-Paley blocks satisfy the inequality

$$\|\hat{\Delta}_j(\varphi)\|_{L^1} \ \leq\ C\,2^{-(\alpha-\varepsilon)j},\ \varepsilon > 0,\ \alpha = -\log(|\lambda|)/\log 2 \tag{2.75}$$

$$\|\hat{\Delta}_j(\varphi)\|_{L^1} \ \leq\ C\,2^{-\alpha j}, \quad \text{if } T_m|_{F_\lambda} \text{ is purely diagonal.} \tag{2.75'}$$

Since $\|\Delta_j(\varphi)\|_{L^\infty} \leq \|\hat{\Delta}_j(\varphi)\|_{L^1}$ we obtain the announced regularity.

To prove that these estimates are optimal, we need to reverse all the inequalities which have been used. First, note that since $m(\omega)$ and $\hat{\varphi}(\omega)$ are positive, we have $\|\Delta_j(\varphi)\|_{L^\infty} = \|\hat{\Delta}_j(\varphi)\|_{L^1}$.

Let f_λ be an eigenfunction in F_λ. If $\int f_\lambda(\omega)\,d\omega > 0$, then

$$\begin{aligned}(2.76)\qquad \int_{-2^n\pi}^{2^n\pi} f_\lambda(2^{-n}\omega)\prod_{k=1}^{n} m(2^{-k}\omega)d\omega &= \int_{-\pi}^{\pi}(T_m)^n f_\lambda(\omega)d\omega \\ &= \lambda^n\int_{-\pi}^{\pi} f_\lambda(\omega)d\omega \geq C\lambda^n.\end{aligned}$$

If $\int_{-\pi}^{\pi} f_\lambda(\omega)\,d\omega < 0$, then we replace f_λ by $-f_\lambda$. If $\int_{-\pi}^{\pi} f_\lambda(\omega)\,d\omega = 0$, then the argument has to be modified slightly; see below (after (2.78)). Since we have supposed that $\hat{\varphi}(\omega)$ does not vanish on $[-\pi,\,\pi]$, we have

$$(2.77)\qquad \hat{\varphi}(\omega) \geq C\prod_{k=1}^{n} m(2^{-k}\omega) \quad \text{for all } n > 0 \text{ and } |\omega| \leq 2^n\pi\ .$$

Note that this hypothesis corresponds to the condition of Theorem 2.1 with $K = [-\pi,\,\pi]$. In a more general setting, we could replace the integrals on $[-2^n\pi,\,2^n\pi]$ by integrals on 2^nK and the same results would hold. Combining (2.76) and (2.77) gives

$$(2.78)\qquad \int_{-2^n\pi}^{2^n\pi} |\hat{\varphi}(\omega)|\ |f_\lambda(2^{-n}\omega)|d\omega \geq C|\lambda|^n\ .$$

(If $\int_{-\pi}^{\pi} f_\lambda(\omega)\ d\omega = 0$, then a slightly more sophisticated argument will do the trick. Lemma 2.5 still holds if the measure $d\omega$ is replaced by any other measure of the type $g(\omega)\,d\omega$ where g is a 2π-periodic, strictly positive, continuous function. We can always choose g such that

$$\int_{-\pi}^{\pi} f_\lambda(\omega)\ g(\omega)\,d\omega > 0\ ;$$

(2.76) then holds if $d\omega$ is replaced everywhere by $g(\omega)\,d\omega$. Since g is strictly positive, this modified version of (2.76) combined with (2.77), still implies (2.78)).

Since f_λ has a zero of order $2N$ at the origin, the function $\gamma(x)$, defined by $\hat{\gamma}(\omega) = |f_\lambda(\omega)|\,\chi_{[-\pi,\pi]}(\omega)$ is convenient for the Littlewood-Paley analysis of Hölder regularity less than $2N$. This is the case for φ since $2N+1$ vanishing moments would be necessary for a higher Hölder exponent than $2N$ (see [FS], [DL] or [DyL]). Consequently (2.78) tells us that φ cannot be more regular than C^α. To prove the optimality of $C^{\alpha-\varepsilon}$ when $T_m|_{F_\lambda}$ is not purely diagonal, it suffices to replace f_λ by a function g_λ such that $T_m g_\lambda = \lambda g_\lambda + \mu f_\lambda$ with $\mu \neq 0$. This leads to

$$\int_{-2^n\pi}^{2^n\pi} |\hat{\varphi}(\omega)|\,|g_\lambda(2^{-n}\omega)|\,d\omega \geq C\,n\lambda^n \tag{2.78'}$$

which proves the optimality of $C^{\alpha-\varepsilon}$.

The theorem is thus completely proved.

REMARKS.

- The estimates (2.75) and (2.75′) can be found by an equivalent technique, using the transition operator T_p corresponding to the factor $p(\omega)$ in (2.65). We simply consider the eigenvalue λ_p with largest $|\lambda|_p$ and iterate T_p on $f \equiv 1$. This leads to

$$\begin{aligned}\int_{2^{j-1}\pi\leq\omega\leq 2^j\pi} \hat{\varphi}(\omega)d\omega &\leq C\int_{2^{j-1}\pi\leq|\omega|\leq 2^j\pi} |\omega|^{-2N}\left[\prod_{k=1}^{j} p(2^{-k}\omega)\right]d\omega \\ &\leq C\,2^{-2Nj}\int_{-\pi}^{\pi}(T_p)^j 1\;d\omega \\ &\leq C\,(|\lambda_p|+\varepsilon)^j\;2^{-2Nj} \\ &\quad (\text{or } C|\lambda_p|^j\;2^{-2Nj} \quad \text{if } T_p/F_{\lambda_p} \;=\; \lambda_p I)\end{aligned}$$

and thus $\varphi \in C^{\alpha-\varepsilon}$ with $\alpha = 2N - \log|\lambda_p|/\log 2$. This estimate is in fact the same as (2.75). Indeed, if μ is an eigenvalue of T_m in F_N, then its associated eigenfunction can be written as

$$f_\mu \;=\; \left(\sin^2\left(\frac{\omega}{2}\right)\right)^N\; g_\mu(\omega)\,. \tag{2.79}$$

Replacing $m(\omega)$ by its factorized form in

$$\mu f_\mu(\omega) \;=\; f_\mu\left(\frac{\omega}{2}\right)m\left(\frac{\omega}{2}\right) + f_\mu\left(\frac{\omega}{2}+\pi\right)m\left(\frac{\omega}{2}+\pi\right) \tag{2.80}$$

we obtain, after dividing by $\left[\sin^2(\omega/2)\ \cos^2(\omega/2)\right]^N$,

$$\mu\, 2^{2N} g_\mu(\omega) \;=\; g_\mu\left(\frac{\omega}{2}\right) p\left(\frac{\omega}{2}\right) + g_\mu\left(\frac{\omega}{2}+\pi\right) p\left(\frac{\omega}{2}+\pi\right) \;. \tag{2.81}$$

We see here that the eigenvalues of T_p are exactly given by $\mu_p = 2^{2N}\mu$. This proves the equivalence between the two techniques.

- In general $m(\omega)$ is not a positive function. One can then define $M(\omega) = |m(\omega)|^2$ and use the operator T_M associated to $M(\omega)$. The result is an estimate of the L^2 norms of $\Delta_j(\varphi)$. Using the Cauchy-Schwarz inequality, we derive the following corollary,

Corollary 2.8. *Suppose that $M(\omega) = |m(\omega)|^2$ has a zero of order $2N$ at $\omega = \pi$. Define λ, the largest eigenvalue of T_M on F_N and $\alpha = -\log\lambda/(2\log 2)$. Then, $\varphi \in H^{\alpha-\varepsilon} \subset C^{\alpha-1/2-\varepsilon}$ where H^s is the Sobolev space of index s. The value α is attained if $T_M|_{F_\lambda} = \lambda I$.*

Note that the Hölder exponent has no chance of being optimal because we have used the Cauchy-Schwarz inequality and $\hat{\varphi}(\omega)$ is not a positive function. The Sobolev exponent however is optimal. The regularity of compactly supported wavelets was estimated with this method in [Dau1].

The transition operator plays also a crucial role in the biorthogonal wavelet theory: we show in Appendix A how it can be used to prove that the families $\{\psi_k^j\}_{j,k\in\mathbb{Z}}$ and $\{\tilde{\psi}_k^j\}_{j,k\in\mathbb{Z}}$ are unconditional bases, with weaker assumptions than the boundedness of $(1+|\omega|)^{1/2+\varepsilon}(|\hat{\varphi}(\omega)| + |\hat{\tilde{\varphi}}(\omega)|)$ imposed in Theorem 2.2.

The optimal estimate for the global and local Hölder regularity of any wavelet can be estimated by another method developed by I. Daubechies and J. Lagarias in [DL]. We now recall its main points.

II.4.b. The time domain approach.

Let $m(\omega) = \sum_{n=0}^{N} c_n e^{in\omega}$ be a trigonometric polynomial such that $m(0) = 1$ and $m(\pi) = 0$. We do not require that $m(\omega)$ be positive. Let $\varphi(x)$ be the scaling function defined by the infinite product (2.71). It is at least a compactly supported distribution in $[0, N]$.

In the time domain approach, we represent $\varphi(x)$ by its "vector" form $w(x) : [0,1] \to \mathbb{R}^N$

$$[w(x)]_n = \varphi(x+n-1), \qquad n = 1, \dots, N. \tag{2.82}$$

From the two scale difference equation (1.5) we get

$$w(x) = \begin{cases} T_0 \; w(2x) & \text{if } x \le 1/2, \\ T_1 \; w(2x-1) & \text{if } x \ge 1/2, \end{cases} \tag{2.83}$$

where T_0 and T_1 are $N \times N$ matrices defined by

$$(T_0)_{i,j} = c_{2i-j-1} \qquad 1 \le i,j \le N, \tag{2.84}$$

$$(T_1)_{i,j} = c_{2i-j} \qquad 1 \le i,j \le N. \tag{2.84'}$$

Using the notations

$$d_n(x) = n^{\text{th}} \text{ binary digit of } x \in [0,1]$$

$$\tau(x) = \begin{cases} 2x & \text{if } x \le 1/2 \\ 2x-1 & \text{if } x \ge 1/2 \end{cases} \qquad \text{(binary shift)},$$

we can rewrite (2.83) as a "fixed point" equation

$$w(x) = T_{d_1(x)} \; w(\tau(x)). \tag{2.85}$$

This leads to an evaluation of $w(x)$ and its derivative by an iterative process. The regularity of the result depends of course on the spectral properties of T_0 and T_1. Note that when $m(\omega)$ has a zero of order L (as for the transition operator studied in the previous section), then the space F_L orthogonal to the vector $p_j = (n^j)_{n=1,\dots,N}$ for $j = 0, \dots, L-1$ is invariant by T_0 and T_1. This method gives sharp estimates on the local regularity in x by considering the products $T_{d_1(x)} \cdots T_{d_n(x)}$ for all $n \ge 0$. The main result on global regularity proved in [DL; Theorem 3.1] is the following

Theorem 2.9. *Suppose that there exist $\rho < 1$ such that, for all binary sequence $(d_j)_{j \in \mathbb{Z}}$ and all $m > 0$, we have*

$$\|T_{d_1} T_{d_2} \cdots T_{d_m}|_{F_L}\| \le C\rho^m. \tag{2.86}$$

Define $\alpha = -\log \rho / \log 2$. *Then,*

- *if* α *is not an integer,* φ *belongs to* C^α*,*
- *if* α *is an integer,* $\varphi^{\alpha-1}$ *is almost Lipschitz: for almost all* x, t*,*

$$|\varphi^{\alpha-1}(x+t) - \varphi^{\alpha-1}(x)| \leq C|t| \; |\log|t|| \; .$$

REMARK.

- The "generalized spectral norm"

$$\rho(T_0, T_1) = \limsup_{m\to\infty} \max_{\substack{d_j=0 \text{ or } 1 \\ j=1,\ldots,m}} \|T_{d_1} T_{d_2} \cdots T_{d_m}|_{F_L}\|^{1/m} \tag{2.87}$$

  gives a sharp estimate of the global regularity. Note that it is in general superior to the spectral radius of T_0 and T_1. When N is not too large it is possible to compute the exact value of $\rho(T_1, T_2)$. For example, in the case of orthonormal wavelets, the optimal Hölder exponent was found in [DL] for $N = 4$, 6 and 8. The same evaluation becomes more difficult for larger filters.
- The generalization of this approach in higher dimensions is not trivial. In particular, it involves nonstandard binary expansions depending on the dilation matrix which is used. We describe these techniques in Appendix B.

As a conclusion of this review of regularity estimators, we could say that these three approach are complementary: the time domain method gives sharp results but it is only practicable for small filters, the Littlewood-Paley estimates can be derived for longer filters but they will be optimal only if $m(\omega)$ is a positive function and finally, the Fourier approach is less precise but appropriate to asymptotical results on very large filters. Let us also mention that another method recently developed by O. Rioul [Ri] and based on $\ell^1(\mathbb{Z})$ norms estimates of the iterated filters leads to interesting results; in particular, it is still manageable for larger filters than the time domain method of [DL].

We are now ready to deal with the bidimensional wavelets. We start by examining the different subband coding schemes that can be used to build these non-separable multiscale bases.

III. Two channel bidimensional subband coding schemes.

As mentioned previously, we shall concentrate on the dilation matrices of determinant equal to 2 or -2. In such conditions, the subband coding scheme that we consider split the signal in two channels (instead of four in the separable case) and only one wavelet is then necessary to characterize the detail coefficients at each scale. We first present a short summary of the equations satisfied by these filter. They are immediate generalizations of the results presented in II.1.

III.1. General conditions for exact reconstruction.

As in the one dimensional case, the scheme that we are considering here is based on four fundamental operations:

- The action of two analyzing filters, one low pass

$$\tilde{M}_0(\omega) = \tilde{M}_0(\omega_1, \omega_2)$$

  and one high pass $\tilde{M}_1(\omega) = \tilde{M}_1(\omega_1, \omega_2)$,
- Decimation on each channel by keeping only the samples on the sublattice $\Gamma = D\mathbb{Z}^2$,
- Insertion of zero values at the intermediate points of $\mathbb{Z}^2/\Gamma$,
- Interpolation by two synthesis filters, one low pass

$$M_0(\omega) = M_0(\omega_1, \omega_2)$$

  and one high pass $M_1(\omega) = M_1(\omega_1, \omega_2)$, followed by reconstruction of the original signal by summation.

We see here that the conditions for perfect reconstruction will not depend on the dilation matrix D but only on the sublattice $\Gamma = D\mathbb{Z}^2$ that is generated (different matrices may lead to the same Γ). More precisely, there exist only two types of grid corresponding to a decimation of a factor 2 in $\mathbb{Z}^2$:

- The quincunx sublattice, shown on figure 5, is generated by the integer combinations of $(1,1)$ and $(1,-1)$.
- The column sublattice, shown on figure 6, is generated by the integer combinations of $(0,1)$ and $(2,0)$. It is of course equivalent to the row sublattice, by exchange of the coordinates.

The same arguments that were used in II.1.b show that perfect reconstruction is achieved by FIR filters, if and only if they satisfy (up to a shift) the following equations, which are similar to (2.13) and (2.14).

- In the quincunx case,

$$\overline{M_0(\omega)}\,\tilde{M}_0(\omega) + \overline{M_0(\omega + (\pi,\pi))}\,\tilde{M}_0(\omega + (\pi,\pi)) = 1 \tag{3.1}$$

and

$$\begin{aligned} M_1(\omega) &= e^{-i(\omega_1+\omega_2)}\,\overline{\tilde{M}_0(\omega + (\pi,\pi))}, \\ \tilde{M}_1(\omega) &= e^{-i(\omega_1+\omega_2)}\,\overline{M_0(\omega + (\pi,\pi))}. \end{aligned} \tag{3.2}$$

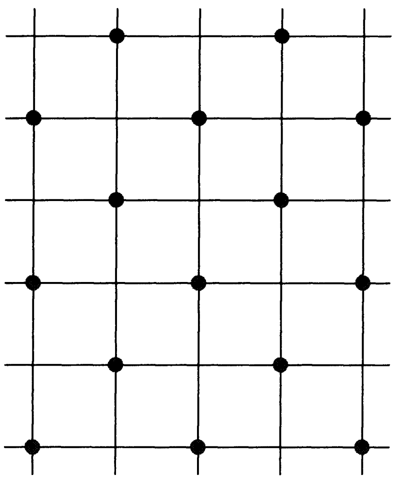


Figure 5
Quincunx decimation.

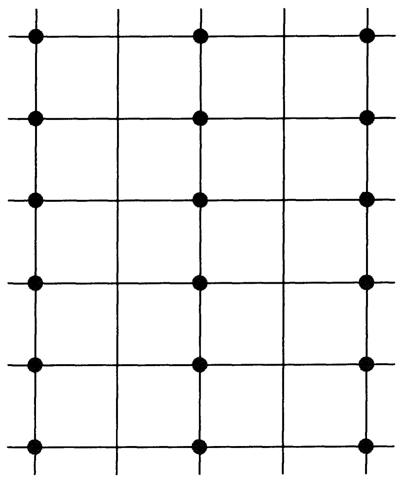


Figure 6
Column decimation.

- In the column case,

$$\overline{M_0(\omega)}\,\tilde{M}_0(\omega) + \overline{M_0(\omega + (\pi,\, 0))}\tilde{M}_0(\omega + (\pi,\, 0)) = 1 \tag{3.3}$$

and

$$\begin{aligned} M_1(\omega) &= e^{-i\omega_1}\,\overline{\tilde{M}_0(\omega + (\pi,\, 0))}\,, \\ \tilde{M}_1(\omega) &= e^{-i\omega_1}\,\overline{M_0(\omega + (\pi,\, 0))}\,. \end{aligned} \tag{3.4}$$

If the analysis and synthesis filters are equal, we find two generalizations of the CQF condition (2.5). The formulas (3.1) and (3.2) become

$$\begin{aligned} |M_0(\omega)|^2 \;+\; |M_0(\omega + (\pi,\, \pi))|^2 &= 1\,, \\ M_1(\omega) &= e^{-i(\omega_1+\omega_2)}\,\overline{M_0(\omega + (\pi,\, \pi))}\;; \end{aligned} \tag{3.5}$$

whereas (3.3) and (3.4) become

$$\begin{aligned} |M_0(\omega)|^2 \;+\; |M_0(\omega + (\pi,\, 0))|^2 &= 1\,, \\ M_1(\omega) &= e^{-i\omega_1}\,\overline{M_0(\omega + (\pi,\, 0))}\,. \end{aligned} \tag{3.6}$$

As in the one dimensional situation, we want to build from these schemes the associated scaling function which can be viewed as the limit of the cascade-reconstruction algorithm.

