ON THE THERMODYNAMIC FORMALISM FOR MULTIFRACTAL FUNCTIONS

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The thermodynamic formalism for "multifractal" functions $\varphi(x)$ is a heuristic principle that states that the singularity spectrum $f(\alpha)$ (defined as the Hausdorff dimension of the set S_α of points where φ has Hölder exponent α) and the moment scaling exponent $\tau(q)$ (giving the power law behavior of $\int |\varphi(x+t) - \varphi(x)|^q \, dx$ for small $|t|$) should be related by the Legendre transform, $\tau(q) = 1 + \inf_{\alpha \geq 0} [q\alpha - f(\alpha)]$. The range of validity of this heuristic principle is unknown. Here this principle is rigorously verified for a family of "toy examples" that are solutions of refinement equations. These example functions exhibit oscillations on all scales, and correspond to multifractal signed measures rather than multifractal measures; moreover, their singularity spectra $f(\alpha)$ are not concave.

1. Introduction

"Multifractal" models were originally proposed to describe the intermittent behavior of fully-developed turbulence (Mandelbrot (1974), Frisch, Sulem and Nelkin (1978), Benzi, Paladin, Parisi and Vulpiani (1984), Frisch (1985), Frisch and Vergassola (1991)); see Frisch (1991) for a historic review. In recent years such models have been applied as well to describe chaotic features in dynamical systems (Eckmann and Ruelle (1985), Halsey *et al.* (1986)); see Amritkar and Gupte (1990) for an extensive review.

A basic descriptive quantity underlying the multifractal models is the "singularity spectrum". In Frisch and Parisi (1985) it is denoted $d(\alpha)$ and it represents the Hausdorff dimension of sets of points in which the velocity field is not Hölder continuous of order α. It is suggested that $d(\alpha)$ is a "universal" quantity of such turbulence. Frisch and Parisi propose a procedure to estimate this quantity from measurements of exponents ζ_p describing the asymptotic power law behavior of moments of velocity increments,

$$\langle |v(\boldsymbol{x}) - v(\boldsymbol{y})|^p \rangle \underset{|\boldsymbol{x}-\boldsymbol{y}|\to 0}{\sim} C|\boldsymbol{x} - \boldsymbol{y}|^{\zeta_p} . \tag{1.1}$$

They present a heuristic argument (see also p. 255 in Falconer (1990) or the introduction of Brown, Michon and Peyrière (1992)) showing that ζ_p is the Legendre transform of the "co-dimension" $3 - d(\alpha)$, i.e.

$$\zeta_p = \inf_{\alpha} \; [p\alpha + 3 - d(\alpha)] . \tag{1.2}$$

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If $d(\alpha)$ is concave (as it is often assumed to be), then it can be recovered from ζ_p by the inverse Legendre transform.

In Halsey *et al.* (1986) the singularity spectrum is denoted by $f(\alpha)$ and is associated with a measure $\mu(\boldsymbol{x})$ on the attractor of a dynamical system in $\mathbb{R}^n$. For each point $\boldsymbol{x}$ they define a *local singularity exponent* $\alpha(\boldsymbol{x})$ by

$$\alpha(\boldsymbol{x}) = \limsup_{\substack{\boldsymbol{x}\in I_\epsilon \\ \epsilon\to 0}} \frac{\log \mu(I_\epsilon)}{\log \epsilon} , \tag{1.3}$$

where I_ϵ denotes a cube of side ϵ. Then for $\alpha \geq 0$ the *singularity spectrum* $f(\alpha)$ of the measure $\mu(\boldsymbol{x})$ is the Hausdorff dimension of the set

$$S_\alpha = \{x \in \mathbb{R}^n;\ \alpha(x) = \alpha\} . \tag{1.4}$$

Again $f(\alpha)$ is proposed as a new universal invariant, now for strange attractors and routes to chaos. It is also related to an associated generalized dimension D_q, proposed by Hentschel and Procaccia (1983), and essentially based on the information measure of Renyi (1970). One way of defining the *generalized dimension* D_q is

$$D_q = (q-1)\ \tau(q) , \tag{1.5}$$

where

$$\tau(q) := \lim_{\epsilon\to 0} \frac{\log\left[\int_{\mathbb{R}^n} \mu(B_\epsilon(\boldsymbol{x}))^{q-1} d\mu(\boldsymbol{x})\right]}{\log \epsilon} , \tag{1.6}$$