III.2. Non-separable scaling function and wavelets.

If c_{mn} are the Fourier coefficients of $M_0(\omega)$, *i.e.*

$$M_0(\omega) \;=\; M_0(\omega_1,\, \omega_2) \;=\; \sum_{m,n} c_{mn}\; e^{-i(m\omega_1+n\omega_2)}\,, \tag{3.7}$$

then the associated scaling function $\phi(x) = \phi(x_1,\, x_2)$ satisfies a two scale difference equation,

$$\phi(x) \;=\; 2\sum_{m,n} c_{mn}\; \phi(Dx - (m,n)) \tag{3.8}$$

and its Fourier transform can be expressed as an infinite product

$$\hat{\phi}(\omega) \;=\; \prod_{k=1}^{+\infty} M_0(D^{-k}\omega) \tag{3.9}$$

which is convergent if and only if $M_0(0) = 1$.

This scaling function has compact support if and only if $M_0(\omega)$ is an FIR filter. We see from (3.9) that ϕ will be highly dependent on the choice of D. For the same sublattice and the same filter, the results can be completely different for different D. The column sublattice for example is generated by both matrices $D_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, but the first one cannot lead to an L^2 scaling function. Indeed, we would have

$$\hat{\phi}_1(0, 2n\pi) = \prod_{k=1}^{+\infty} M_0(D_1^{-k}(0, 2n\pi)) = 1 ,$$

for all $n \geq 0$. But since ϕ_1 is compactly supported and belongs to $L^2(\mathbb{R})$, it is also in $L^1(\mathbb{R})$ and its Fourier transform should tend to zero at infinity. We can also remark that only the eigenvalues of D_2 have their modulus strictly superior to 1.

The choice of the dilation matrix is thus very important. In fact, although the equations (3.1)-(3.2) are different from (3.3)-(3.4), the choice of the sublattice is less important: Indeed, for any dilation matrix D_1 such that $D_1\mathbb{Z}^2$ is the column sublattice, we can define

$$D_2 = P\, D_1 P^{-1} \quad \text{with } P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{3.10}$$

Clearly, the image of $\mathbb{Z}^2$ by D_2 is now the quincunx sublattice. Then, for any filter $M_0^1(\omega)$ satisfying the column-CQF condition (3.6), the corresponding scaling function ϕ_1 can be written in the following way,

$$\hat{\phi}_1(\omega) = \prod_{k=1}^{+\infty} M_0^1(D_k^{-k}\omega) = \prod_{k=1}^{+\infty} M_0^1(P^{-1}D_2^{-k}P\omega) = \hat{\phi}_2(P\omega)$$

where $\hat{\phi}_2$ is also a scaling function defined by

$$\begin{cases} \hat{\phi}_2(\omega) = \prod_{k=1}^{+\infty} M_0^2(D_2^{-k}\omega), \\ M_0^2(\omega) = M_0^1(P^{-1}\omega) . \end{cases} \tag{3.11}$$

Since $P^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we have

$$|M_0^2(\omega)|^2 + |M_0^2(\omega + (\pi, \pi))|^2 = |M_0^1(\omega_1, \omega_1 + \omega_2)|^2 + |M_0^1(\omega_1 + \pi, \omega_1 + \omega_2 + 2\pi)|^2 = 1 .$$

And thus M_0^2 satisfies the quincunx-CQF condition (3.5). A similar result holds of course if we start from two dual filters M_0^1 and $\tilde{M}_0^1$ which satisfy (3.3). This shows that the scaling functions associated to D_1 and D_2 are linked by the simple relation $\phi_2(x) = \phi_1(Px)$. Consequently we can restrain our study to the quincunx case. More generally, if D_1 and D_2 satisfy

$$D_2 = PD_1P^{-1} \tag{3.12}$$

where P is a matrix having integer entries and determinant equal to 1, then we also have the same type of equivalence between the scaling functions. For this reason, we shall only consider the two simplest dilation matrices of determinant 2, which cannot be related as in (3.12) since they do not have the same eigenvalues:

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(\text{Rotation of } \frac{\pi}{4} \text{ and dilation of } \sqrt{2}\right) \tag{3.13}$$

and

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(\text{Symmetry with respect to } (\sqrt{2}+1, 1) \text{ and dilation of } \sqrt{2}\right) . \tag{3.13'}$$

In both of these cases the image of $\mathbb{Z}^2$ is the quincunx sublattice. The wavelet ψ is then defined by

$$\hat{\psi}(D\omega) = M_1(\omega)\hat{\phi}(\omega) \qquad \text{with} \quad D = R \text{ or } S , \tag{3.14}$$

where $M_1(\omega)$ is defined by (3.5) in the orthogonal case, and by (3.2) in the biorthogonal case where we also have a dual wavelet defined by

$$\hat{\tilde{\psi}}(D\omega) = \tilde{M}_1(\omega)\, \hat{\tilde{\phi}}(\omega) \qquad \text{with} \quad D = R \text{ or } S . \tag{3.15}$$

The goal is now to design filters leading to regular scaling functions and wavelets. We end this section by presenting two important families of filters. The regularity of the associated ϕ, ψ, $\tilde{\phi}$ and $\tilde{\psi}$ will be estimated in sections IV and V by different techniques which all are natural generalizations of the one dimensional tools that we introduced previously.

III.3. Filter design.

III.3.a. The orthonormal case.

Recall (see [Dau1]) that in $1D$, the CQF filter can be designed in the following way, in order to obtain wavelets with an arbitrarily high regularity:

1) For a given number N of vanishing moments, define m_0 by

$$|m_0(\omega)|^2 = \left[\cos^2\left(\frac{\omega}{2}\right)\right]^N P_N\left[\sin^2\left(\frac{\omega}{2}\right)\right] \tag{3.16}$$

where $P_N(y)$ is a polynomial, solution of the Bezout problem

$$y^N P_N(1-y) + (1-y)^N P_N(y) = 1 . \tag{3.17}$$

The minimal degree choice is given by

$$P_N(y) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} y^j .$$

2) Find the function $m_0(\omega)$ by using the Riesz lemma which guarantees that there exist a trigonometric polynomial solving (3.16).

Unfortunately, this last result does not generalize to higher dimensions. We thus have to find other means to build trigonometric polynomials which satisfy (3.5). One possible method is the "polyphase component" construction used by Vaidyanathan [Va] and M. Vetterli [Ve], [VK]. It is based on the remark that $M_0(\omega)$ satisfies (3.5) if and only if the polyphase matrix

(3.18)

$$H_0(\omega) = \frac{1}{\sqrt{2}} \begin{pmatrix} M_0(\omega) + M_0(\omega + (\pi,\, \pi)) & M_1(\omega) + M_1(\omega + (\pi,\, \pi)) \\ M_0(\omega) - M_0(\omega + (\pi,\, \pi)) & M_1(\omega) - M_1(\omega + (\pi,\, \pi)) \end{pmatrix}$$

is unitary for all ω. Since the product of two polyphase matrices is also a polyphase matrix for a third pair of filter, infinite families can be constructed by multiplying elementary building blocks of the type (3.18) as soon as we know some simple filters which satisfy (3.5). The disadvantage of this method is that it does not furnish the vanishing moments in a natural way. Recall (see [Mel]) that the N times differentiability of the function ψ implies

$$|\hat{\psi}(\omega)| \leq C\left(|\omega_1|^{N+1} + |\omega_2|^{N+1}\right), \qquad (|\omega| \mapsto 0) \tag{3.19}$$

and thus $M_0(\omega)$ has necessarily a zero or order $N+1$ at the frequency $\omega = (\pi, \pi)$. This can also be viewed as the Fix-Strang condition (see [SF]) for the regularity of the scaling function ϕ.

The simplest way to build such filters with N arbitrarily high is to remark that if $m_0(\omega)$ is a $1D$ solution of the CQF equation (2.5), then the $2D$ filter defined by

$$M_0(\omega) = M_0(\omega_1, \omega_2) = m_0(\omega_1) \tag{3.20}$$

satisfies the equation (3.5). It is apparently a good candidate for building regular wavelets since it has the same order of cancellation in (π, π) as $m_0(\omega)$ in π. This allows us to build an infinite family of filters with an arbitrarily high number of vanishing moments by posing

$$M_0^N(\omega) = m_0^N(\omega_1) \tag{3.21}$$

where $\{m_0^N(\omega)\}_{N>0}$ is the family of filters designed in [Dau1], defined by (2.35), (2.37) and (2.38). Note that the filter (3.21) has a unidimensional structure but since the dilation D contains either a rotation or a symmetry, the final analysis (using iterates of the filter) is performed in all the directions of the plane. In Section IV, we shall take a closer look at the associated wavelets and their regularity. If $D = R$, then one can also derive another family of "almost" one-dimensional filters M_0 from unidimensional m_0 (they get again fanned out to other directions by applying R^{-1}). Explicitly,

$$\begin{aligned} M_0(\omega_1, \omega_2) = {} & \frac{1}{2}\left[m_0\left(\frac{\omega_1-\omega_2}{2}\right) + m_0\left(\frac{\omega_1-\omega_2}{2}+\pi\right)\right] \\ & + \frac{1}{2}\left[m_0\left(\frac{\omega_1-\omega_2}{2}\right) - m_0\left(\frac{\omega_1-\omega_2}{2}+\pi\right)\right] e^{i(\omega_1+\omega_2)/2}. \end{aligned}$$

This construction corresponds to a filter with taps on two diagonals, $h_{n_1,n_2} = 0$ if $n_1 \neq -n_2$ and $n_1 \neq -n_2 + 1$. It is easy to check that this M_0 satisfies (3.5) if m_0 satisfies $|m_0(\omega)|^2 + |m_0(\omega + \pi)|^2 = 1$. If $m_0(0) = 1$, $m_0(\pi) = 0$, then $M_0(\pi, \pi) = 0$ follows, so that M_1, as defined in (3.5), satisfies $M_1(0, 0) = 0$, as it should. One easily checks, however, that $\partial_{\omega_1} M_0(\pi, \pi)$ and $\partial_{\omega_2} M_0(\pi, \pi)$ cannot both be zero for these examples, so that the corresponding bases cannot possibly be C^1. Only the small examples are therefore of any interest; it seems possible (numerical experiment) to construct a continuous ϕ corresponding to a 4-tap filter in this way.

III.3.b. The biorthogonal case.

The filter design is clearly easier in the biorthogonal situation. One can start from a given filter $M_0(\omega)$ and find the dual $\tilde{M}_0(\omega)$ by solving linear equations.

In particular we can look for filters which have more isotropy than those of the family (3.21). Here, again, the one dimensional theory can help us to build families of filters in a simple way. Several examples of real and symmetrical dual filters have been designed by the authors and J. C. Feauveau in [CDF].

In these one dimensional construction the symmetry allows us to use the variable $y = \sin^2(\omega/2)$ and to write the transfer functions as

$$m_0(\omega) = p(y) \quad \text{and} \quad \tilde{m}_0(\omega) = \hat{p}(y) \tag{3.22}$$

where p and $\hat{p}$ are two polynomial satisfying

$$p(y)\hat{p}(y) + p(1-y)\hat{p}(1-y) = 1 \ . \tag{3.23}$$

In two dimensions, consider the variables $y_1 = \sin^2(\omega_1/2)$ and $y_2 = \sin^2(\omega_2/2)$. If the filters are symmetrical with respect to the vertical and the horizontal axes, the duality condition in (3.3) can be rewritten as

$$P(y_1, y_2)\hat{P}(y_1, y_2) + P(1-y_1, 1-y_2)\hat{P}(1-y_1, 1-y_2) = 1 \ , \tag{3.24}$$

where

$$P(y_1, y_2) = M_0(\omega_1, \omega_2) \ , \qquad \hat{P}(y_1, y_2) = \tilde{M}_0(\omega_1, \omega_2) \ .$$

We see that a possible choice for P and $\hat{P}$ is given by

$$P(y_1, y_2) = p(\alpha y_1 + (1-\alpha)y_2) \tag{3.25}$$

$$\hat{P}(y_1, y_2) = \hat{p}(\alpha y_1 + (1-\alpha)y_2) \tag{3.25'}$$

where α is in $[0,1]$. For an optimal isotropy it is natural to choose $\alpha = 1/2$; in this case the diagonals are also symmetry axes. This choice is known in signal processing as the McClellan transform of the $1D$ filters p and $\tilde{p}$. Using the variable $z = (y_1 + y_2)/2$ we can thus write

$$M_0(\omega) = p(z) \quad \text{and} \quad \tilde{M}_0(\omega) = \hat{p}(z) \tag{3.26}$$

where p and $\hat{p}$ are polynomials satisfying (3.24). These polynomials must also satisfy

$$p(0) = \hat{p}(0) = 1 \quad \text{and} \quad p(1) = \hat{p}(1) = 0 \tag{3.27}$$

which are necessary for the construction of wavelet bases. Note that we have

$$\begin{aligned} z &= \frac{1}{2}\left(\sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right)\right) \\ &= \frac{1}{8}\left(4 - e^{i\omega_1} - e^{i\omega_2} - e^{-i\omega_1} - e^{-i\omega_2}\right) \end{aligned} \tag{3.28}$$

and thus z can be regarded as the transfer function of the filter which computes the discrete Laplacian with the formula

$$\begin{aligned} (\Delta_d x)_{m,n} = \frac{1}{8}(\, 4\, x_{m,n} - x_{m-1,n} - x_{m+1,n} \\ - x_{m,n-1} - x_{m,n+1}\,)\,. \end{aligned} \tag{3.29}$$

Since a Laplacian scheme has frequently been proposed in image processing to detect the edges with a maximum isotropy (see [AB], [M]), it seems tempting to use z or one of its powers as a high pass analyzing filter (and thus $1 - z$ as the corresponding low pass synthesis filter). This can be achieved in a very simple way, by a method already used to build biorthogonal bases in $L^2(\mathbb{R})$. Recall that

$$P_N(z) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} z^j$$

is the lowest degree solution of the Bezout problem

$$z^N P_N(1-z) + (1-z)^N P_N(z) = 1 \ . \tag{3.30}$$

If we fix the reconstruction low pass as $M_0^N(\omega) = (1-z)^N$ (so that the analyzing high pass is, up to a shift, the N-th power of the Laplacian), then a possible choice for the dual filter is given by

$$\tilde{M}_0^{N,L}(\omega) \;=\; (1-z)^L \; P_{N+L}(z) \tag{3.31}$$

where L is a positive integer indicating the cancellation order of $\tilde{M}_0$ at $\omega = (\pi,\, \pi)$. L has to be chosen large enough so that both functions $\varphi(x)$ and $\tilde{\varphi}(x)$ satisfy the necessary conditions to generate a pair $\{\psi_k^j, \tilde{\psi}_k^j\}_{j\in\mathbb{Z}, k\in\mathbb{Z}^2}$ of unconditional Riesz bases (see Theorem 2.1 and Appendix A). We shall examine the properties of these functions and give an estimate of the minimal value of L in Section V.

We have now at hand two families of filters, orthonormal and biorthogonal, with an arbitrarily high number of vanishing moments. We still have to know if these filters allow us to build wavelet bases with an arbitrarily high regularity as in the one dimensional case ([Dau1], [Co2]). As we shall see in the next two sections, the results of our investigations are very surprising and show that the multidimensional situation contains a lot of new difficulties from this point of view.

IV. Orthonormal bases of non-separable wavelets.

Let us consider the family of CQF filters defined by

$$M_0^N(\omega_1, \omega_2) \;=\; m_0^N(\omega_1) \tag{4.1}$$

with

$$|m_0^N(\omega)|^2 = \left[\cos^2\left(\frac{\omega}{2}\right)\right]^N \sum_{j=0}^{N-1} \binom{N-1+j}{j} \left[\sin^2\left(\frac{\omega}{2}\right)\right]^j \tag{4.2}$$

and the associated scaling functions for the dilations S and R,

$$\hat{\phi}_{N,S}(\omega) = \prod_{k=1}^{\infty} \; M_0^N(S^{-k}\omega)\,, \tag{4.3}$$

$$\hat{\phi}_{N,R}(\omega) = \prod_{k=1}^{\infty} M_0^N(R^{-k}\omega)\,. \tag{4.4}$$

IV.1. Orthonormality of the translates.

A first requirement is that the $\mathbb{Z}^2$-translates of $\phi_{N,S}$ or $\phi_{N,R}$ are orthonormal. This is a necessary and sufficient condition to generate multiresolution analyses and orthonormal bases of wavelets.

Theorem 4.1. *For all $N > 0$, the functions $\phi_{N,D}$ have orthonormal translates and generate wavelet bases of the type*

$$2^{-j/2}\psi(D^{-j}x - k)\,, \qquad j \in \mathbb{Z},\ k \in \mathbb{Z}^2\,,$$

where $D = S$ or R.

PROOF. By a trivial generalization of Theorem 2.1, this orthonormality is ensured if and only if $|\hat{\phi}(\omega)| \geq C > 0$ on a compact set K congruent to $[-\pi,\, \pi]^2$ modulo $2\pi\mathbb{Z}^2$ which contains a neighbourhood of the origin.

It is clear that $M_0^N(\omega)$ vanishes only on the vertical lines $\omega_1 = (2k+1)\pi,\ k \in \mathbb{Z}$. Consequently we see that the simple choice $K = [-\pi,\, \pi]^2$ is not convenient since for both dilations, we have

$$D^{-1}(\pi,\, \pi) = (\pi, 0) \tag{4.5}$$

and thus

$$\hat{\phi}(\pi, \pi) = 0\,. \tag{4.6}$$

Recall that in the one dimensional case, the trivial choice $K = [-\pi,\, \pi]$ was convenient for the family $m_0^N(\omega)$. Here we have to use a compact set K slightly different from $[-\pi,\, \pi]^n$ so that $D^{-j}K \cap \{\omega_1 = (2k+1)\pi\}$ is empty for all $j > 0$ and for all k in $\mathbb{Z}$. This can be done very easily by removing small neighbourhoods of $(\pi,\, \pi)$ and $(-\pi,\, -\pi)$ and translating them by $(-2\pi,\, 0)$ and $(2\pi,\, 0)$ as shown in figure 7.