in which $B_\epsilon(\boldsymbol{x})$ is the ball of radius ϵ centered at $\boldsymbol{x}$. (In fact D_q is defined differently in Halsey *et al.* and (1.6), (1.7) are given as derivations.) $\tau(q)$ is automatically a concave function of q. Halsey *et al.* (1986) proposed estimating $f(\alpha)$ using the Legendre transform of $\tau(q)$. The similarity between Frisch and Parisi (1985) and Halsey *et al.* (1986) is clear. In particular, if one works in dimension $n = 1$, then $B_\epsilon(\boldsymbol{x}) = [x-\epsilon,\ x+\epsilon]$, and $\tau(q)$ can be rewritten as

$$\tau(q) := \lim_{\epsilon\to 0} \frac{\log\left[\int (\varphi(x+\epsilon) - \varphi(x-\epsilon))^{q-1} d\mu(x)\right]}{\log \epsilon} , \tag{1.7}$$

where

$$\varphi(x) = \int_{-\infty}^{x} d\mu(y) . \tag{1.8}$$

(We implicitly assume that μ has no discrete part, meaning that $\mu(\{z\}) = 0$ for any singleton $\{z\}$.) The function $\varphi(x)$ plays the role of the velocity function $v(\boldsymbol{x})$ in (1.1), and (1.7) shows that $\tau(q)$ is analogous to ζ_p.

The Halsey *et al.* (1986) framework does not include the Frisch and Parisi (1985) framework: the one-dimensional analog of (1.1) would involve the power law scaling of a function (the velocity field) that is not nondecreasing, unlike the cumulative distribution function of a measure such as φ in (1.8). To include such functions one must allow *signed* measures. Our formulas (1.11)–(1.13) below give extensions of the concepts above to (integrable) signed measures. In both frameworks the singularity spectrum is viewed as the fundamental quantity, and the "moment scaling exponent" ζ_p or $\tau(q)$ is considered merely as an auxiliary quantity, possibly useful in the estimation of the singularity spectrum. However, $\tau(q)$ can also be interpreted as a generalization of the entropy, and can thus be viewed as an ergodic-theoretic invariant in the special case of a dynamical system ergodic with respect to an invariant measure, cf. Eckmann and Procaccia (1986).

The Frisch and Parisi (1985) procedure of determining a moment scaling exponent $\tau(q)$ from experimental data and then taking its inverse Legendre transform is now often called (by some authors) the "moment method". It was applied in Jensen, Kadaroff and Libchaber (1985), who briefly argue that it should be more stable than a direct evaluation of $\alpha(\boldsymbol{x})$ and $f(\alpha)$ because $\tau(q)$ smooths the data. Since then the moment method has been used in many studies as a tool to estimate $f(\alpha)$; see e.g. Amritkar and Gupte (1990). Note that there is another general approach to estimate the singularity spectrum from experimental data, without recourse to moments; this method uses a direct double histogram approach described in Meneveau and Sreenivasan (1989) and long advocated by Mandelbrot (see Mandelbrot (1993)).

Yet another way of linking $f(\alpha)$ and $\tau(q)$ is the "thermodynamic formalism", which goes back to Bohr and Rand (1987). In this approach $\tau(q)$ is analogous to free energy, and its Legendre transform $S(\alpha)$ is analogous to entropy; moreover, $f(\alpha)$ and $S(\alpha)$ can be derived from each other. This derivation implicitly assumes that $f(\alpha)$ is concave; as observed in Bohr and Jensen (1987), $f(\alpha)$ need not be concave in general, and the equation relating $S(\alpha)$ and $f(\alpha)$ becomes an inequality when f is not concave (see also below). For some multifractal models, the function $\tau(q)$ exhibits different regimes, separated by "critical values" of q; the thermodynamic formalism interprets these as analogues to phase transitions (see Csordás and Szépfalusy (1989)). See Tel (1988) for a review of multifractals and thermodynamics.

In one form or other, most of the approaches summarized above seem to support the following heuristic principle:

Thermodynamic Formalism for Multifractals. *The moment scaling exponent $\tau(q)$ is equal to the Legendre transform*

$$\tau(q) = \inf_{\alpha \geq 0} \; [\alpha q + n - f(\alpha)] \,, \tag{1.9}$$

where $f(\alpha)$ is the singularity spectrum, and n is the dimension of the model or system under study.