One checks easily that all the sets $D^{-j}K$ for $j > 0$ are contained in the strip $|\omega_1| \leq \pi - \varepsilon,\ \varepsilon > 0$ where $M_0^N(\omega)$ does not vanish.

We now have to check the regularity of the scaling functions which have been obtained. We shall see that the results are completely different depending on whether one chooses S or R as the dilation matrix.

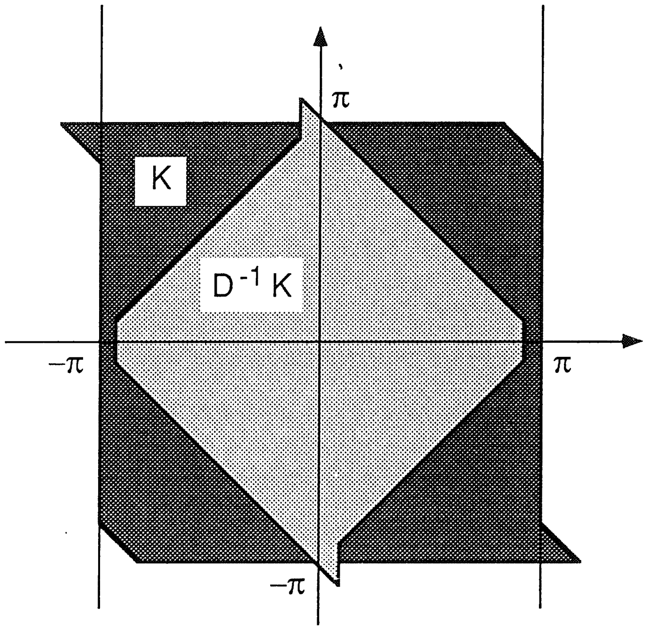


Figure 7
The convenient compact set K congruent to $[-\pi,\, \pi]^2$:
Neighbourhoods of $(\pi,\, \pi)$ and $(-\pi,\, -\pi)$ have been
shifted so that $\hat{\varphi}$ does not vanish on K.

IV.2. The symmetry dilation case.

In this case the dilation matrix is $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and its inverse is $S^{-1} = \frac{1}{2}S$. Since $M_0^N(\omega) = m_0^N(\omega_1)$, we have to consider the sequence $\{[S^{-j}\omega]_1\}_{j>0}$ for a given $\omega = (\omega_1,\, \omega_2)$. Clearly, it has the following form:

$$\frac{1}{2}(\omega_1 + \omega_2),\ \frac{1}{2}\omega_1,\ \frac{1}{4}(\omega_1 + \omega_2),\ \frac{1}{4}\omega_1, \ldots, 2^{-j}(\omega_1 + \omega_2),\ 2^{-j}\omega_1, \ldots$$

Since $S^{-2} = \frac{1}{2}I$, the odd and the even parts are simple dyadic sequences and this leads to

$$\hat{\phi}_{N,S}(\omega) = \hat{\varphi}_N(\omega_1 + \omega_2)\ \hat{\varphi}_N(\omega_1) \tag{4.7}$$

or

$$\phi_{N,S}(x) \ = \ \varphi_N(x_2)\ \varphi_N(x_1 - x_2) \tag{4.8}$$

where φ_N is the one dimensional scaling function. The associated wavelet is defined by

$$\hat{\psi}_{N,S}(\omega) = M_1(\omega)\hat{\phi}_{N,S}(\omega) = \hat{\psi}_N(\omega_1 + \omega_2)\ \hat{\varphi}_N(\omega_1) \tag{4.9}$$

or

$$\psi_{N,S}(x) = \psi_N(x_2)\ \varphi_N(x_1 - x_2)\ . \tag{4.10}$$

We see here that the scaling function and wavelet are in this case separable in the sense that they can be expressed directly in terms of the one dimensional functions φ_N and ψ_N. This separability can be explained by the fact that S is similar to the matrix $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, which is simply a dilation by a factor 2 (in one) direction, followed by an exchange of the axes. The regularity can of course be made arbitrarily high since it is directly given by the Hölder exponent of φ_N.

REMARK. Theorem 4.1 is not necessary here to prove the orthonormality of the translates since it is a trivial consequence of the separability formulas (4.7) and (4.8).

We now consider the case of the matrix R which is by far less trivial.

IV.3. The rotation dilation case.

We now have $R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The sequence $\left\{[R^{-j}\omega]_1\right\}_{j>0}$ is then,

$$\frac{1}{2}(\omega_1 + \omega_2),\ \frac{1}{2}\omega_2,\ \frac{1}{4}(\omega_2 - \omega_1),\ -\frac{1}{4}\omega_1,\ -\frac{1}{8}(\omega_1 + \omega_2),$$
$$-\frac{1}{8}\omega_2,\ \frac{1}{16}(\omega_1 - \omega_2),\ \frac{1}{16}(\omega_1),\ \frac{1}{32}(\omega_1 + \omega_2),\ \frac{1}{32}\omega_2\ ,\ldots$$

Here the first power of R^{-1} proportional to the identity is $R^{-4} = -\frac{1}{4}I$. Consequently, it is not possible to use the one dimensional scaling functions and wavelets to express the ϕ_N and ψ_N in a separable way. We first consider the case $N = 1$ which corresponds to the Haar filter. The result of the cascade algorithm with this filter shows how different the situation is when R is used instead of S.

IV.3.a. The twin dragon.

For $M_0^1(\omega) = (1 + e^{-i\omega_1})/2$, the function $\phi_{1,R}$ satisfies

$$\phi_{1,R}(x) = \phi_{1,R}(Rx) + \phi_{1,R}(Rx - (1,0)) \tag{4.11}$$

and

$$\hat{\phi}_{1,R} = \prod_{k=1}^{\infty} M_0^1(R^{-k}\omega) \ . \tag{4.12}$$

By iteration of the cascade algorithm, one finds that ϕ is the characteristic function of a well known fractal set called the "twin dragon" (see [K]) shown in figure 8. This set can be defined directly in the complex plane as

$$\Delta = \left\{ \sum_{n=1}^{\infty} \varepsilon_n \left(\frac{1-i}{2}\right)^n : \quad \{\varepsilon_n\}_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \right\} \tag{4.13}$$

and it is clear that $\phi_{1,R} = \chi_\Delta$ solves (3.41) since we have

$$\begin{aligned} \Delta &= \left(\frac{1-i}{2}\right)\Delta \cup \left(\frac{1-i}{2}\right)(\Delta + 1) \\ &\sim R^{-1}\Delta \cup R^{-1}(\Delta + (0,1)) . \end{aligned} \tag{4.14}$$

The self-similarity of Δ is thus expressed by the two scale difference equation (4.11), but furthermore, since the family $\{\phi_{1,R}(x-k)\}_{k\in\mathbb{Z}^2}$ is orthonormal (by Theorem 4.1) and since $|\Delta| = \hat{\phi}_{1,R}(0) = 1$, these integer translates constitute a fractal tiling of the whole plane $\mathbb{R}^2$ (similarly to the squares obtained in the tensor product situation with the same filter). This beautiful property has been observed independently by W. Madych and K. Gröchenig [MG] and W. Lawton and H. Resnikoff

[LR]. More generally, such tilings can be derived by considering a two scale difference equation of the type

$$\phi(x) = \sum_{i=1}^{d} \phi(Dx + e_i) \tag{4.15}$$

where D is a dilation matrix and $\{e_i\}_{i=1,\ldots,d}$ are d representatives of $\mathbb{Z}^n/D\mathbb{Z}^n$ ($d = |\det D|$). This scaling function and the corresponding wavelet do not seem however of great interest for image processing: not only are they discontinuous but the set of discontinuity is a very chaotic fractal curve. Nevertheless the twin dragon is important in estimating the regularity (local and global) of the wavelets with dilation matrix R.

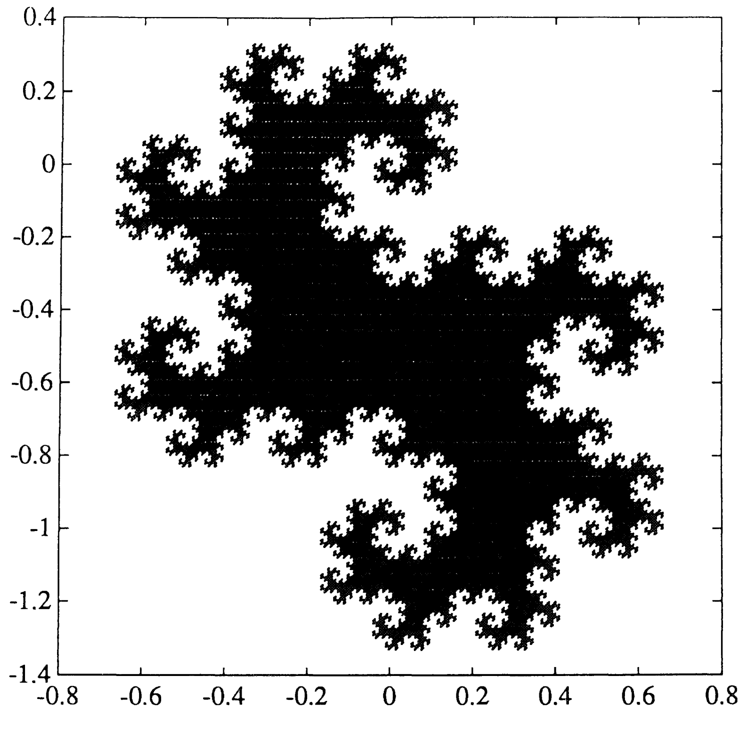


Figure 8
The "twin dragon" set Δ.

Indeed, if we want to generalize the method of [DL] (see Section II.4.6), it is necessary to consider the expansion of any point in $\mathbb{C}$ in terms of the power of $(1+i)/2$ ($\sim R^{-1}$), which also means that the point is considered as the limit of a "dragonic sequence" $\{\Delta_j\}_{j\in\mathbb{Z}}$ with $\Delta_j \subset \Delta_{j-1}$ and $|\Delta_j| = 2^{-j}$. These "dragonic expansion" techniques are described in Appendix B.

Let us now examine the functions obtained with higher order filters which have more vanishing moments.

IV.3.b. Higher order filters.

We are interested in the family of scaling function $\phi_{N,R}$, $N > 1$.

Recall that in the one dimensional case, the asymptotic result ensuring arbitrarily high regularity (Theorem 2.4, Section II.3.6) is based on the value of $|m_0(\pm 2\pi/3)|$ since $\{-2\pi/3, 2\pi/3\}$ is a cyclic orbit of $\omega \mapsto 2\omega$ modulo 2π. In the present case similar considerations for a fixed orbit of $\omega \mapsto R\omega$ modulo $2\pi\mathbb{Z}^2$, lead to an opposite result: arbitrarily high regularity cannot be obtained by increasing the number of vanishing moments. More precisely, we have

Theorem 4.2. *For all $N > 0$, the function $\phi_{N,R}$ is not in $C^1(\mathbb{R}^2)$.*

PROOF. This is of course true for $N = 1$ since we obtain the twin dragon. For $N > 1$, we shall prove a stronger result: the decay at infinity of $\hat{\phi}_{N,R}(\omega)$ cannot be majorated by $C\,|\omega|^{-1}$ (which is a necessary condition for $\phi_{N,R}$ to be in C^1 because it is a compactly supported function). For this we consider the orbit of $\omega \mapsto R\omega$ modulo $2\pi\mathbb{Z}^2$ given by the four points $(2\pi/5, 4\pi/5)$, $(2\pi/5, -4\pi/5)$, $(-2\pi/5, -4\pi/5)$ and $(-2\pi/5, 4\pi/5)$. Let us denote $v_0 = (2\pi/5, 4\pi/5)$ and $v_j = R^j v_0$. One checks easily that

$$|\hat{\phi}_{N,R}(v_0)| = C_N \neq 0 \quad \text{for all } N > 0\ . \tag{4.16}$$

We then have, for all $N > 0$,

$$|\hat{\phi}_{N,R}(v_j)| = C_N \left| m_0^N\left(\frac{2\pi}{5}\right)\right|^j . \tag{4.17}$$

From the definition of m_0^N we have

$$\left| m_0^N\left(\frac{2\pi}{5}\right)\right|^2 = \left[\cos^2\left(\frac{\pi}{5}\right)\right]^N P_N\left(\sin^2\left(\frac{\pi}{5}\right)\right) \tag{4.18}$$

and we know from (2.51) that

$$P_N(y) \le (4y)^{N-1}, \quad \text{if } \frac{1}{2} \le y \le 1 \,. \tag{4.19}$$

Because $\cos^2(\pi/5) > \cos^2(\pi/4) = 1/2$, we can write

$$\begin{aligned} \left| m_0^N\left(\frac{2\pi}{5}\right)\right|^2 &= 1 - \left[\sin^2\left(\frac{\pi}{5}\right)\right]^N P_N\left[\cos^2\left(\frac{\pi}{5}\right)\right] \\ &\ge 1 - \left[\sin^2\left(\frac{\pi}{5}\right)\right]^N \left[4\cos^2\left(\frac{\pi}{5}\right)\right]^{N-1} \\ &= 1 - \sin^2\left(\frac{\pi}{5}\right)\left[\sin^2\left(\frac{2\pi}{5}\right)\right]^{N-1} \end{aligned}$$

and thus, since $|v_j| \ge 2^{j/2}$,

$$\begin{aligned} |\hat{\phi}_{N,R}(v_j)| &\ge C_N \left[1 - \sin^2\left(\frac{\pi}{5}\right)\left[\sin^2\left(\frac{2\pi}{5}\right)\right]^{N-1}\right]^{j/2} \\ &\ge C_N |v_j|^{-\alpha_N} \end{aligned}$$

with $\alpha_N = \left|\log\left(1 - \sin^2(\pi/5)\left(\sin^2(2\pi/5)\right)^{N-1}\right)\right| / \log 2$. Clearly α_N is decreasing with N. Since $\alpha_1 \simeq 0.6115 < 1$, this ends the proof.

In fact, these wavelets do not even seem continuous although we have no mathematical proof for this. A simple look at the result of the cascade algorithm for the 4 tap filter (which corresponds to a .55 Hölder continuous one dimensional wavelet) shows how chaotic the functions $\phi_{R,N}$ can be (figure 9). The design of FIR filters leading to regular wavelet bases with R as the dilation matrix seems to be a difficult problem. Using a polyphase component approach M. Vetterli and J. Kovacevic ([KV], p. 32) have constructed a filter for which the result of the cascade looks continuous but no infinite family with arbitrarily high regularity has been designed so far.

The main difficulty which makes this design unpracticable is the absence of the Riesz lemma in more than one dimension and thus the impossibility to start by designing the square modulus of $M_0(\omega)$ in an appropriate way. Apart from this problem, the CQF filters (in particular the family (3.21) that we have introduced) cannot be symmetrical. We must keep in mind that one of the interests of the quincunx grid decimation is to have a more isotropic analysis; this is only achieved if the filter coefficients are themselves symmetrical around the horizontal, vertical and diagonal directions.

These two reasons encourage us to construct biorthogonal bases of wavelets from dual filters for which the Riesz lemma is not necessary and linear phase can be achieved.

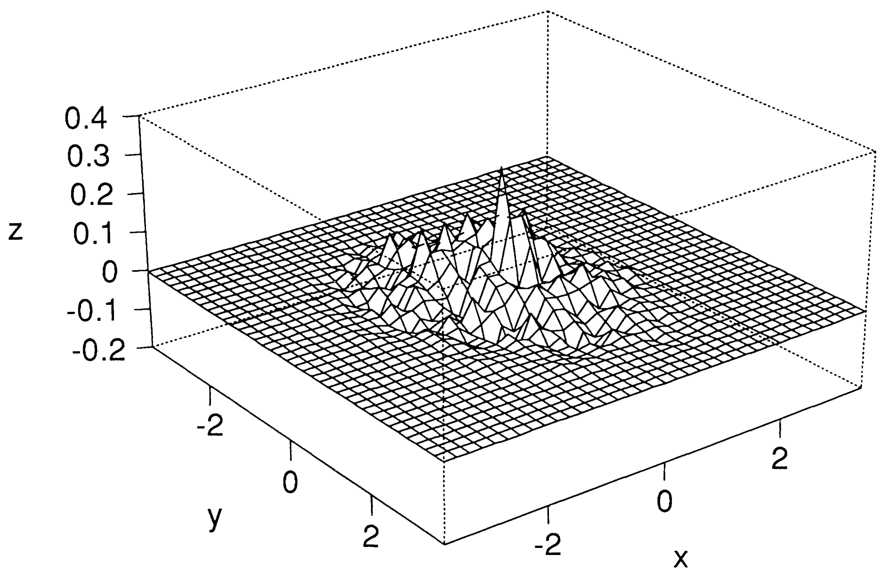


Figure 9
Approximation of the scaling function ϕ_{2R} .

V. Biorthogonal bases of nonseparable wavelets.

Let us recall the family of dual filters introduced in III.3.b. It is based on the variable

$$z = \frac{1}{2}\left(\sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right)\right) .$$

We have chosen

$$M_0^N(\omega) = (1-z)^N \,, \tag{5.1}$$

and

$$\tilde{M}_0^{N,L}(\omega) = (1-z)^L \; P_{N+L}(z) \,, \tag{5.2}$$

where L is still to be fixed.

A first remark is that the action of the dilation matrices R and S on the variable z are equivalent. This is due to the fact that z is invariant if we exchange ω_1 and ω_2 or if we change the sign of one of these variables. We shall thus consider a dilation matrix D which can be equal to R or S. To express its action on z we still need the two variables

$$y_1 = \sin^2\left(\frac{\omega_1}{2}\right) \quad \text{and} \quad y_2 = \sin^2\left(\frac{\omega_2}{2}\right) . \tag{5.3}$$

We then have

$$\begin{aligned} z = \frac{1}{2}(y_1+y_2) &\overset{D}{\longmapsto} z = \frac{1}{2}(y_1 + y_2 - 2\,y_1y_2) \\ &\overset{D}{\longmapsto} z = \frac{1}{2}(4\,y_1(1-y_1) + 4\,y_2(1-y_2)) = \frac{1}{2}(y_1' + y_2') \\ &\overset{D}{\longmapsto} z = \frac{1}{2}(y_1' + y_2' - 2\,y_1'y_2') \ldots \end{aligned}$$

We shall start by studying the scaling function ϕ_1 associated to the filter $M_0^1(\omega) = 1 - z$, because it is the elementary building block for the family ϕ_N $(= (*)^N \phi_1)$.

V.1. The quincunx Laplacian scheme.

The coefficients of $M_0^1(\omega)$ are centered around the origin and have the following form:

$$\frac{1}{8}\begin{pmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{pmatrix}. \tag{5.4}$$

Note that this is the simplest symmetrical filter (with respect to the horizontal, vertical and diagonal directions) which satisfies the cancellation condition $M_0^1(\pi,\pi) = 0$. To estimate the decay of $\hat{\phi}_1(\omega)$ we could hope for a bidimensional formula equivalent to

$$\prod_{k=1}^{+\infty} \cos(2^{-k}\omega) \;=\; \frac{\sin\omega}{\omega}\,, \tag{5.5}$$

used in the one dimensional case. Note that (5.5) is based on the iteration of $\sin\omega = 2\sin(\omega/2)\,\cos(\omega/2)$. Unfortunately, similar relations do not exist in the bidimensional case for the dilation matrix D. In particular the infinite product

$$\hat{\phi}_1(\omega) \;=\; \prod_{k=1}^{+\infty} M_0(D^{-j}\omega) \tag{5.6}$$

has no simple expression and one checks easily that, unlike (5.5), it does not have uniform decay at infinity. Indeed, let us consider the sets $\{(2\pi/5,\, 4\pi/5)\}$ and $\{(2\pi/3,\, 2\pi/3),\, (2\pi/3,\, 0)\}$. These are two cyclic orbits of $\omega \mapsto D\omega$ modulo $2\pi\mathbb{Z}^2$ and modulo the exchange of coordinates and sign changes which do not affect the variable z. Consequently, if we define $v_j = D^j(2\pi/5,\, 4\pi/5)$ and $\mu_j = D^j(2\pi/3,\, 2\pi/3)$, we have, when j goes to $+\infty$,

$$\hat{\phi}_1(v_j) \sim C\left[\frac{\cos^2(\pi/5) + \cos^2(2\pi/5)}{2}\right]^j \;\sim\; C\,|v_j|^{-\alpha_v} \tag{5.7}$$

and

$$\hat{\phi}_1(\mu_j) \sim C\left[\left(\frac{\cos^2(\pi/3) + 1}{2}\right)\cos^2\left(\frac{\pi}{3}\right)\right]^{j/2} \;\sim\; C\,|\mu_j|^{-\alpha_\mu} \tag{5.8}$$

with

$$\alpha_v = -\frac{2}{\log 2} \log \left[\frac{\cos^2(\pi/5) + \cos^2(2\pi/5)}{2} \right] \simeq 2.83 \tag{5.9}$$

and

$$\alpha_\mu = -\frac{1}{\log 2} \log \left[\left(\frac{\cos^2(\pi/3) + 1}{2} \right) \cos^2\left(\frac{\pi}{3}\right) \right] \simeq 2.68 \neq \alpha_v \; . \tag{5.10}$$

Still we would like to find a global exponent for the decay of $\hat{\phi}_1(\omega)$ at infinity. For this we shall introduce an "artificial" function which will play the same role as $\cos\omega$ in (5.5). We define

$$C(\omega) = \frac{\sin^2\left(\frac{\omega_1 + \omega_2}{2}\right) + \sin^2\left(\frac{\omega_1 - \omega_2}{2}\right)}{2\left[\sin^2\left(\frac{\omega_1}{2}\right) + \sin^2\left(\frac{\omega_2}{2}\right)\right]} \, , \quad C(0) = 1 \; . \tag{5.11}$$

Contrarily to $M_0^1(\omega)$, $C(\omega)$ is not a trigonometric polynomial, but it is a bounded regular function which vanishes at the point (π, π) with the same order of cancellation as $M_0^1(\omega)$. Moreover, it satisfies by construction

$$\prod_{k=1}^{+\infty} C\left(D^{-j}\omega\right) = \frac{2\left[\sin^2(\omega_1/2) + \sin^2(\omega_2/2)\right]}{\omega_1^2 + \omega_2^2} \leq C\,(1 + |\omega|)^{-2} \; . \tag{5.12}$$

The decay of this infinite product is now uniform and, for this reason, $C(\omega)$ will play an important role in the construction of our dual bases. For the moment, by comparing $C(\omega)$ and $M_0^1(\omega)$, we obtain the following result:

Proposition 5.1. *The decay of $\hat{\phi}_1(\omega)$ at infinity is controlled by*

$$|\hat{\phi}_1(\omega)| \leq C\,(1 + |\omega|)^{-2} \; . \tag{5.13}$$

Furthermore, this exponent is globally optimal, i.e. there exists a sequence $\{\omega_j\}_{j>0}$ such that $\lim_{j\to+\infty} |\omega_j| = +\infty$ and $|\hat{\phi}_1(\omega_j)| \sim C|\omega_j|^{-2}$.

PROOF. Using the variables $y_1 = \sin^2(\omega_1/2)$ and $y_2 = \sin^2(\omega_2/2)$ we can rewrite $C(\omega)$ as

$$C(\omega) = \frac{y_1 + y_2 - 2\,y_1 y_2}{y_1 + y_2} = \frac{(1 - y_1)y_2 + (1 - y_2)y_1}{y_1 + y_2} \; . \tag{5.14}$$

We thus have

$$\begin{aligned} C(\omega) - M_0^1(\omega) &= \frac{(1-y_1)y_2 + (1-y_2)y_1}{y_1+y_2} - \frac{(1-y_1)+(1-y_2)}{2} \\ &= \frac{(1-y_1)(y_2-y_1) + (1-y_2)(y_1-y_2)}{2(y_1+y_2)} \\ &= \frac{(y_1-y_2)^2}{2(y_1+y_2)} \geq 0 . \end{aligned}$$

Thus $M_0^1(\omega) \leq C(\omega)$ and by (5.12) $|\hat{\phi}_1(\omega)| \leq C(1+|\omega|)^{-2}$. To prove that this exponent is optimal we consider a small vector $\rho \neq 0$ in $\mathbb{R}^2$ and define

$$\omega_j = D^j(\pi, \pi) + \rho , \tag{5.15}$$

so that

$$\hat{\phi}_1(\omega_j) = \prod_{k=1}^{+\infty} M_0^1 \left(D^{j-k}(\pi, \pi) + D^{-k}\rho\right) . \tag{5.16}$$

Let us divide this product in three parts

$$\begin{aligned} \hat{\phi}_1(\omega_j) &= \left[\hat{\phi}_1 \left((\pi, \pi) + D^{-j}\rho\right)\right] \\ &\quad \cdot \left[\prod_{k=1}^{j-1} M_0^1 \left(D^{j-k}(\pi, \pi) + D^{-k}\rho\right)\right] \\ &\quad \cdot \left[M_0^1 \left((\pi, \pi) + D^{-j}\rho\right)\right] \\ &= A(j)\, B(j)\, C(j) . \end{aligned} \tag{5.17}$$

One checks easily that $\hat{\phi}_1(\pi, \pi) \neq 0$ and thus, for j large enough or sufficiently small ρ, we have $0 < C_1 \leq A(j) \leq 1$. It is also clear that for $1 \leq k \leq j-1$, $M_0^1\left(D^{j-k}(\pi, \pi)\right) = 1$ and that for $\ell \geq 1$, $M_0^1(D^\ell(\pi, \pi) + \sigma) \geq 1 - C\,\|\sigma\|$ for σ small enough, with $C > 0$. Consequently, if ρ has been chosen small enough, $1 \geq B(j) \geq \prod_{\ell=1}^{\infty} [1 - C\,2^{-\ell}\,\|\rho\|] \geq C_2 > 0$. Finally since (π, π) is a second order zero of $M_0(\omega)$, the third factor satisfies

$$\begin{aligned} 2^{-j} C_3\, \|\rho\|^2 &= C_3\, \|D^{-j}\rho\|^2 \leq C(j) \\ &\leq C_4\, \|D^{-j}\rho\|^2 = 2^{-j} C_4\, \|\rho\|^2 . \end{aligned} \tag{5.18}$$

This shows that $\hat{\phi}_1(\omega_j)$ behaves like $2^{-j} \sim |\omega_j|^{-2}$ when j goes to $+\infty$ and the proposition is proved.

Note that from the decay of $\hat{\phi}_1(\omega)$ we cannot even conclude that it belongs to $L^1(\mathbb{R}^2)$ or that $\phi_1(x)$ is a continuous function. Yet both are true; we are going to prove this by the Littlewood-Paley method explained in II.4.a. The filter $M_0^1(\omega)$ and the scaling function $\hat{\phi}_1(\omega)$ are particularly well adapted for this approach since they are positive so that the regularity estimation is optimal (because $\|\Delta_j(\phi_1)\|_{L^\infty} \sim \|\widehat{\Delta_j}(\phi_1)\|_{L^1}$; see Section II.4.a).

Proposition 5.2. *The optimal global Hölder exponent for $\phi_1(x)$ is*

$$\alpha = \frac{2}{\log 2} \log \left(\frac{1+\sqrt{5}}{4} \right) \simeq .61$$

PROOF. We consider the transition operator defined by

$$TF(D\omega) = M_0^1(\omega)F(\omega) + M_0^1(\omega + (\pi,\, \pi))F(\omega + (\pi,\, \pi)). \tag{5.19}$$

As in the one dimensional case T can be studied in a finite dimensional space but this subspace cannot be defined as simply as $E(N_1, N_2)$ in (2.61). One way of finding an invariant subspace is to apply T to the constant 1 and then iterate it on the characters $e^{i(k_1\omega_1+k_2\omega_2)}$ which are obtained until a stable set is attained. With M_0^1 corresponding to (5.4), this subspace is trivial, since $T_1 = 1$. Lemma 2.5 then guarantees the integrability of $\hat{\phi}_1$, hence the continuity of ϕ_1. To estimate the Hölder exponent of ϕ_1 we need a larger subspace, which we obtain by iterating T on 1 and on $\cos\omega_1 + \cos\omega_2$. The size of the matrix representing the action of T on this subspace can be seriously reduced by exploiting the symmetries, *i.e.* the invariance under $\omega_1 \longleftrightarrow -\omega_1$, $\omega_2 \longleftrightarrow -\omega_2$ and $\omega_1 \longleftrightarrow \omega_2$.

Using the subspace E generated by the basis

$$e_1 = 1,\ e_2 = \cos\omega_1 + \cos\omega_2,\ e_3 = \cos(\omega_1+\omega_2) + \cos(\omega_2 - \omega_1) \tag{5.20}$$

we obtain the following matrix

$$T = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1/2 & 1 \\ 0 & 1/4 & 0 \end{pmatrix} \tag{5.21}$$

which has the eigenvalues $\{1,\ (1+\sqrt{5})/4,\ (1-\sqrt{5})/4\}$. The two last eigenvalues correspond to the subspace $E_0 \subset E$ defined by

$$E_0 \;=\; \{F(\omega) \in E:\ F(0) = 0\}\ . \tag{5.22}$$

Similarly to the one dimensional case, we iterate T on the positive function $e_1 - \frac{1}{2}e_2$ which is clearly in E_0 and this leads us to

$$\left\|\Delta_{j/2}(\phi_1)\right\|_{L^\infty} \;\sim\; \left\|\hat{\Delta}_{j/2}(\phi_1)\right\|_{L^1} \;\sim\; C\left(\frac{1+\sqrt{5}}{4}\right)^j , \tag{5.23}$$

where $\Delta_{j/2}(\phi_1)$ is the Littlewood-Paley block corresponding to the region $D^j([-\pi,\,\pi]^2)/D^{j-1}\ ([-\pi,\,\pi]^2)$, situated at a distance $2^{j/2}$ of the origin. Consequently, if we define

$$\alpha \;=\; -\frac{2}{\log 2}\ \log\left(\frac{1+\sqrt{5}}{4}\right) \;\simeq\; 0.61\,, \tag{5.24}$$

then it follows from (5.23) that

$$(1+|\omega|)^\alpha\ \hat{\phi}_1(\omega) \in L^1(\mathbb{R}^2) \quad \text{and} \quad \phi_1(x) \in C^\alpha(\mathbb{R}^2)\ . \tag{5.25}$$

Consequently ϕ_1 is Hölder continuous with regularity 0.61.

This property appears in the graph of ϕ_1 on figure 10 (obtained by the cascade algorithm) which presents a smooth aspect with several pointwise cusps. Note that this regularity is not sufficient to derive a better decay of $\hat{\phi}_1(\omega)$ than $|\omega|^{-0.61}$; Propositions 5.1 and 5.2 are thus complementary.

REMARKS.

- Note that, since we have

$$M_0^1(\omega) \;+\; M_0^1(\omega + (\pi,\,\pi)) \;=\; 1\ , \tag{5.26}$$

we can derive the L^1 convergence of the truncated products $\hat{\phi}_{1n} = \prod_{j=1} M_0(D^{-j}\omega)\chi_{D^n([-\pi,\,\pi]^2)}(\omega)$ with the same method as in the orthonormal case for the L^2 convergence (Theorem 2.1). This leads us to a Poisson summation formula

$$\sum_{k\in\mathbb{Z}^2} \hat{\phi}_1(\omega + 2k\pi) \;=\; 1 \tag{5.27}$$

which is equivalent to

$$\phi_1(n_1, n_2) \;=\; 1 \text{ if } n_1 = n_2 = 0,\; 0 \text{ if } (n_1,\, n_2) \in \mathbb{Z}^2/\{0\}\ . \tag{5.28}$$

This interpolating property of ϕ_1 has been noticed in approximation theory by Deslaurier and Dubuc [DD]. It explains the four cusps surrounding the center at the points $(0,1)$, $(1,0)$, $(0,-1)$ and $(-1,0)$ which are visible on figure 10. However, a sharper analysis shows that the isolated points where $\phi_1(x) = 0$ are an infinite family.

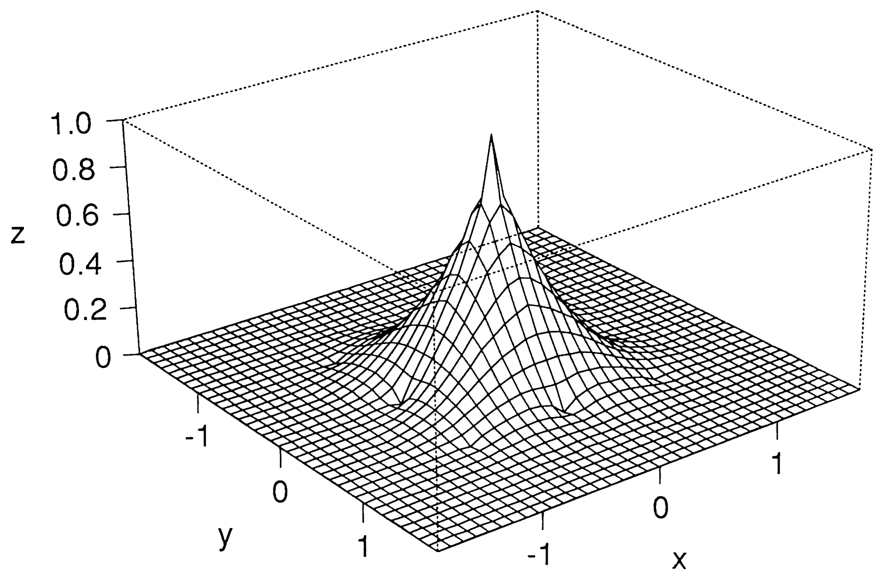


Figure 10
The scaling function $\phi_1(x)$.

- As mentioned in Section III.3.b, the variable $z = (y_1 + y_2)/2$ can be replaced by, more generally, $z_\lambda = \lambda y_1 + (1-\lambda) y_2$ with $\lambda \in [0,1]$; $M_0^1(\omega) = 1 - z_\lambda$ is still positive. Let us now distinguish the dilation matrices R and S. Then, a similar analysis in the case of $D = R$ leads to a 5×5 matrix in the basis

$$(e_1, e_2, e_3, e_4, e_5) \;=\; (1, \cos\omega_1, \cos\omega_2, \cos(\omega_1+\omega_2), \cos(\omega_1-\omega_2))$$

$$\text{(5.29)} \qquad T_\lambda \;=\; \frac{1}{2}\begin{pmatrix} 2 & \lambda & 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & \lambda & 0 & 2 \\ 0 & 1-\lambda & \lambda & 2 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 \end{pmatrix}$$

and numerical computations show that the "isotropic value" $\lambda = 1/2$ gives the highest index of regularity. The lowest index of regularity is attained for $\lambda = 0$ or 1. Note that $\lambda = 1$ corresponds to the convolution product $g(x) = \chi_\Delta * \chi_\Delta$ where Δ is the twin dragon introduced in IV.3.a. The Hölder exponent is then $\alpha \simeq 0.47$.
- To estimate the decay of $\hat{g}(\omega)$ $(= (\hat{\chi}_\Delta(\omega))^2)$, one can again use the function $C(\omega)$ of Proposition 5.1, in a slightly different way. Remark that, if we define $G(\omega) = 1 - z_1 = 1 - y_1$, then

$$\begin{aligned} C(\omega) - G(\omega) &= \frac{(1-y_1)y_2 + (1-y_2)y_1}{y_1+y_2} \;-\; (1-y_1) \\ &= \frac{y_1(y_1-y_2)}{y_1+y_2} \geq 0 \end{aligned}$$

if $y_1 \geq y_2$, and

$$\begin{aligned} 2\,C(\omega) - G(\omega) &= \frac{2\,[(1-y_1)y_2 + (1-y_2)y_1]}{y_1+y_2} \;-\; (1-y_1) \\ &= \frac{(1-y_1)(y_2-y_1) + 2\,y_1(1-y_2)}{y_1+y_2} \geq 0 \end{aligned}$$

if $y_2 \geq y_1$. On the other hand

$$\text{(5.30)} \qquad |\hat{g}(\omega)| \;=\; \prod_{k=1}^{\infty} G(R^{-k}\omega) \;;$$

to majorate $|\hat{g}(\omega)|$ for $2^{j/2} \leq |\omega| \leq 2^{(j+1)/2}$ we only need to majorate the j first factors in (5.30). Since R rotates by $\pi/4$, half of the factors can be majorated by $C(\omega)$ and the others by $2C(\omega)$. This leads to

$$(5.31)\quad |\hat{g}(\omega)| \leq C\, 2^{\log(1+|\omega|)/\log 2} \prod_{1\leq k\leq 2\log(1+|\omega|)/\log 2} C(R^{-k}\omega)$$

and thus

$$(5.32)\qquad \hat{g}(\omega) \leq C\,(1+|\omega|)^{-1}\ .$$

It is easy to check (in a similar way as for $\hat{\phi}_1(\omega)$) that this estimate is optimal. An immediate consequence is that the Fourier transform of the twin dragon characteristic function χ_Δ satisfies

$$(5.33)\qquad \hat{\chi}_\Delta(\omega) \leq C\,(1+|\omega|)^{-1/2}$$

which was not obvious since we did not have a formula similar to (5.5) for $\hat{\chi}_\Delta$.