In most papers, smoothness of $f(\alpha)$ is presupposed, in which case (1.9) simplifies to the form

$$\tau(q) = \alpha(q)q + n - f(\alpha(q))$$

where $\alpha(q)$ is the value of α such that

$$f'(\alpha) = q .$$

The more general form (1.9) is needed for discontinuous $f(\alpha)$.

The usefulness of this heuristic principle as applied to data has generally been to recover $f(\alpha)$ when $\tau(q)$ was computed or given. This is possible only if $f(\alpha)$ is concave, in which case if (1.9) holds, we have

$$f(\alpha) = S(\alpha) := n - \sup_q [\tau(q) - q\alpha] . \tag{1.10}$$

If f is not concave, then (1.9) only implies $f^*(\alpha) = S(\alpha)$, where $f^*(\alpha) \geq f(\alpha)$ is the concave hull of $f(\alpha)$.

This paper is concerned with rigorous results. Most of the papers mentioned above are not mathematically rigorous, and the exact range of validity of the heuristic principle above is not known. In the form in which we have stated it, no counterexamples are known; in the stronger form in which it is often used, asserting (1.10) rather than (1.9), every situation with a nonconcave $f(\alpha)$ is of course a counterexample. (One such counterexample is Example 1 in Brown, Michon and Peyrière (1992).) In the framework closest to Halsey *et al.* (1986) the thermodynamic heuristic principle has been justified rigorously in a number of situations, including Gibbs measures of hyperbolic attractors (Grassberger, Badii and Politi (1988)), cookie-cutter Cantor measures (Bedford (1988, 1991), Rand (1989), Falconer (1990)), certain probability measures (Brown, Michon and Peyrière (1992), Peyrière (1993)), Moran fractals (Cawley and Mauldin (1992)), digraph recursive fractals (Edgar and Mauldin (1992)), various routes to chaos including Feigenbaum period-doubling (Collet, Lebowitz and Porzio (1987)) and self-affine fractals (Schmeling and Siegmund-Schultze (1993)). As far as we know the existing rigorous results for the thermodynamic heuristic principle only apply to multifractal measures μ. The only papers we know of that discuss related questions for possibly oscillating functions (which can be viewed as associated with signed measures), are Jaffard (1992), which gives an inequality for the Hausdorff dimension of the set where a $W^{s,p}(\mathbb{R}^n)$ function is not in C^α (with $\alpha < s$), and Eyink (1993), which pushes Jaffard's arguments further to prove that $\tau(q)$ is bounded above by the right-hand side of (1.9) if φ lies in some appropriate Besov spaces.

The object of this paper is to present a family of one-dimensional examples coming from intrinsically signed measures for which the thermodynamic formalism can be verified rigorously. These examples are interesting for several reasons. First, the singularity spectrum of signed measures is intrinsically more difficult to analyze, and it is useful to have a verified example closer to the Frisch and Parisi (1986) framework. Even though our constructions do not derive from a physical example, they can still be viewed as a caricature of a velocity field, in that they exhibit oscillations at all scales. Second, our examples have a non-concave singularity spectrum and exhibit a "phase transition". Third, the sets S_α in our examples are all dense (for an appropriate range of α), implying that their box counting dimension is 1, independently of α. This shows that the use of the Hausdorff dimension in (1.9) is crucial: it cannot be replaced by the box counting dimension. It also shows that our examples fall outside the framework considered in Corollary 1 in Eyink (1993), where (1.9) is proved for those functions for which the Hausdorff dimensions and the box counting dimensions of the S_α coincide.

We now introduce some precise definitions adapted to our point of view. For $\varphi \in L^1(\mathbb{R})$ we define its singularity spectrum $f_\varphi(\alpha)$ and its moment scaling exponent $\tau_\varphi(q)$ as follows. The *singularity spectrum* $f(\alpha)$ for $\alpha \geq 0$ measures the Hausdorff dimension of the set of points $S_\alpha(\varphi)$ where φ has lower Hölder exponent exactly α. More formally, we set

$$\alpha(x) := \liminf_{|t|\to 0} \frac{\log|\varphi(x+t)-\varphi(t)|}{\log|t|} , \tag{1.11}$$

and let $S_\alpha = \{x;\ \alpha(x)=\alpha\}$, $f(\alpha) = \dim_{\text{Hausd.}}(S_\alpha)$. We shall also say that $\varphi \in C^\alpha(x)$ ("φ is *lower Hölder continuous in x with Hölder exponent α*") if $\alpha(x)=\alpha$. For the moment scaling exponent, consider the *Renyi information measure of order* q,

$$I(t,q) = \int |\varphi(x+t)-\varphi(x)|^q \, dx . \tag{1.12}$$

Now $\tau(q)$ should measure the scaling exponent in the asymptotic behavior of $I(t,q) \sim c_q|t|^{\tau(q)}$ as $|t| \to 0$, so we formally define the *scaling exponent* $\tau(q)$ by

$$\tau(q) := \liminf_{|t|\to 0} \frac{\log|I(t,q)|}{\log|t|} , \tag{1.13}$$

i.e. $\tau(q)$ is the lower Hölder exponent of $I(t,q)$ in $t=0$. Note that the domain of $\tau(q)$ may have to be restricted to values q where $I(t,q)$ is defined. The same problem already presents itself for φ which are cumulative distribution functions of a measure μ; in that case one can however (and usually does) restrict the domain of integration in

$$I(t,q) = \int [\varphi(x+t)-\varphi(x)]^q \, dx = \int \mu(\,]x,x+t])^q \, dx$$

to those x for which $\mu(\,]x,x+t]) \neq 0$ for all $t>0$; this leads then to a meaningful definition of $\tau(q)$ for all q, positive and negative. For non monotone φ things are not that simple, as shown by the very innocuous function $\varphi(x) = x(1-x)$ on $[0,1]$ for which $I(t,q)$ diverges for $q \leq -1$. In this simple case it would be easy to excise the offending point $x=\frac{1}{2}$, but for more complicated functions, which present oscillations at every scale, this is not feasible.

We shall prove that

$$\tau(q) = \inf_\alpha \ [\alpha q + 1 - f(\alpha)] \tag{1.14}$$

holds for a family of "toy examples" $\varphi(x)$ which are solutions of particular refinement equations (also called two-scale difference equations or dilation equations), i.e.

$$\varphi(x) = \sum_{j=0}^{L} c_j \ \varphi(2x - j), \quad \text{with} \quad \sum_{j=0}^{L} c_j = 2 \,. \tag{1.15}$$

Such functions arise in subdivision schemes in computer aided design (see e.g. Cavaretta, Dahmen and Micchelli (1991) for a review and many references) and in the construction of orthonormal wavelets (see Daubechies (1988, 1992)). Under special conditions equations of type (1.15) have a unique L^1-solution (up to a multiplicative constant) which necessarily has compact support. They are always only finitely many times differentiable (Micchelli and Prautzsch (1989)) and exhibit a complicated "multifractal" set of Hölder exponents in general (Daubechies and Lagarias (1992)). In particular, the regularity of φ is governed by the spectral properties of the matrices $T(n;d) = T_{d_1} \cdots T_{d_n}$, where $d_j = 0$ or 1 and T_0, T_1 are two matrices constructed from the c_j, see Daubechies and Lagarias (1992) and Micchelli and Prautzsch (1989). The Hölder exponent $\alpha(x)$ of f in x depends on the relative frequency of the digits 0 and 1 in the binary expansion of x. The Hausdorff dimension of the sets containing all the x in $[0,1]$ with a fixed preassigned density of the digit 1 in their binary expansion is well-known (Besicovitch (1934) and Eggleston (1949)), so that we can determine $f(\alpha)$ explicitly. On the other hand, detailed knowledge of T_0, T_1 and their products can be used to compute $\tau(q)$. This enables us to prove formula (1.14) directly. The method to estimate local exponents is an extension of that given in Daubechies and Lagarias (1992); it requires however more detailed estimates, and, in particular, the determination of optimal pointwise Hölder exponents. The method of calculation of $f(\alpha)$ and $\tau(q)$ given here can be done in principle for any refinable function, but since we had to make explicit and very detailed estimates, we only carried it out on specific examples.