We now return to the construction of our biorthogonal bases and attack the problem of obtaining isotropic wavelet bases with arbitrarily high regularity.

V.2. Biorthogonal wavelet bases with arbitrarily high regularity.

We now consider the whole family of filters

$$\left\{M_0^N(\omega),\ \tilde{M}_0^{N,L}(\omega)\right\}_{N,L>0}$$

defined by (5.1) and (5.2).

A first remark is that the regularity of the functions ϕ_N increases linearly with N. More precisely, since

$$(5.34)\qquad \phi_N(x)\ =\ (*)^N\phi_1(x)\ ,$$

we can use the characterization of the optimal decay exponent for $\hat{\phi}_1(\omega)$ established in Proposition 5.1 to estimate the regularity index $\alpha(N)$ of $\phi_N(x)$. This leads to

$$(5.35)\qquad 2N-2 \leq \alpha(N) \leq 2N$$

and thus to

$$(5.36) \qquad \lim_{N \to +\infty} \frac{\alpha(N)}{N} = 2 .$$

The estimate (5.35) is of course more interesting for large values of N than for small values where the error is comparable with the regularity.

For $N = 1$, we have seen that $\alpha \simeq 0.61$.

For $N = 2$, the Littlewood-Paley approach is still reasonable; using the symmetries reduces the size of the matrix to 9×9. Analyzing the eigenvalues, one finds that ϕ_2 is in C^α with $\alpha \simeq 2.93$. The function $\phi_2 = \phi_1 * \phi_1$ looks very smooth indeed on figure 13.

For $N \geq 3$, the matrix becomes too large to tackle by hand. In all cases the regularity of the wavelet $\psi_{N,L}$ will of course be the same as that of ϕ_N. The problem is now to find the appropriate dual function for the analysis. More precisely we want to design the filter

$$(5.37) \qquad \tilde{M}_0^{N,L}(\omega) = (1-z)^L \, P_{N+L}(z)$$

by choosing the number L in such way that the hypotheses of Theorem 2.2 (in its bidimensional generalization) are satisfied, *i.e.* that we have at least

$$(5.38) \qquad \left|\hat{\tilde{\phi}}_{N,L}(\omega)\right| \leq C\,(1+|\omega|)^{-1-\varepsilon}, \qquad \varepsilon > 0 .$$

To show that such a choice is possible for any value of N (*i.e.* for an arbitrarily regular synthesis function), we need an asymptotical result of the same nature as Theorem 2.4. We want to be sure that the regularizing action of the factor $(1-z)^L$ can compensate the inverse effect of P_{N+L} if L is large enough.

Using a similar approach, we consider the simplest fixed point of $\omega \longmapsto D\omega$ modulo $2\pi\mathbb{Z}^2$, and modulo sign changes and the exchange of ω_1 and ω_2. This fixed point is $\omega_0 = (2\pi/5, 4\pi/5)$ which corresponds to $z_0 = z(\omega_0) = 5/8$.

We now decompose $\tilde{M}_0^{N,L}$ into three factors, by introducing the function $C(\omega)$ defined by (5.11):

$$(5.39) \quad \tilde{M}_0^{N,L}(\omega) = [C(\omega)]^L \, [Q(\omega)]^L \, P_{N+L}(z) = [C(\omega)]^L \, B_{N,L}(\omega)$$

with

$$Q(\omega) = \frac{M_0^1(\omega)}{C(\omega)} = \frac{(y_1+y_2)(2-y_1-y_2)}{2\,(y_1+y_2-2y_1y_2)} .$$

We already know from Section II.3.b that

$$(5.40)\qquad P_N(z) \le (4z)^{N-1} \qquad \text{if } z \ge \frac{1}{2}\,.$$

From the Bezout relation (3.30), we also have

$$(5.41)\qquad P_N(z) \le \left(\frac{1}{1-z}\right)^N \,.$$

Consequently, we can roughly majorate $P_N(z)$ by

$$(5.42)\qquad P_N(z) \le \left[\min\{\frac{1}{1-z},\ \max\{4z,2\}\}\right]^N \qquad \text{if } z \in [0,1]\,.$$

Defining $H(\omega) = \min\{1/(1-z),\ \max\{4z,2\}\}$ and $G(\omega) = H(\omega)Q(\omega)$, (5.39) leads us to

$$(5.43)\qquad \tilde{M}_0^{N,L}(\omega) \le [C(\omega)]^L\ [G(\omega)]^L\ [H(\omega)]^N\,.$$

We are now facing a similar situation as in Theorem 2.4 where we had shown that the function $g(y) = \max\{2,4y\} = h(\omega)$ satisfied

$$(5.44)\qquad \begin{cases} h(\omega) = g(y) \le g\,(3/4) & \text{if } y \le 3/4\,, \\ h(\omega)h(2\omega) = g(y)\,g(4y(1-y)) \le [g\,(3/4)]^2 & \text{if } 3/4 \le y \le 1\,. \end{cases}$$

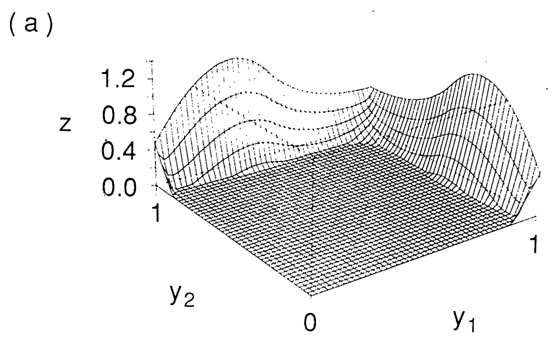
In the present case, although we do not dispose of any simple mathematical proof, numerical evidence shows that we have

$$(5.45)\qquad \begin{cases} G(\omega)G(D\omega) \le [G(\omega_0)]^2 & \text{or if not,} \\ G(\omega)G(D\omega)G(D^2\omega) \le [G(\omega_0)]^3 & \end{cases}$$

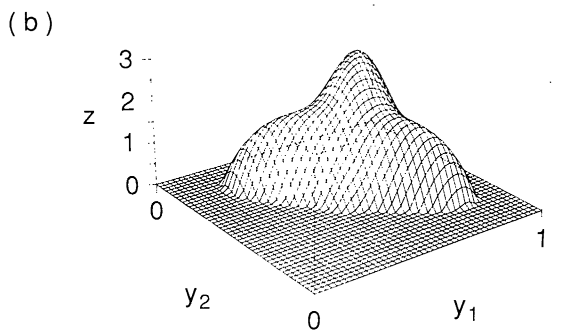
and similarly

$$(5.46)\qquad \begin{cases} H(\omega)H(D\omega) \le [H(\omega_0)]^2 & \text{or if not,} \\ H(\omega)H(D\omega)H(D^2\omega) \le [H(\omega_0)]^3 & \end{cases}$$

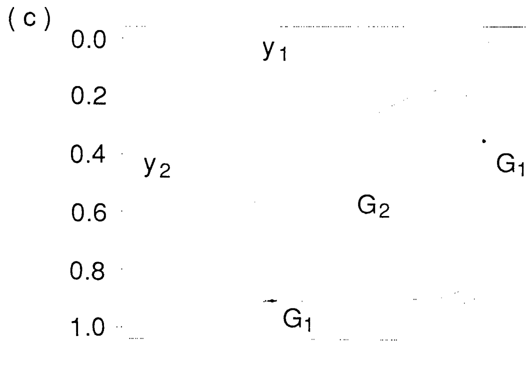
These two statements are illustrated respectively in figures 11 and 12.



a) Graph of $G_1(y_1, y_2) = \max\{G(\omega)G(D\omega), [G(\omega_0)]^2\} - [G(\omega_0)]^2$



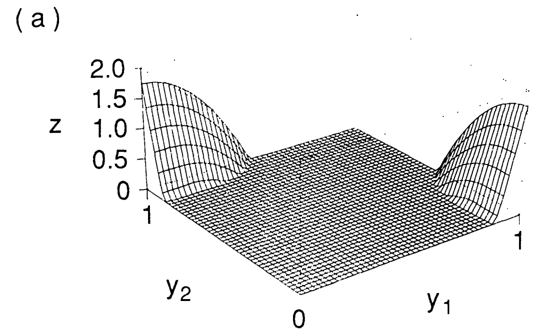
b) Graph of $G_2(y_1, y_2) = \max\{G(\omega)G(D\omega)G(D^2\omega), [G(\omega_0)]^3\} - [G(\omega_0)]^3$



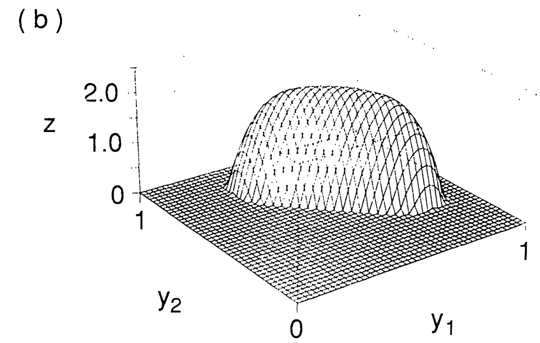
c) Compared supports of $G_1(y_1, y_2)$ and $G_2(y_1, y_2)$

Figure 11

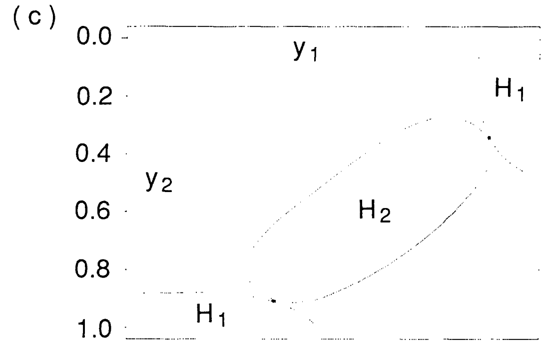
Graphic proof of (5.45)



a) Graph of $H_1(y_1, y_2) = \max\{H(\omega)H(D\omega), [H(\omega_0)]^2\} - [H(\omega_0)]^2$



b) Graph of $H_2(y_1, y_2) = \max\{H(\omega)H(D\omega)H(D^2\omega), [H(\omega_0)]^3\} - [H(\omega_0)]^3$



c) Compared supports of $H_1(y_1, y_2)$ and $H_2(y_1, y_2)$

Figure 12

Graphic proof of (5.46)

On a) and b) of each of these figures we have plotted the functions

$$\max\{F(\omega)F(D\omega), [F(\omega_0)]^2\} \; - \; [F(\omega_0)]^2$$

and

$$\max\{F(\omega)F(D\omega)F(D^2\omega), [F(\omega_0)]^3\} \; - \; [F(\omega_0)]^3$$

for $F = G$ and H (the coordinates are $(y_1,\, y_2) \in [0,1]^2$). On c) the supports of a) and b) are shown to be disjoint regions in $[0,1]^2$.

We now estimate $\hat{\tilde{\phi}}_{N,L}(\omega)$. From(5.39) and (5.43) we get

$$\begin{aligned}\hat{\tilde{\phi}}_{N,L}(\omega) &= \prod_{k=1}^{+\infty} \left[C(D^{-k}\omega)\right]^L B_{N,L}(D^{-k}\omega)\\ &\leq C\,(1+|\omega|)^{-2L} \prod_{1\leq k\leq 2\log(1+|\omega|)/\log 2} B_{N,L}(D^{-k}\omega)\\ &\leq C\,(1+|\omega|)^{-2L}\Big[\prod_{1\leq k\leq 2\log(1+|\omega|)/\log 2} G(D^{-k}\omega)\Big]^L\\ &\quad\cdot\Big[\prod_{1\leq k\leq 2\log(1+|\omega|)/\log 2} H(D^{-k}\omega)\Big]^N .\end{aligned}$$

Using (5.45) and (5.46) to divide these products in groups of two of three factors which satisfy one of the inequalities, this leads to

$$\text{(5.47)} \qquad \hat{\tilde{\phi}}_{N,L}(\omega) \leq C\,(1+|\omega|)^{-2L+2(L\log G(\omega_0)\; +\; N\log H(\omega_0))/\log 2}$$

or

$$\text{(5.47')} \qquad \hat{\tilde{\phi}}_{N,L}(\omega) \leq C\,(1+|\omega|)^{2L(\alpha-1)+2N\beta}$$

with

$$\alpha = \frac{\log(G(\omega_0))}{\log 2} \simeq 0.907 \quad \text{and} \quad \beta = \frac{\log(H(\omega_0))}{\log 2} \simeq 1.322 \;.$$

Fortunately $\alpha < 1$. This means $\hat{\phi}_{L,N}(x)$ can be made arbitrarily regular by choosing L large enough. In particular, (5.38) will be satisfied if we have

$$\text{(5.48)} \qquad 2L(\alpha-1) + 2N\beta \;<\; -1 \;.$$

The smallest L such that M_0^N and $\tilde{M}_0^{N,L}$ generate unconditional multiscale bases is therefore given asymptotically by

$$L(N) \simeq \frac{\beta N}{1-\alpha} = \frac{\log 5 - \log 2}{\log 16 - \log 15} N \simeq 14.2\, N \,. \tag{5.49}$$

This asymptotical estimate is moreover optimal. Indeed define $\omega_j = D^j \omega_0 = D^j(2\pi/5,\, 4\pi/5)$. Because of the fixed point property of ω_0, we clearly have

$$\tilde{\phi}^{N,L}(\omega_j) \sim C\left[\tilde{M}_0^{N,L}(\omega_0)\right]^j \sim C\,|\omega_j|^{-\gamma} \tag{5.50}$$

with

$$\gamma = \frac{-2\,\log \tilde{M}_0^{N,L}(\omega_0)}{\log 2} \,. \tag{5.51}$$

From the definition (5.2) of $\tilde{M}_0^{N,L}$, we get

$$\begin{aligned}
\tilde{M}_0^{N,L}(\omega_0) &= \left(\frac{3}{8}\right)^L \; P_{N+L}\left(\frac{5}{8}\right) \\
&\geq \left(\frac{3}{8}\right)^L \binom{2(N+L-1)}{N+L-1} \left(\frac{5}{8}\right)^{N+L-1} \\
&\geq C \left(\frac{3}{8}\right)^L \left(\frac{5}{2}\right)^{N+L} = C\left(\frac{15}{16}\right)^L \left(\frac{5}{2}\right)^N
\end{aligned}$$

and thus

$$\begin{aligned}
\gamma &\leq C + 2L\,\frac{\log 16/15}{\log 2} - 2N\,\frac{\log 5/2}{\log 2} \\
&= C + 2L(1-\alpha) - 2N\beta \,.
\end{aligned}$$

It follows that the estimate (5.49), if true, is certainly optimal. While we expect (5.45), (5.46), hence (5.49), to be true, we have unfortunately no rigorous proof. However, we *can* prove inequalities which are slightly less strong than (5.45), (5.46), leading to a non-optimal but rigorous estimate for $L(N)$. More precisely, we can prove that $\Omega = [-\pi,\, \pi]^2$ can be split up as $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, with

$$\begin{cases}
G(\omega) \leq \xi & \omega \in \Omega_1 \,, \\
G(\omega)\; G(D\omega) \leq \xi^2 & \omega \in \Omega_2 \,, \\
G(\omega)\; G(D\omega)\; G(D^2\omega) \leq \xi^3 & \omega \in \Omega_3 \,,
\end{cases} \tag{5.52}$$

with $\xi/2 \simeq .9588 < 1$, resulting in (5.47′) with

$$\alpha = \frac{\log \xi}{\log 2} \simeq .93982 .$$

If we use the crude estimate $H(\omega) \leq 4$ for all $\omega \in [-\pi, \pi]^2$, corresponding to $\beta = 2$, then this leads to

$$L(N) \simeq \frac{\beta}{1-\alpha} N \simeq 32.959\ N ;$$

this factor is about twice as large as in (5.49). The detailed proof of this estimate is in Appendix C.

All these results can be summarized in the following theorem.

Theorem 5.3. *The family of dual filters* $\{M_0^N(\omega),\ \tilde{M}_0^{N,L}(\omega)\}_{N,L>0}$ *generates biorthogonal bases of compactly supported wavelets with arbitrarily high regularity. For large values of* N*, the Hölder exponent of* $\phi_N(x)$ *is equivalent to* $2N$ *and the minimal choice for* L *is asymptotically proportional to* N*,*

$$L(N) \simeq \mathcal{K} N , \tag{5.53}$$

with $14.215 \leq \mathcal{K} \leq 32.959$.

Here the upper bound on $\mathcal{K}$ is not tight, and we expect $\mathcal{K} = 14.215$ to hold, as indicated above.

REMARK. By taking L larger than $L(N)$, $\hat{\phi}^{N,L}$ can also be made arbitrarily regular. However, in many applications such as coding, approximation, data storage and compression, we do not really care about the regularity of the analyzing functions $\tilde{\psi}$ and $\tilde{\phi}$; only the synthesis function ψ and ϕ have to be smooth since this property is important for the cascade-reconstruction algorithm. This justifies the choice of the minimal value $L(N)$ such that the families $\{2^{j/2}\ \psi_{N,L}(D^j x - k)\}_{j\in\mathbb{Z},\ k\in\mathbb{Z}^2}$ and $\{2^{j/2}\ \tilde{\psi}_{N,L}(D^j x - k)\}_{j\in\mathbb{Z},\ k\in\mathbb{Z}^2}$ are unconditional dual bases of $L^2(\mathbb{R})$. Recall that the existence of frame bounds is essential for the stability of the subband coding scheme.

We end this section by taking a closer look at the size of these dual filters.

V.3. Size and optimal implementation of the dual filters.

The asymptotical ratio $L(N)/N \simeq 14.2$ is big in the sense that the filter $\tilde{M}_0^{NL(N)}$ may have a very large number of taps. More precisely, a polynomial $P(z)$ of degree p corresponds to a filter with $p^2 + (p+1)^2$ nonzero coefficients. For example, if $N = 3$,

$$\tilde{M}_0^{NL(N)}(\omega) = (1-z)^{L(N)}\ P_{N+L(N)}(z)$$

is according to (5.49) a polynomial of degree $p = N + 2L(N) \simeq 87$ in z. Consequently it is the transfer function of a filter with approximately 1350 taps!

It seems thus that the dual filter is, even for small values of N, much too large for a realistic implementation. This is not quite true for several reasons.

First, one can factorize the polynomial $P_{N+L(N)}(z)$ and express $\tilde{M}_0^{N,L}$ as a product of p monomials in z. By applying successively these monomial filters instead of using directly their product, the number of multiplications per sample in the filtering process is reduced from order p^2 to p. Note that this complexity reduction associated with the factorization is due to the multidimensional situation and does not occur in the $1D$ case.

Second, the filter corresponding to the variable z, *i.e.* the laplacian discrete scheme, has coefficients $c_{0,0} = 1/2$ and $c_{1,0} = c_{-1,0} = c_{0,1} = c_{0,-1} = -1/8$. It can thus be implemented by using binary shifts instead of multiplications. This is very important since a binary shift is usually performed 10 times faster than an addition and 100 times faster than a multiplication in most processors. This shows that only the additions count here. If t is the time for one multiplication, each monomial filter will generate one sample in approximately $3t/5$ and the same operation will take $3pt/5$ for the whole filter. For $N = 3$ and $p = 87$, this corresponds to the complexity of a 52 tap filter which is much more reasonable than the first estimation.

Finally, for small values of N, it is clear that the asymptotical estimate (5.49) of $L(N)$ is far from sharp, just as, in $1D$, the asymptotical estimate on regularity of Section II.3.b was ill-suited to small filters. A better estimate for $L(N)$ can be found by checking that the optimal decay exponent for $\hat{\tilde{\phi}}(\omega)$ is exactly determined by the value of $\tilde{M}_0^{N,L}$ at $\omega_0 = (2\pi/5,\ 4\pi/5)$. More precisely recall that we have

$$\tilde{M}_0^{N,L}(\omega) \ = \ [C(\omega)]^L\ B_{N,L}(\omega)\ .$$

For the small values of N and L considered below, one can check by the same graphical arguments that the inequalities (5.45) or (5.46) are also satisfied by $B_{N,L}(\omega)$, *i.e.*

$$\begin{cases} B_{N,L}(\omega)\, B_{N,L}(D\omega) \leq [B_{N,L}(\omega_0)]^2 \quad \text{or if not,} \\ B_{N,L}(\omega)\, B_{N,L}(D\omega)\, B_{N,L}(D^2\omega) \leq [B_{N,L}(\omega_0)]^3 \,. \end{cases} \tag{5.54}$$

In order for (5.38) to be satisfied, we therefore only need

$$\tilde{M}_0^{N,L}(\omega_0) \leq \frac{\sqrt{2}}{2} \tag{5.55}$$

and this will be sufficient for these small values of N and L for which (5.52) holds. Using the definition (5.2) of $\tilde{M}_0^{N,L}$ we obtain

- for $N = 1$, $L(1) = 3$,
- for $N = 2$, $L(2) = 12$,
- for $N = 3$, $L(3) = 22$.

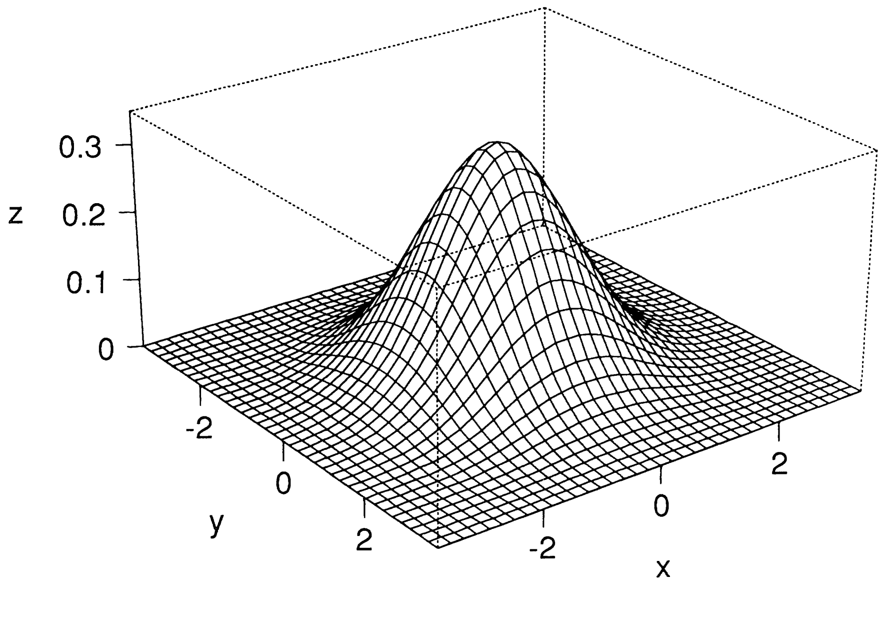
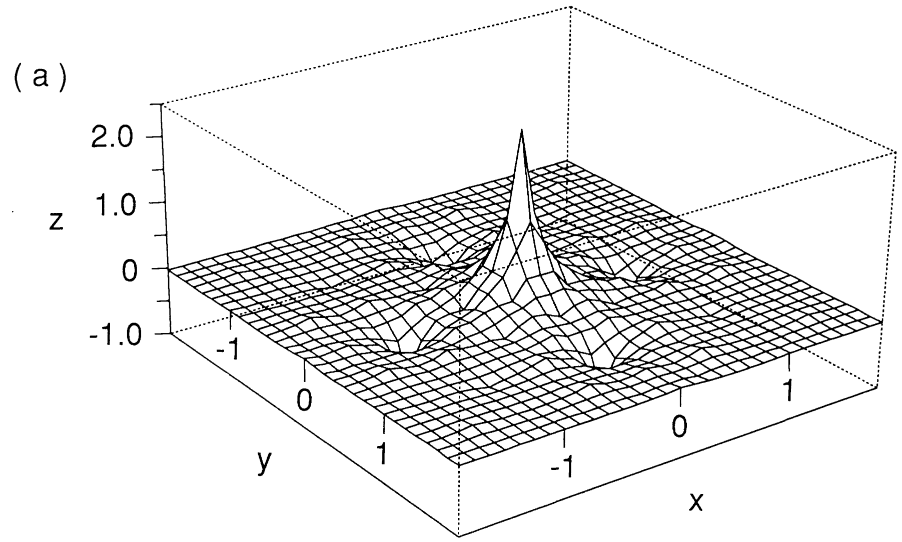
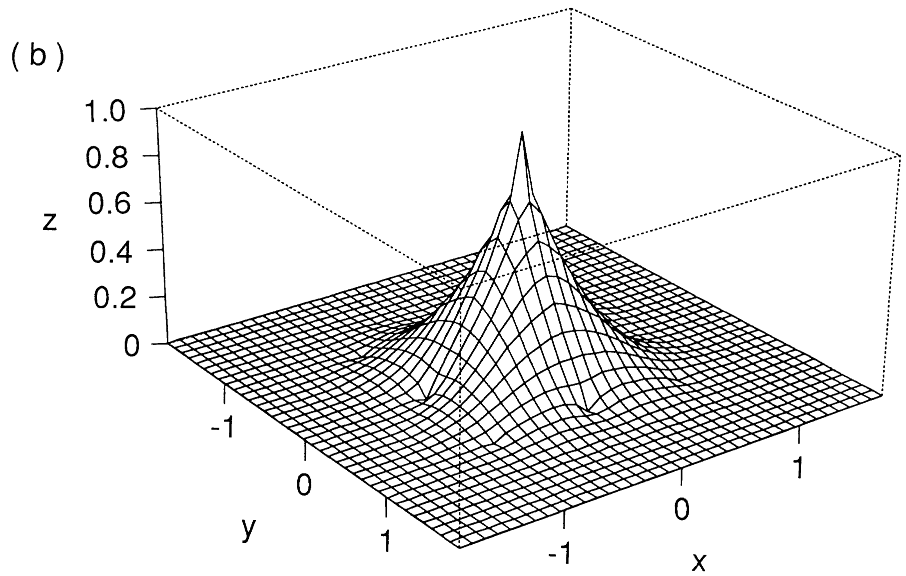


Figure 13
The scaling function ϕ_2 $(= \phi_1 * \phi_1)$.



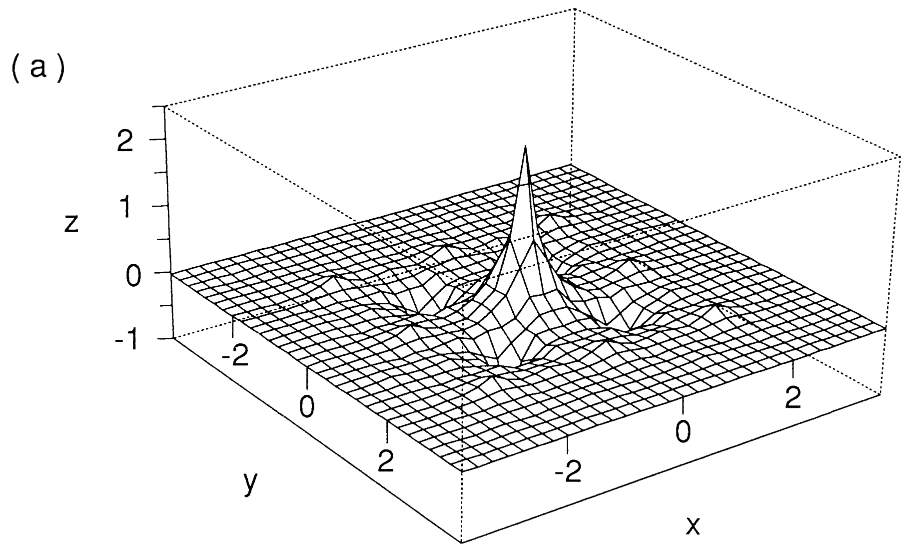
a) $\hat{\phi}_{12}$



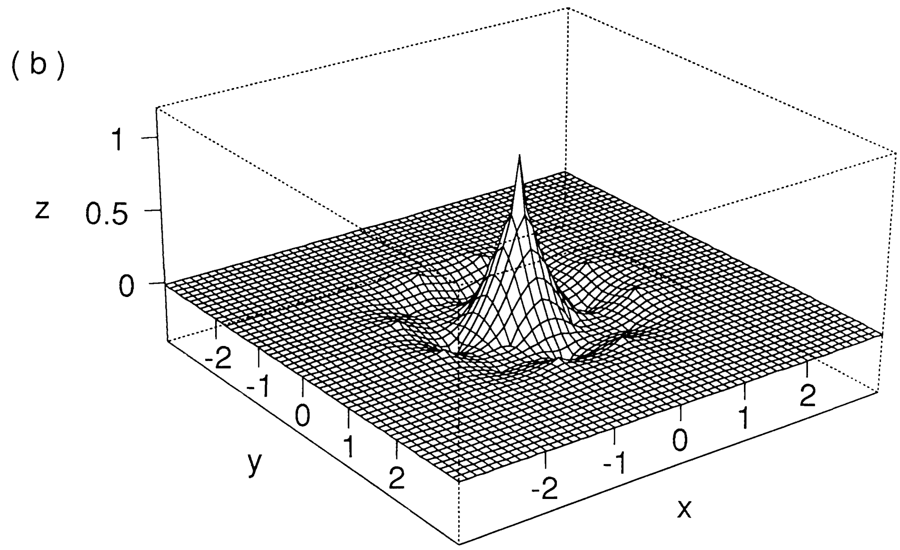
b) ϕ_1

Figure 14

Analysis and synthesis scaling function for $N = 1$ and $L = 2$.



a) $\tilde{\psi}_{12}$



b) ψ_{12}

Figure 15

Analysis and synthesis wavelets for $N = 1$ and $L = 2$.

Clearly these estimates are much better than (5.49). Finally, $L(N)$ can be even more reduced, for small values of N, if an even sharper criterion that the frequency decay (5.38) is used to ensure the existence of frame bounds. We show indeed in Appendix A that the spectral analysis of the transition operators T_0 and $\tilde{T}_0$ corresponding to the functions $|M_0|^2$ and $|\tilde{M}_0|^2$ can be used to derive both the frame property and the L^2 convergence needed to have a pair of dual Riesz bases. In [CD] we prove that this criterion is sharp so that the value of $L(N)$ is here optimal. Unfortunately the matrices of T_0 and $\tilde{T}_0$ can be very big, even for small N and L.

For $N = 1$, we now obtain $L(N) = 2$ so that the two filters M_0^1 and $\tilde{M}_0^{12}$ are of small size. We show on figure 14 and 15 the scaling functions and wavelets obtained from such a choice.

Appendix A: A sharp criterion for frame bounds.

We want to give here a better result than Theorem 2.2 to characterize the dual filter pair $(m_0, \tilde{m}_0)$ which lead to biorthogonal Riesz bases of wavelets. The method that we show here uses the transition operators associated to the positive functions $|m_0|^2$ and $|\tilde{m}_0|^2$ (see Section II.4.a).

First, recall that the φ, $\tilde{\varphi}$, ψ and $\tilde{\psi}$ are defined by

$$\text{(A.1)} \qquad \begin{cases} \hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_0(2^{-k}\omega) \,, & \hat{\psi}(2\omega) = m_1(\omega)\,\hat{\varphi}(\omega) \,, \\ \hat{\tilde{\varphi}}(\omega) = \prod_{k=1}^{+\infty} \tilde{m}_0(2^{-k}\omega) \,, & \hat{\tilde{\psi}}(2\omega) = \tilde{m}_1(\omega)\,\hat{\tilde{\varphi}}(\omega) \,. \end{cases}$$

As mentioned in Theorem 2.2 the duality relations (2.19), (2.20) and the decomposition formula (2.21) are ensured as soon as the partial products

$$\hat{\varphi}_n(\omega) = \prod_{k=1}^{n} m_0(2^{-k}\omega)\chi_{[-2^n\pi,\, 2^n\pi]}(\omega) \,,$$

$$\hat{\tilde{\varphi}}_n(\omega) = \prod_{k=1}^{n} \tilde{m}_0(2^{-k}\omega)\chi_{[-2^n\pi,\, 2^n\pi]}(\omega)$$

converge in $L^2(\mathbb{R})$ respectively to $\hat{\varphi}(\omega)$ and $\hat{\tilde{\varphi}}(\omega)$.

The main difficulty is then to obtain the frame bounds A, B, $\tilde{A}$ and $\tilde{B}$ all strictly positive such that for all f in $L^2(\mathbb{R})$,

$$(A.2) \qquad \begin{cases} A\,\|f\|^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_k^j\rangle|^2 \leq B\,\|f\|^2\,, \\ \tilde{A}\,\|f\|^2 \leq \sum_{j,k\in\mathbb{Z}} |\langle f, \tilde{\psi}_k^j\rangle|^2 \leq \tilde{B}\|f\|^2\,. \end{cases}$$

It is sufficient to obtain the two upper bounds of (A.2) because the lower bounds are then obtained by using (2.21) and the Schwarz inequality which give

$$(A.3) \qquad \|f\|^2 \leq \left(\sum_{j,k} |\langle f, \psi_k^j\rangle|^2\right)^{1/2} \left(\sum_{j,k} |\langle f, \tilde{\psi}_k^j\rangle|^2\right)^{1/2}$$

In [CDF] we used the following assumption

$$(A.4) \qquad |\hat{\psi}(\omega)|^2 \; + \; |\hat{\tilde{\psi}}(\omega)|^2 \leq C\,(1+|\omega|)^{-1/2-\varepsilon}$$

which can also be formulated with φ and $\tilde{\varphi}$ instead of ψ and $\tilde{\psi}$. Here, we shall prove the L^2-convergence of $\{\varphi_n,\,\tilde{\varphi}_n\}_{n>0}$ and the frame inequalities (A.2) using weaker assumptions. More precisely let T_0 and $\tilde{T}_0$ be the two transition operators associated to the functions $|m_0|^2$ and $|\tilde{m}_0|^2$, as defined in Section II.4.a. They both operate in two spaces of trigonometric polynomials E_N and $E_{\tilde{N}}$. We have proved in Lemma 2.2 that the subspaces $F_N \;=\; \{f \in E_N \,:\, f(0) = 0\}$ and $F_{\tilde{N}} \;=\; \{f \in E_{\tilde{N}} \,:\, f(0) = 0\}$ are invariant under the action of T_0 and $\tilde{T}_0$. The following result gives a sharp characterization of the dual filter pairs associated to biorthogonal wavelet bases.

Theorem A.1. *Let λ (respectively $\tilde{\lambda}$) be the largest eigenvalue of T_0 (respectively, $\tilde{T}_0$) in the subspace F_N (respectively, $F_{\tilde{N}}$). Then if $|\lambda|$ and $|\tilde{\lambda}|$ are both strictly inferior to 1, the functions ψ and $\tilde{\psi}$ defined by (A.1) generate biorthogonal Riesz bases of wavelets $\{\psi_k^j,\, \tilde{\psi}_k^j\}_{j,k\in\mathbb{Z}}$.*

PROOF. We shall prove here that this condition on the eigenvalues of T_0 and $\tilde{T}_0$ is sufficient to obtain biorthogonal wavelet bases. In fact, it is also a necessary condition. This result is detailed in [CD].