The simplest family of examples is given by $c_0 = \gamma$, $c_1 = 1$, $c_2 = 1 - \gamma$, with $\frac{1}{2} < \gamma < 1$, and all other $c_j = 0$. The resulting function is continuous and supported on $[0,2]$, and is monotonically increasing on $[0,1]$ and monotonically decreasing on $[1,2]$. Because of the monotonicity properties of φ, these examples essentially correspond to positive measures, and formula (1.14) can already be deduced from theorems proved in Falconer (1990). In this example we have in fact, for $0 \le x \le 1$,

$$\varphi(x+1) = 1 - \varphi(x), \quad \text{and} \quad \varphi(x) = \alpha \sum_{k=1}^{\infty} d_k(x) \prod_{j=1}^{k-1} \gamma_{d_j(x)} \ ,$$

where $d_k(x)$ is the k-th digit in the binary expansion of x, and $\gamma_0 = \alpha$, $\gamma_1 = 1 - \alpha$. Then one can check that, for $x \in [0,1]$, $\varphi(x) = \mu_\alpha([0,x])$, where μ_α is the singular probability measure on $[0,1]$ obtained as the limit of a Cantor-like splitting process in which intervals are halved, and the first half is given weight α, the second weight $1 - \alpha$. This is in fact one of the first measures for which (1.9) was established.

This paper studies in detail a more complex family of examples, which is given by the choice $c_0 = \beta$, $c_1 = \beta + \frac{1}{2}$, $c_2 = 1 - \beta$, $c_3 = \frac{1}{2} - \beta$, with $\frac{1}{2} < \beta < 1$, and all other $c_j = 0$. (For technical reasons, we shall eventually restrict β to the range

$\frac{1}{2} < \beta < \frac{3}{4}$.) That is, the corresponding function φ is a solution of

$$\begin{aligned}\varphi(x) = \beta\varphi(2x) + \left(\beta + \frac{1}{2}\right)\varphi(2x-1) + (1-\beta)\varphi(2x-2) \\ + \left(\frac{1}{2} - \beta\right)\varphi(2x-3)\ .\end{aligned} \tag{1.16}$$

For $\frac{1}{2} < \beta < 1$, the function φ is continuous and supported on $[0,3]$, but it is non monotone on any interval $[a,b]$ contained in $[0,3]$ (with $a < b$). This example does therefore not fit into any of the categories for which the thermodynamic heuristic principle was previously proved rigorously to hold. We shall show that (1.14) does indeed hold for $q \geq 0$. We do not consider $q \leq 0$, because for most negative values of q, the expression (1.12) for $I(t,q)$ is not defined.

For $\frac{1}{2} < \beta < \frac{3}{4}$, the range of β on which we shall concentrate, the singularity spectrum $f(\alpha)$ for our functions φ will be discontinuous and non-concave; more precisely, f is nicely concave and increasing for $\alpha < 1$, but $\lim_{\alpha \gtrsim 1} f(\alpha) < f(1) = 1$. Thus $f(\alpha)$ exhibits a "phase transition" and this provides therefore yet another example where (1.10) is not true; the inverse Legendre transform of $\tau(q)$ can only give the convex hull of the graph of $f(\alpha)$.

On the basis of these examples, it seems reasonable to conjecture that the thermodynamic formalism heuristic is true for all continuous functions $\varphi(x)$ arising as the solution of a refinement equation of type (1.15), and possibly also of larger classes of refinement equations.

We call our examples "toy examples" because they do not stem from a physically motivated dynamical system. Nevertheless, because they are nontrivial and at the same time sufficiently simple to be understood in great detail, they may be useful as a laboratory to test out further developments such as the impact of noise in the "data" $\varphi(x)$ on the computation of $\tau(q)$ and of (the convex hull of) $f(\alpha)$, or proposals to replace $I(t,q)$ by related but different "averages" (e.g. using wavelets) so as to define $\tau(q)$ also for $q < 0$.

This paper is organized as follows. In Sec. 2 we recall the definitions of the matrices T_0, T_1 corresponding to (1.16), and we quickly review those of their properties that we will need. We also derive detailed bounds on φ that will be needed further. In Sec. 3 we show how to compute the Hölder exponent of φ in any x (including points for which the "density of the digit 1" is not well defined). In Sec. 4 we use the results of Sec. 3 to compute $f(\alpha)$ and its Legendre transform. Then, in Sec. 5, we compute $\tau(q)$ for our examples, and we show that (1.14) holds, at least for $q \geq 0$.