We first show that φ and ψ are in $L^2(\mathbb{R})$. As in Theorem 2.7, we apply T_0^n to the function $C_1(\omega) = 1-\cos\omega$ which is in F_N and by using Lemma 2.5, we obtain

$$\begin{aligned}\int_{2^{n-1}\pi\le|\omega|\le 2^n\pi} |\hat{\varphi}(\omega)|^2 d\omega &\le C\int_{2^{n-1}\pi\le|\omega|\le 2^n\pi} |\hat{\varphi}_n(\omega)|^2 d\omega \\ &\le C\int_{-2^n\pi}^{2^n\pi} C_1(2^{-n}\omega)|\hat{\varphi}_n(\omega)|^2 d\omega \\ &\le C\left(\frac{\gamma+1}{2}\right)^n\end{aligned}$$

because $(\gamma+1)/2 > \gamma$. Since we also have $(\gamma+1)/2 < 1$, it follows that the dyadic blocks in the Littlewood-Paley decomposition of φ satisfy the inequality

$$\|\Delta_j(\varphi)\|_{L^2} \le C\,2^{-\varepsilon j} \quad \text{for some } \varepsilon > 0\ . \tag{A.5}$$

This proves that φ and ψ are even better than L^2: They belong to a Besov space $B_2^{\varepsilon,\infty}$ ($\subset L^2(\mathbb{R})$) for some $\varepsilon > 0$. We shall use this property to prove the frame inequalities. Similarly $\tilde{\varphi}$ and $\tilde{\psi}$ belong to $B_2^{\tilde{\varepsilon},\infty}$ for some $\tilde{\varepsilon} > 0$. To prove the L^2 convergence of the sequence φ_n to φ, we remark that since $m_0(0) = 1$, for α in $]0,\pi]$ small enough we have

$$|\omega| \le \alpha \quad \text{implies} \quad |\hat{\varphi}(\omega)| \ge C > 0\ . \tag{A.6}$$

We now introduce the sequence φ_n^α defined by

$$\hat{\varphi}_n^\alpha(\omega) \;=\; \prod_{k=1}^{n} m_0(2^{-k}\omega)\chi_{[-2^n\alpha,\,2^n\alpha]}(\omega)\ . \tag{A.7}$$

It is clear that $\hat{\varphi}_n^\alpha(\omega)$ converges pointwise to $\hat{\varphi}(\omega)$, but (A.6) also implies $|\hat{\varphi}_n^\alpha(\omega)| \le |\hat{\varphi}(\omega)|/C$ for all $n > 0$. By the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} \|\varphi_n^\alpha - \varphi\|_{L^2} \;=\; 0\ . \tag{A.8}$$

We now use the hypothesis on the eigenvalues to evaluate the L^2 norm of the difference $\varphi_n - \varphi_n^\alpha$

$$\int |\hat{\varphi}_n(\omega) \;-\; \hat{\varphi}_n^\alpha(\omega)|^2 d\omega = \int_{|\omega|>\alpha} |\hat{\varphi}_n(\omega)|^2 d\omega$$

$$\leq \frac{1}{C_1(\alpha)} \int_{-2^n\pi}^{2^n\pi} C_1(2^{-n}\omega)|\hat{\varphi}_n(\omega)|^2 d\omega$$
$$\leq C\, 2^{-\varepsilon n} \underset{n\to\infty}{\longrightarrow} 0\,.$$

Consequently φ_n converges to φ in $L^2(\mathbb{R})$ and the same holds for $\tilde{\varphi}_n$ and $\tilde{\varphi}$.

It remains to establish the upper frame inequalities in (A.2). We shall obtain them by using the following lemma.

Lemma A.2. *Let ψ be a function in $L^2(\mathbb{R})$ such that for some $\sigma > 0$,*

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\omega + 2k\pi)|^{2-\sigma} \leq C_1 \tag{A.9}$$

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^{-j}\omega)|^{\sigma} \leq C_2 \tag{A.10}$$

uniformly in ω. Define, for j, k in $\mathbb{Z}$, $\psi_k^j(x) = 2^{-j/2}\psi(2^{-j}x - k)$. Then, for all f in $L^2(\mathbb{R})$,

$$\sum_{j,k\in\mathbb{Z}} |\langle f, \psi_k^j\rangle|^2 \leq C_1 C_2 \|f\|^2\ . \tag{A.11}$$

Let us first assume that this result is true to conclude the proof of the theorem. We thus have to check that there exist a $\sigma > 0$ such that (A.9) and (A.10) are satisfied for ψ and $\tilde{\psi}$.

To check (A.10), we define $I_j = [-2^{j+1}\pi,\ -2^j\pi] \cup [2^j\pi,\ 2^{j+1}\pi]$. For $j \leq 1$, we can use the cancellation of $\hat{\psi}(\omega)$ at the origin to obtain

$$\max_{\omega\in I_j} |\hat{\psi}(\omega)| \leq C\, 2^j \qquad \text{for } j \leq 1\ . \tag{A.12}$$

For $j \geq 2$, we know that $\hat{\psi}(\pm 2^j\pi) = 0$ since $\hat{\varphi}(2k\pi) = 0$ for $k \in \mathbb{Z}\backslash\{0\}$. We thus have

$$\max_{\omega\in I_j} |\hat{\psi}(\omega)|^2 \leq \int_{I_j} \frac{d}{d\omega}\left(|\hat{\psi}|^2\right)\ d\omega$$
$$\leq 2\int_{I_j} \left|\hat{\psi}(\omega)\frac{d\hat{\psi}}{d\omega}(\omega)\right|\ d\omega$$

$$\leq 2\Big[\int_{I_j} |\hat{\psi}(\omega)|^2 \, d\omega\Big]^{1/2} \Big[\int_{\mathbb{R}} \Big|\frac{d\hat{\psi}}{d\omega}(\omega)\Big|^2 \, d\omega\Big]^{1/2} .$$

The first factor can be majorated by $2^{-\varepsilon j}$ because we have proved that ψ belongs to $B_2^{\varepsilon,\infty}$. The second factor is finite since it is proportional to $\int |x\psi(x)|^2 \, dx$ and ψ is a compactly supported L^2 function. Consequently

$$\max_{\omega \in I_j} |\hat{\psi}(\omega)| \leq C\, 2^{-\varepsilon j/2} \qquad \text{for } j \geq 2 . \tag{A.13}$$

Combining (A.12) and (A.13) we see that (A.10) holds for all $\sigma > 0$, since we have

$$\begin{aligned}\max_{\omega\in\mathbb{R}} \sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\omega)|^\sigma &\leq \sum_{j\in\mathbb{Z}} \Big[\max_{\omega\in I_j} |\hat{\psi}(\omega)|\Big]^\sigma \\ &\leq C\Big[\sum_{j\leq 1} 2^{\sigma j} + \sum_{j\geq 2} 2^{-\varepsilon\sigma j/2}\Big] \\ &\leq C_2(\sigma) .\end{aligned}$$

We now check that (A.9) is satisfied for some $\sigma > 0$. Because the wavelet satisfies $\hat{\psi}(4k\pi) = 0$ for all $k \in \mathbb{Z}$, we can derive

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |\hat{\psi}(\omega + 2k\pi)|^{2-\sigma} &\leq \sum_{k\in\mathbb{Z}} \int_{2k\pi}^{2k\pi+2\pi} \Big|\frac{d}{d\omega}(|\hat{\psi}|^{2-\sigma})\Big| \, d\omega \\ &\leq \int_{\mathbb{R}} \Big|\frac{d}{d\omega}[|\hat{\psi}|^2]^{1-\sigma/2}\Big| \, d\omega \\ &\leq |2-\sigma| \int_{\mathbb{R}} |\hat{\psi}(\omega)|^{1-\sigma} \Big|\frac{d\hat{\psi}}{d\omega}\Big| \, d\omega \\ &\leq |2-\sigma| \Big[\int_{\mathbb{R}} |\hat{\psi}(\omega)|^{2-2\sigma}\Big]^{1/2} \Big[\int_{\mathbb{R}} \Big|\frac{d\hat{\psi}}{d\omega}\Big|^2 \, d\omega\Big]^{1/2} .\end{aligned}$$

We already saw that the second factor was finite (in the proof of (A.10)). The first factor is also finite for σ small enough. Indeed, using $\psi \in B_2^{\varepsilon,\infty}$ and the Hölder inequality, we obtain

$$\int_{I_j} |\hat{\psi}(\omega)|^{2-2\sigma} d\omega \leq \Big[\int_{I_j} |\hat{\psi}(\omega)|^2 \, d\omega\Big]^{1-\sigma} (2^{j+1}\pi)^\sigma$$

$$\leq C\ 2^{j(\sigma-2\varepsilon(1-\sigma))}\ .$$

We thus have to choose $\sigma > 0$ such that $\sigma - 2\varepsilon(1-\sigma) < 0$, *i.e.* $\sigma < 2\varepsilon/(1+2\varepsilon)$. Since the same results also hold for the dual wavelet $\tilde{\psi}$, the theorem is proved modulo Lemma A.2 that we tackle now . Using the Plancherel and the Poisson formulas, we derive for any f in $L^2(\mathbb{R})$

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\langle f, \psi_k^j\rangle|^2 &= \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}} 2^j \left| \int_{\mathbb{R}} \hat{f}(\omega)\ \overline{\hat{\psi}(2^j\omega)}\ e^{-i2^j k\omega}\ d\omega \right|^2 \\
&= \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}} 2^{-j} \left| \int_{\mathbb{R}} \hat{f}(2^{-j}\omega)\ \overline{\hat{\psi}(\omega)}\ e^{-ik\omega}\ d\omega \right|^2 \\
&= \frac{1}{2\pi} \int_0^{2\pi} 2^{-j} \left| \sum_{\ell\in\mathbb{Z}} \hat{f}(2^{-j}(\omega+2\ell\pi))\ \overline{\hat{\psi}(\omega+2\ell\pi)} \right|^2\ d\omega \\
&\leq \frac{2^{-j}}{2\pi} \int_0^{2\pi} \Bigg(\sum_{\ell\in\mathbb{Z}} |\hat{f}(2^{-j}(\omega+2\ell\pi)|\ |\hat{\psi}(\omega+2\ell\pi)|^{\sigma/2} \\
&\qquad \cdot |\hat{\psi}(\omega+2\ell\pi)|^{1-\sigma/2} \Bigg)^2\ d\omega \\
&\leq \frac{2^{-j}}{2\pi} \int_0^{2\pi} \left(\sum_{\ell\in\mathbb{Z}} |\hat{f}(2^{-j}(\omega+2\ell\pi))|^2\ |\hat{\psi}(\omega+2\ell\pi)|^{\sigma} \right) \\
&\qquad \cdot \left(\sum_{\ell\in\mathbb{Z}} |\hat{\psi}(\omega+2\ell\pi)|^{2-\sigma} \right)\ d\omega \\
&\leq\ C_1\ \frac{2^{-j}}{2\pi} \int_{\mathbb{R}} |\hat{f}(2^{-j}\omega)|^2\ |\hat{\psi}(\omega)|^{\sigma}\ d\omega \\
&= \frac{1}{2\pi}\ C_1 \int_{\mathbb{R}} |\hat{f}(\omega)|^2\ |\hat{\psi}(2^j\omega)|^2\ d\omega\ .
\end{aligned}$$

Summing on all the scales $j \in \mathbb{Z}$ and using (A.10), we get

$$\text{(A.14)} \qquad \sum_{j,k\in\mathbb{Z}} |\langle f, \psi_k^j\rangle|^2 \leq \frac{C_1C_2}{2\pi} \int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega\ =\ C_1C_2\ \|f\|^2$$

and this concludes the proof.

Appendix B. Dragonic expansions.

In this Appendix we want to show how the one-dimensional techniques in [DL] can be extended to multidimensional situations. As an example we discuss the two-dimensional case, with the dilation matrix $R = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

A first multidimensional extension of [DL] can be found in [Mo]. Even though he looks at general matrices, Mongeau effectively reduces his analysis to pure dilations by considering the smallest n such that $\overline{D} = D^n$ is a multiple of the identity, and rewriting (by iteration) the two-scale equation so that it involves only $\overline{D}$. This procedure can drastically increase the number of different terms in the equation. We choose here to work directly with $D = R$ itself.

When the two-scale equation is one-dimensional, and the dilation factor is 2, the regularity at x of the function ϕ solving the equation is regulated by the binary expansion of x (for dilation factor p, the same role is played by the p-ary expansion). Moreover, $\mathbb{R}$ and in particular $\operatorname{supp}\phi$ is tiled with integer translates of the interval $[0,1]$, which can be viewed as the set of numbers equal to the decimal part only of their dyadic expansion; if N such tiles are needed to cover the support of ϕ, then the two-scale functional equation can be rewritten as an equation for an N-dimensional vector-valued function involving two matrices T_0 and T_1. The spectral properties of T_0, T_1 then determine the regularity of ϕ, both local and global [DL].

In the two-dimensional case with dilation matrix R, the role of elementary tile is now played by the twin dragon set Δ. It is defined by

$$\text{(B.1)} \qquad \left\{ x \in \mathbb{R}^2 : \quad x = \sum_{j=1}^{\infty} R^{-j} p_j \quad \text{where} \quad p_j \in L = \mathbb{Z}^2 / R\mathbb{Z}^2 = \{(0,0),(1,0)\} \right\}.$$

Under the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$, with $(x,y) \simeq x + iy$, Δ can also be written as

$$\text{(B.2)} \quad \Delta = \left\{ z \in \mathbb{C} : z = \sum_{j=1}^{\infty} d_j \left(\frac{1+i}{2} \right)^j \text{ where } d_j = 0 \text{ or } 1 \right\}.$$

This set Δ is compact, has fractal boundary, is selfsimilar, and its $\mathbb{Z}^2$-translates tile the plane. The indicator function of Δ is the solution to the two-scale equation

$$\phi(x) = \phi(Rx) + \phi(Rx - (1,0))$$

(see [GM]). Δ is called the twin dragon set [K]. We shall give the name *dragonic expansions* to expansions of x or z as in (B.1), (B.2). Note that (as in the binary case) some points may have two different dragonic expansions, *e.g.* $.01000\ldots = ((1+i)/2)^2 = i/2 = .1011111\ldots$ (This example also illustrates that addition follows rules very different from the binary case, since $.0100\ldots + .0100\ldots = .1111\ldots$)

Suppose we are interested in various regularity properties of L^1-solutions ϕ of

$$\phi(x) = \sum_{k\in\Lambda} c_k\, \phi(Rx - k)\ , \tag{B.3}$$

where Λ is a finite subset of $\mathbb{Z}^2$. Such solutions are uniquely defined up to normalization and have necessarily compact support. One can determine the minimal set $\Gamma \subset \mathbb{Z}^2$ so that $R^{-1}(\Gamma + \Lambda - L) \subset \Gamma$; then $\operatorname{supp}\phi \subset \cup_{\ell\in\Gamma}(\Delta + \ell)$. The equation (B.3) for ϕ can be rewritten by defining the $|\Gamma|$-dimensional vector $v(x)$ by

$$v_j(x) = \phi(x+j)\,, \qquad j \in \Gamma,\ x \in \Delta\ , \tag{B.4}$$

we have

$$v_j(x) = \sum_k c_{Rj+d_1(x)-k}\ v_k(\tau x)$$

where $d_1(x)$ is the first digit in the dragonic expansion of x, and τx is the point obtained by dropping $d_1(x)$ from the same dragonic expansion of x,

$$\tau x = \sum_{j=1}^{\infty} d_{j+1}(x) \left(\frac{1+i}{2}\right)^j .$$

Equation (B.4) can be recast as

$$v(x) = T_{d_1(x)}\ v(\tau x)\ , \tag{B.5}$$

where $\qquad (T_0)_{jk} = c_{Rj-k}\ , \qquad (T_1)_{jk} = c_{Rj-k+(1,0)}\ .$

We have completed a setup analogous to that of [DL]. The question is now whether the proof techniques of [DL] still work in this case. The answer is basically yes. For instance, we still have

Theorem B.1. *Assume that the c_k in* (B.3) *satisfy*

$$\sum_n c_{Rn} = \sum_n c_{Rn+(1,0)} = 1 .$$

Then $e_1 = (1, 1, \ldots, 1)$ is a common lefteigenvector of T_0, T_1 with eigenvalue 1 for both matrices. Define E_1 to be the one-dimensional subspace orthogonal to e_1. If there exist $\lambda < 1$, $C > 0$ so that

$$\|T_{d_1} \cdots T_{d_m}|_{E_1}\| \leq C\,\lambda^m \tag{B.6}$$

for all possible $d_j = 0$ or 1, all $m \in \mathbb{N}$, then the L^1-solution ϕ to (B.3) *is Hölder continuous with exponent $\alpha = |\log\lambda|/\log\sqrt{2}$.*

This is the analog of Theorem 2.3 in [DL]. Two different strategies of proof are given in [DL]. The first one involves piecewise linear spline approximants; this technique would be hard to generalize here because of the fractal boundary of our domain building blocks $\Delta + k$. A second strategy, which does not use splines at all, but leads to longer proofs, is explained in the Appendix in [DL]; this strategy generalizes to the present case. The main point we have to check to make sure the proof carries over is whether elements that are close necessarily have dragonic expansions with the same starting digits. In the one-dimensional, binary case, if two dyadic rationals x, y are closer than 2^{-m}, $|x-y| < 2^{-m}$, then x and y have binary expansions with coinciding first m digits. (If *e.g.* $x \leq y < x + 2^{-m}$, then the expansion "from above" of x – ending in all zeros – has the same first m digits as the expansion "from below" of y – ending in all ones.) This is crucial in the proof, and allows to extract Hölder continuity from the condition (B.6). We therefore have to check whether a similar property holds in the "dragonic" case.

By analogy we shall call *dragonic rationals* all the points in Δ for which a terminating dragonic expansion can be written. Typically dragonic rationals also have other, non-terminating dragonic expansions. For each dragonic rational x the terminating expansion is unique; we denote its digits by $d_j^0(x)$, $j \in \mathbb{N}$.

Let us also introduce the notations R_0, R_1,

$$R_0 y = Ry, \qquad R_1 y = Ry + (1,0) ,$$

or $$R_d y = Ry + d(1,0), \qquad \text{with } d = 0 \text{ or } 1 .$$

Take now a fixed dragonic rational x, and assume that $d_j^0(x) = 0$ for $j > J$. All the $y \in \Delta$ that have the same first J digits $d_j^0(x)$, $j \leq J$, constitute a little dragon $\Delta_J(x)$ themselves,

$$\Delta_J(x) = R_{d_J(x)}^{-1} \cdots R_{d_1(x)}^{-1} \Delta ;$$

x itself is the image of $(0,0)$ under the same map $R_{d_J(x)}^{-1} \cdots R_{d_1(x)}^{-1}$. The set Δ is tiled by 2^J little dragons of the same size as Δ_J, all translates of Δ_J. For every such little dragon, we call the point corresponding to $(0,0)$ the "bottom", and the point corresponding to $(0,1)$ (the only other point in $\mathbb{Z}^2 \cap \Delta$) the "top". If x is a dragonic rational with at most N nonzero digits, then x is the bottom of $\Delta_J(x)$ for all $J > N$. (But note that the "orientation" of $\Delta_J(x)$, as indicated by the line connecting bottom and top, changes with J!). It follows that x is on the border of these $\Delta_J(x)$. If x is not at the edge of Δ itself, then there must exist another little dragon $\Delta_J(y)$ so that x is the top of $\Delta_J(y)$ (since Δ is the union of all the 2^J possible dragons Δ_J). Since the top $(0,1)$ of Δ is given by the expansion $.111111\ldots$, we can therefore find another dragonic expansion for x, ending in all ones, and with the same J first digits as y,

$$d_j^1(x) = d_j^0(y) \quad \text{for } j \leq J, \quad d_j^1(x) = 1 \quad \text{for } j > J .$$

We have seen how to obtain the two expansions for a dragonic rational x. We now want to show that if another dragonic rational y is "close" to x, then at least one of its expansions starts with the same digits as one of the expansions for x. Define

$$\rho = \max \{r : B((0,0); r) \subset \Delta \cup (\Delta - (0,1))\} ,$$

where $B(y;\lambda)$ is the open Euclidean ball centered at y with radius λ. Suppose x is a dragonic rational with $d_j^0(x) = 0$ for $j > J$. Take $m > J$, and consider the set

$$B_m = \{y \in \Delta : |y - x| \leq \rho\, 2^{-m/2}\} .$$

There are two possibilities: either x is on the border $\partial\Delta$ of Δ, or it isn't. If $x \in \partial\Delta$, then

$$R_{d_m^0(x)}^{-1} \cdots R_{d_1^0(x)}^{-1} (\Delta - (0,1))$$

has no common interior points with Δ, so that $B_m \subset \Delta_m(x)$, and the terminating expansions of all $y \in B_m$ have the same first m digits $d_j^0(x)$, $j = 1, \ldots, m$. If $x \notin \partial\Delta$, then $R^{-1}_{d^0_m(x)} \cdots R^{-1}_{d^0_1(x)} (\Delta - (0,1)) \subset \Delta$; this set is then a little dragon $\Delta_m(z)$ of which x is the top. In this case $B_m \subset \Delta_m(x) \cup \Delta_m(z)$, so that every point $y \in B_m$ has a dragonic expansion with either the same first m digits as $d^0(x)$ (if $y \in \Delta_m(x)$) or as $d^1(x)$ (if $y \in \Delta_m(z)$). This is the main ingredient needed to make the proof of Theorem 2.3, as sketched in the Appendix in [DL], work in the present case.

One other point that needs checking is whether the existence of two different dragonic expansions for x does not lead to inconsistencies for the definition of $v(x)$. If $d_j^0(x) = 0$ for $j > J$, then $d^0(x)$, $d^1(x)$ are linked by

$$x = \sum_{j=1}^{N} d_j^0(x)\, R^{-j}(1,0) = \sum_{j=1}^{N} d_j^1(x)\, R^{-j}(1,0) + R^{-N}(0,1)$$

for $N \geq J$ arbitrary. One can then compute $v(x)$ in two ways, using the two expansions. The following computation shows that they lead to the same result: for $k \in \Gamma$,

$$\begin{aligned}
&\left[v\left(\sum_{j=1}^{N} d_j^0(x) R^{-j}(1,0) \right) \right]_k = \left[T_{d_1^0(x)} \cdots T_{d_n^0(x)} v(0,0) \right]_k \\
&= \sum_{j_1 \cdots j_N} c_{Rk+d_1^0(x)(1,0)-j_1}\, c_{Rj_1+d_2^0(x)(1,0)-j_2} \cdots c_{Rj_N+d_N^0(x)(1,0)-j_N} \\
&\qquad \cdot [v(0,0)]_{j_N} \\
&= \sum_{m_1 \ldots m_N} c_{Rk+d_1^1(x)(1,0)-m_1} \cdots c_{Rm_{N-1}+d_n^1(x)(1,0)-m_N} \\
&\qquad \cdot [v(0,0)]_{m_N + \sum_{k=1}^{N} d_k^0(x) R^{N-k}(1,0) - \sum_{k=1}^{N} d_k^1(x) R^{N-k}(1,0)} \\
&= \sum_{m_1 \ldots m_N} c_{Rk+d_1^1(x)(1,0)-m_1} \cdots c_{Rm_{N-1}+d_n^1(x)(1,0)-m_N}\, [v(0,0)]_{m_N+(0,1)} \\
&= \left[T_{d_1^1(x)} \cdots T_{d_N^1(x)}\, v(0,1) \right]_k .
\end{aligned}$$

The reader can now check that the proof in [DL] indeed carries over to prove Theorem B.1. Similarly, one can prove differentiability of ϕ under stronger conditions on T_0, T_1, similar to Theorem 3.1

in [DL]. Finally, the same techniques can also be used for local regularity estimates, but these are a bit more tricky, and require further study of the properties of dragonic expansions. In practice, the matrices $T_0|_{E_1}$, $T_1|_{E_1}$ are often too large to permit a rigorous estimate of λ in (B.6). However, λ is bounded below by the quantities $\rho(T_{d_1}\cdots T_{d_m}|_{E_1})^{1/m}$, and this leads to upper bounds for the Hölder exponent α.

EXAMPLES.

1. $g(x) = \frac{1}{2}\, g(Rx+(1,0)) + g(Rx) + \frac{1}{2}\, g(Rx-(1,0))$
   The solution to this equation is the convolution $\chi_\Delta * \chi_\Delta$, where χ_Δ is the indicator function of the dragon set Δ (see also the second remark following Proposition 5.2). In this case Γ has 10 elements. The largest spectral radius of $T_d|_{E_1}$ is obtained for $d=0$, $\rho(T_0|_{E_1}) = .847810\ldots$, corresponding to a lower bound $\lambda \geq \rho(T_0|_{E_1})$ in (B.6), or a Hölder exponent $\alpha \leq .47637\ldots$ Via other methods (using the transition operator T of (5.19)) one also derives that this value is a lower bound. This global Hölder exponent is attained in dragonic rationals, in particular in $(0,0)$.
   Note that when M_0 is positive, as in this case, the transition operator T is already known to give optimal results. One easily checks that the matrix representing T is in fact a submatrix of T_0, so that it is not surprising that they have a common eigenvalue!
2. $\phi(x) = h_0\phi(Rx)+h_1\phi(Rx-(1,0))+h_2\phi(Rx-(-1,1))+h_3\phi(Rx-(0,1))$, with $h_0 = .506970418225$, $h_1 = -.207072424345$, $h_2 = .493029581775$, $h_3 = 1.20707242435$. This is an example from the family described at the very end of Section III.3.a. It leads to an orthonormal wavelet basis. In this case $|\Gamma| = 14$; the parameters have been chosen so that $\rho(T_0|_{E_1}) \simeq \rho(T_1|_{E_1}) \simeq .714$. Plots of approximations to ϕ seem to suggest that ϕ might be continuous, but we have no proof. If it is, then its Hölder exponent is bounded above by $\log[\rho(T_0T_1|_{E_1})^{1/2}]/\log\sqrt{2} \simeq \log(.90649)/\log\sqrt{2} \simeq .28327$.

Appendix C. Proof of the inequalities (5.52) for $G(\omega)$.

The function G is defined as

$$G(\omega) \;=\; \left[\cos^2\frac{\omega_1}{2} \;+\; \cos^2\frac{\omega_2}{2}\right]\left[\sin^2\frac{\omega_1}{2} \;+\; \sin^2\frac{\omega_2}{2}\right]$$

$$\cdot \left[\sin^2 \frac{\omega_1 + \omega_2}{2} + \sin^2 \frac{\omega_1 - \omega_2}{2} \right]^{-1} H(\omega) ,$$

with $$H(\omega) = h\left(\frac{1}{2}\left(\sin^2 \frac{\omega_1}{2} + \sin^2 \frac{\omega_2}{2}\right)\right) ,$$

and $$h(t) = \begin{cases} \dfrac{1}{1-t} & 0 \le t \le 1/2 , \\ 4t & 1/2 \le t \le 1 . \end{cases}$$

We want to prove inequalities for $G(\omega)$, $G(\omega)G(D\omega)$ and $G(\omega)G(D\omega)$ $G(D^2\omega)$, where $D(\omega_1, \omega_2)$ is either $(\omega_1 + \omega_2,\ \omega_1 - \omega_2)$ or $(\omega_1 - \omega_2,\ \omega_1 + \omega_2)$. (Since G is invariant for the interchange of ω_1, ω_2, it does not matter which definition of D is taken, $D = R$ or $D = S$.) To prove these inequalities it is convenient to use different variables,

$$s = s(\omega) = \frac{1}{2}\left(\sin^2 \frac{\omega_1}{2} + \sin^2 \frac{\omega_2}{2}\right) ,\ p = p(\omega) = \sin^2 \frac{\omega_1}{2} \sin^2 \frac{\omega_2}{2} .$$

We then have

$$G(\omega) = \frac{s(1-s)}{s-p} h(s) = 2\,\eta(s,p) .$$

Moreover,

$$s(D\omega) = 2(s-p), \qquad p(D\omega) = 4\,(s^2 - p) .$$

As ω ranges over $[-\pi,\ \pi]$, (s,p) fill out the domain Δ defined by

$$\Delta = \{(s,p) :\ 0 \le s \le 1,\ \max\{0, 2s-1\} \le p \le s^2\} .$$

In terms of these new variables, we therefore want to study $\eta(s,p)$, $\eta(s,p)\ \eta(\tilde{D}(s,p))$ and $\eta(s,p)\ \eta(\tilde{D}(s,p))\ \eta(\tilde{D}^2(s,p))$, for all $(s,p) \in \Delta$, where $\tilde{D}$ is defined by

$$\tilde{D}(s,p) = (\tilde{s}, \tilde{p}) = (2(s-p),\ 4(s^2 - p)) .$$

Note that $\tilde{D}$ maps Δ twice onto itself (both $\Delta \cap \{s \le 1/2\}$ and $\Delta \cap \{s \ge 1/2\}$ get mapped to all of Δ). Moreover $\tilde{D}$ has one fixed point, $(s_0, p_0) = (5/8,\ 5/16)$, corresponding to $\eta(s_0, p_0) = 15/16$.

We shall prove that $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where

$$\eta(s,p) \le \zeta \quad \text{on } \Delta_1 \,, \tag{C.1}$$

$$\eta(s,p)\; \eta(\tilde{D}(s,p)) \le \zeta^2 \quad \text{on } \Delta_2 \,, \tag{C.2}$$

$$\eta(s,p)\; \eta(\tilde{D}(s,p))\; \eta(\tilde{D}^2(s,p)) \le \zeta^3 \quad \text{on } \Delta_3 \,. \tag{C.3}$$

The value of ζ will be fixed by our estimates below; our goal is to obtain $\zeta < 1$.

Choose $\alpha = \sqrt{.9}$, and define the region Δ_1 by

$$\Delta_1 = \{(s,p) \in \Delta : p \le \left(1 - \frac{1}{2\alpha}\right) s \quad \text{if } s \le 1/2 \,, \qquad p \le s - \frac{2}{\alpha}\, s^2(1-s) \quad \text{if } s \ge 1/2 \Big\} \,.$$

Since

$$\eta(s,p) = \frac{s}{2(s-p)} \quad \text{if } s \le 1/2 \,, \qquad \frac{2s^2(1-s)}{s-p} \quad \text{if } s \ge 1/2 \,,$$

we automatically have

$$\eta(s,p) \le \alpha \text{ on } \Delta_1 \,. \tag{C.4}$$

By the definition of η and $\tilde{D}$, we have to distinguish four different regions when studying $\eta_2(s,p) = \eta(s,p)\eta(\tilde{D}(s,p))$:

$$\eta_2(s,p) = \begin{cases} \dfrac{s}{2(\tilde{s}-\tilde{p})} = \dfrac{s}{4(s-2s^2+p)} & \text{if } s \le 1/2, \quad p \le s - 1/4 \,, \\[2ex] \dfrac{2s\tilde{s}(1-\tilde{s})}{\tilde{s}-\tilde{p}} & \text{if } s \le 1/2, \quad p \ge s - 1/4 \,, \\[2ex] \dfrac{s^2(1-s)}{\tilde{s}-\tilde{p}} = \dfrac{s^2(1-s)}{s-2s^2+p} & \text{if } s \ge 1/2, \quad p \ge s - 1/4 \,, \\[2ex] \dfrac{4s^2(1-s)(1-\tilde{s})}{\tilde{s}-\tilde{p}} = \dfrac{8s^2(1-s)(s-p)(1-2(s-p))}{s+p-2s^2} & \\ & \text{if } s \ge 1/2, \quad p \le s - 1/4 \,. \end{cases}$$

We define Δ_2 by

$$\begin{aligned}\Delta_2 = &\left\{(s,p) \in \Delta:\ s \le \frac{1}{2}\,,\ p \ge \left(1-\frac{1}{2\alpha}\right)s\right\}\\ &\bigcup \left\{(s,p) \in \Delta:\ s \ge \frac{1}{2}\,,\ p \ge 1.8\,s - .81\right\}\\ = &\ \Delta_{2,1} \cup \Delta_{2,2}\ .\end{aligned}$$

Since $\tilde{s}(1-\tilde{s}) \le 1/4$ for all $\tilde{s} \in [0,1]$, we have

$$\eta_2(s,p) \le \frac{s}{2(\tilde{s}-\tilde{p})} = \frac{s}{4(s-2s^2+p)}$$

on all of $\Delta_{2,1}$. Since moreover $p \ge (1-1/2\alpha)\,s$, we have

$$\eta_2(s,p) \le \left[4\left(2-\frac{1}{2\alpha}-2s\right)\right]^{-1} \le \left[4\left(1-\frac{1}{2\alpha}\right)\right]^{-1} < \alpha^2$$

on $\Delta_{2,1}$.

On $\Delta_{2,2} \cap \{(s,p) \in \Delta:\ p \ge s - 1/4\}$, one easily checks that

$$\eta_2(s,p) = \frac{s^2(1-s)}{s-2s^2+p}$$

satisfies $\partial_p \eta_2 \ne 0$ everywhere. It follows that η_2 achieves its maximum on the boundary of this domain, given by the three pieces $p = s^2$, with $1/2 \le s \le .9$, $p = s - 1/4$ with $1/2 \le s \le .7$, and $p = 2\alpha s - \alpha^2$ with $.7 \le s \le .9$. One easily checks that the maximum value of η_2 on this boundary is $.9$.

Similarly one checks that η_2 achieves its maximum on $\Delta_{2,2} \cap \{(s,p) \in \Delta:\ p \le s - 1/4\}$ on the boundary of this set; again this leads to $\eta_2 \le .9$.

It follows that

$$\text{(C.5)} \qquad \eta_2(s,p) \le .9 = \alpha^2 \text{ on all of } \Delta_2\ .$$

It remains to determine an upper bound on $\eta_3(s,p) = \eta(s,p)\eta(\tilde{D}(s,p))\,\eta(\tilde{D}^2(s,p))$ on $\Delta \setminus (\Delta_1 \cup \Delta_2) = \{(s,p):\ 2s-1 \le p \le s^2,\ p \ge s - 2s^2)(1-s)/\alpha,\ p \le 1.8\,s - .81\}$ Since $s - 2s^2(1-s)/\alpha$ is strictly increasing, we have $\Delta \setminus (\Delta_1 \cup \Delta_2) \subset \Delta_3 = \{(s,p):\ 2s-1 \le p \le s^2,\ p_1 = 1.8\,s_1 - .81 \le p \le 1.8\,s - .81\}$, where s_1 is the solution to $s - 2s^2(1-s)/\alpha = 1.8s - .81$. In Δ_3 one has to distinguish 4 subdomains, corresponding to different expressions for η_3, namely $\Delta_{3,1} = \Delta_3 \cap \{p \geq p_1,\ p \geq 2s-1,\ p \leq 2(s-1/4)^2\}$, $\Delta_{3,2} = \Delta_3 \cap \{p \geq 2(s-1/4)^2,\ p \leq s-1/4\}$, $\Delta_{3,3} = \Delta_3 \cap \{p \geq s-1/4,\ p \geq 2(s-1/4)^2\}$ and $\Delta_{3,4} = \Delta_3 \cap \{p \geq s-1/4,\ p \leq s(s-1/4)^2\}$. On $\Delta_{3,1}$, $\Delta_{3,3}$ and $\Delta_{3,4}$ one checks explicitly that $\partial_p \eta_3 \neq 0$. On $\Delta_{3,2}$, the exact expression for η_3 is too complicated, but one can replace it by an upper bound,

$$\begin{aligned}\eta_3(s,p) &= \frac{2s^2(1-s)}{s-p}\ \frac{2\tilde{s}^2(1-\tilde{s})}{\tilde{s}-\tilde{p}}\ \frac{2\tilde{\tilde{s}}^2(1-\tilde{\tilde{s}})}{\tilde{\tilde{s}}-\tilde{p}} \\ &\leq \frac{2s^2(1-s)}{s-p}\ \frac{\tilde{s}}{\tilde{s}-\tilde{p}}\ \frac{2\tilde{\tilde{s}}^2(1-\tilde{\tilde{s}})}{\tilde{\tilde{s}}-\tilde{p}} \\ &= \frac{16\,s^2(1-s)\tilde{\tilde{s}}(1-\tilde{\tilde{s}})}{\tilde{\tilde{s}}-\tilde{p}} = \overline{\eta}_3(s,p)\ .\end{aligned}$$

This upper bound again satisfies $\partial_p \overline{\eta}_3 \neq 0$ on $\Delta_{3,2}$. It follows that η_3 on Δ_3 is bounded by the maximum of η_3 on the boundaries of $\Delta_{3,1}$, $\Delta_{3,3}$, $\Delta_{3,4}$ and of $\overline{\eta}_3$ on the boundary of $\Delta_{3,2}$. Explicitly, for all $(s,p) \in \Delta_3$,

$$\eta_3(s,p) \leq \overline{\eta}_3(s_1,p_1) = .88145650226\ldots \tag{C.6}$$

This numerical upper bound is larger than $(.9)^{3/2}$; it follows therefore from (C.4) and (C.5) that we have proved (C.1)-(C.2) for

$$\zeta = [\overline{\eta}_3(s_1,p_1)]^{1/3} = .958812370442\ldots$$

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