2. Using the Refinement Equation to Derive Properties and Bounds for φ

We want to study the function φ defined by

$$\varphi(x) = \beta\varphi(2x) + \left(\beta + \frac{1}{2}\right)\varphi(2x-1) + (1-\beta)\varphi(2x-2) + \left(\frac{1}{2} - \beta\right)\varphi(2x-3)\ . \tag{2.1}$$

This function is uniquely determined by the additional requirement that $\varphi \in L^1(\mathbb{R})$ and by the normalization $\int \varphi(x)\,dx = 1$. It is real and compactly supported; its support is $[0,3]$. Moreover, φ is continuous. (All this, and more, immediately follows from the general discussion of refinement equations in e.g. Daubechies and Lagarias (1992), Cavaretta, Dahmen and Micchelli (1991) or Daubechies (1992).) In order to discuss the regularity properties of φ, it is useful to rewrite (2.1) in a vector notation. For $x \in [0,1]$ we define $v(x) = (\varphi(x),\ \varphi(x+1),\ \varphi(x+2)) \in \mathbb{R}^3$; (2.1) implies then

$$v(x) = T_0\ v(2x) \quad \text{if } x \in \left[0, \frac{1}{2}\right] \tag{2.2}$$

$$= T_1\ v(2x-1) \quad \text{if } x \in \left[\frac{1}{2}, 1\right] , \tag{2.3}$$

where

$$T_0 = \begin{pmatrix} \beta & 0 & 0 \\ 1-\beta & \frac{1}{2}+\beta & \beta \\ 0 & \frac{1}{2}-\beta & 1-\beta \end{pmatrix}, \quad T_1 = \begin{pmatrix} \frac{1}{2}+\beta & \beta & 0 \\ \frac{1}{2}-\beta & 1-\beta & \frac{1}{2}+\beta \\ 0 & 0 & \frac{1}{2}-\beta \end{pmatrix} .$$

Using $v(0) = (0, 2\beta, 1-2\beta)$, $v(1) = (2\beta, 1-2\beta, 0)$, one easily checks that there is no inconsistency at $x = \frac{1}{2}$. If we denote by $d_k(x) = 0$ or 1 the k-th digit in the binary expansion of x,

$$x = \sum_{k=1}^{\infty} d_k(x) 2^{-k} ,$$

and by σ the shift operator,

$$d_k(\sigma x) = d_{k+1}(x) , \quad \text{or} \quad \sigma x = \sum_{k=2}^{\infty} d_k(x) 2^{-k+1} ,$$

then the equations (2.2), (2.3) can be rewritten as

$$v(x) = T_{d_1(x)} v(\sigma x) , \tag{2.4}$$

which is now valid for all $x \in [0,1]$. More generally,

$$v(x) = T(n; d(x))\ v(\sigma^n x) , \tag{2.5}$$

where $T(n; d(x)) = T_{d_1(x)} \cdots T_{d_n(x)}$. (For dyadic rationals x, $x = 2^{-N}K$ with $K \in 2\mathbb{N}+1$, there exist two binary expansions, one in which $d_N(x) = 1$ and $d_n(x) = 0$ for $n > N$, and another in which $d_N(x) = 0$ and $d_n(x) = 1$ for $n > N$. We shall denote the first by $d^+(x)$, the second by $d^-(x)$, and call them the expansions "from above" and "from below", respectively. The two right-hand sides of (2.5) corresponding to the two choices for $d(x)$ give identical results; see Daubechies and Lagarias (1992).)

It follows from (2.5) that if $x+t$ and x have the same first n digits in their binary expansions (implying $|t| \le 2^{-n}$), then

$$[v(x+t)-v(x)] = T_{d_1(x)} \cdots T_{d_n(x)}[v(\sigma^n x + 2^n t) - v(\sigma^n x)] \ .$$

This formula shows that bounds on $T(n;d(x))$ are the clue to bounds on $v(x+t) - v(x)$, hence to regularity properties of φ in x.

Note that $e_1 = (1,1,1)$ satisfies $e_1 T_0 = e_1 = e_1 T_1$, implying that

$$e_1 T(n;d) = e_1 \ . \tag{2.6}$$

This gives us already one eigenvector for $T(n;d)$, with eigenvalue 1. We also have that $e_2 = (2-2\beta, 1-2\beta, -2\beta)$ satisfies $e_2 T_0 = \frac{1}{2} e_2$, $e_2 T_1 = \frac{1}{2} e_2 + \frac{1}{2} e_1$, so that

$$e_2 T(n;d) = 2^{-n} e_2 + \left(\sum_{k=1}^{n} 2^{-k} d_k \right) e_1 \ . \tag{2.7}$$

It follows that the second eigenvalue of $T(n;d)$ is 2^{-n}. Finally, $e_3 = (1,0,0)$ satisfies $e_3 T_0 = \beta e_3$ and $e_3 T_1 = \left(\frac{1}{2} - \beta\right) e_3 + \beta e_2 + 2\beta^2 e_1$, implying

$$e_3 T(n;d) = \mu(n;d) e_3 + \lambda(n;d) e_2 + \gamma(n;d) e_1 \tag{2.8}$$

where $\mu(n;d) = \prod_{j=1}^{n} \mu_{d_j}$ and $\mu_0 = \beta$, $\mu_1 = \frac{1}{2} - \beta$ are the respective third eigenvalues of T_0, T_1, and where

$$\begin{aligned}
\lambda(n;d) &= \beta \sum_{k=1}^{n} d_k \ 2^{-n+k} \mu(k-1;d) \\
\gamma(n;d) &= \sum_{k=1}^{n} d_k \left[\frac{1}{2} \lambda(k-1;d) + 2\beta^2 \mu(k-1;d) \right] \\
&= \beta \sum_{k=1}^{n} d_k \ \mu(k-1;d) \left[2\beta + 2^k \sum_{\ell=k+1}^{n} 2^{-\ell} d_\ell \right] \ .
\end{aligned}$$

Here we use the standard conventions that a product $p_{j_1,j_2} = \prod_{j=j_1}^{j_2} \varphi_j$ equals 1 if $j_2 < j_1$, whereas a sum $s_{j_1,j_2} = \sum_{j=j_1}^{j_2} \varphi_j$ equals 0 if $j_2 < j_1$, meaning that $\mu(0,d) = 1$, $\lambda(0;d) = 0$ and $\sum_{\ell=k+1}^{n} 2^{-\ell} d_\ell = 0$ if $k = n$. It follows from (2.8) that the third eigenvalue of $T(n;d)$ is $\mu(n;d)$. In the range $\frac{1}{2} < \beta < 1$ of interest to us, it is easy to find many values of β for which $\mu(n;d) = 2^{-n}$ for appropriately chosen n and d; we will therefore not be able to assume that $T(n;d)$ is diagonalizable in general. This accounts for some of the technicalities in the detailed estimates below.

In Daubechies and Lagarias (1992) it is shown how equations of type (2.6), (2.7) and (2.8) imply the following facts on $\varphi(x)$ and $v(x)$ which we summarize here without detailed proof:

Proposition 2.1. *Assume that φ is the L^1 solution of (2.1) that is normalized so that $\int \varphi(x)\,dx = 1$. Then φ has support $[0,3]$, and it is a continuous real function. For $x \in [0,1]$, $v(x) = (\varphi(x), \varphi(x+1), \varphi(x+2))$ satisfies*

$$\text{(i)} \qquad e_1 \cdot v(x) = 1 \tag{2.9}$$

$$\text{(ii)} \qquad e_2 \cdot v(x) = x \tag{2.10}$$

$$\text{(iii)} \qquad \textit{there exists } C > 0 \textit{ so that } \|v(x) - v(y)\| \le C \mid x - y \mid^{|\log_2 \beta|} . \tag{2.11}$$

Proof. Parts (i) and (ii) are easy consequences of (2.6) and (2.7) and the continuity of φ. (In Daubechies and Lagarias (1992) continuity is not assumed a priori, but proved as well.) Part (iii) is a consequence of $e_1 \cdot [v(x) - v(y)] = 0$, and $\mid e_2 \cdot [v(x) - v(y)] \mid, \mid e_3 \cdot [v(x) - v(y)] \mid \le C\beta^n$ if $\mid x - y \mid < 2^{-n}$. See Daubechies and Lagarias (1992) or Micchelli and Prautzsch (1989) for more details. □

We can write an explicit formula for $\varphi(x)$. From (2.9), (2.10) it already follows that for $x \in [0,1]$

$$\varphi(x+1) = -2\varphi(x) + x + 2\beta \tag{2.12}$$

$$\varphi(x+2) = \varphi(x) - x + 1 - 2\beta . \tag{2.13}$$

Moreover

$$\varphi(x) = e_3 \cdot v(x) = \lim_{n\to\infty} \quad e_3 \cdot T(n; d(x)) v(0) \ ;$$

since $\mid \mu(n;d) \mid \le \beta^n$ and $\mid \lambda(n;d) \mid \le \beta 2^{-n} \sum_{k=1}^{n} (2\beta)^k \le \frac{\beta^{n+1}}{2\beta - 1}$ both tend to zero for $n \to \infty$, we have therefore

$$\begin{aligned} \varphi(x) &= \lim_{n\to\infty} \gamma(n; d(x)) \\ &= \beta \sum_{k=1}^{\infty} d_k \mu(k-1; d(x)) \, [2\beta + \sigma^k x] \ . \end{aligned} \tag{2.14}$$

If $x + t$ and x have the same first ℓ binary digits, then we have also

$$\begin{aligned} \varphi(x+t) - \varphi(x) &= e_3 \cdot T(\ell; d(x)) [v(\sigma^\ell (x+t)) - v(\sigma^\ell x)] \\ &= \lambda(\ell; d(x)) [\sigma^\ell (x+t) - \sigma^\ell x] + \mu(\ell; d(x)) [\varphi(\sigma^\ell (x+t)) - \varphi(\sigma^\ell x)] \\ &= \beta t \sum_{k=1}^{\ell} d_k(x) 2^k \mu(k-1; x) + \mu(\ell; x) [\varphi(\sigma^\ell (x+t)) - \varphi(\sigma^\ell x)] \ , \end{aligned} \tag{2.15}$$

where we have introduced the shorthand $\mu(\ell; x)$ for $\mu(\ell; d(x))$.

The two formulas (2.14) and (2.15) are our main tools in deriving detailed estimates on φ that will be used in Secs. 3 and 5. Let us first compute some special values for $\varphi(x)$,

$$\varphi(1) = 2\beta, \qquad \varphi\left(\frac{1}{2}\right) = 2\beta^2 \ ;$$

more generally,

$$\varphi(2^{-n}) = 2\beta^{n+1} \qquad n \in \mathbb{N}. \tag{2.16}$$

By (2.12), we also have, for $0 \le x \le \frac{1}{2}$,

$$\varphi\left(x + \frac{1}{2}\right) = \beta\varphi(2x+1) + \left(\frac{1}{2} + \beta\right)\varphi(2x)$$

$$= \left(\frac{1}{2} - \beta\right)\varphi(2x) + 2\beta(x+\beta) \tag{2.17}$$

$$= \frac{\frac{1}{2} - \beta}{\beta}\varphi(x) + 2\beta(x+\beta)\ . \tag{2.18}$$

In particular, $\varphi\left(\frac{3}{4}\right) = \beta\left(\frac{1}{2} + 3\beta - 2\beta^2\right)$.

We next prove upper and lower bounds on $\varphi(x)$, in a series of lemmas that establish successively tighter bounds. We start with

Lemma 2.2. *For all* $x \in \left[\frac{1}{2}, 1\right]$,

$$0 < \beta\left(2\beta - 2\beta^2 + \frac{1}{2}\right) \le \varphi(x) \le (2\beta+1)\beta\ .$$

Proof. (i) Define $A = \frac{1}{\beta} \max\limits_{\frac{1}{2} \le x \le 1} \varphi(x)$, $B = \frac{1}{\beta} \min\limits_{\frac{1}{2} \le x \le 1} \varphi(x)$. Because $\varphi(z) = \beta\varphi(2z)$ if $0 \le z \le \frac{1}{2}$, we then also have $A = \frac{1}{\beta} \max\limits_{0 \le x \le 1} \varphi(x)$, $\min(0, B) = \frac{1}{\beta} \min\limits_{0 \le x \le 1} \varphi(x)$.

(ii) Take $x = \frac{1}{2}(1+y) \in \left[\frac{1}{2}, 1\right]$, i.e. $y \in [0,1]$. Then (2.17) gives

$$\varphi(x) = \left(\frac{1}{2} - \beta\right)\varphi(y) + \beta(2\beta + y)\ .$$

Consequently

$$A \le 2\beta + 1 - \left(\beta - \frac{1}{2}\right)\min(0, B) \tag{2.19}$$

$$B \ge 2\beta - \left(\beta - \frac{1}{2}\right)A. \tag{2.20}$$

(iii) Suppose that $B \le 0$. Then it would follow that

$$B \ge 2\beta - \left(\beta - \frac{1}{2}\right)\left[2\beta + 1 - \left(\beta - \frac{1}{2}\right)B\right]$$

or

$$\left[1 - \left(\beta - \frac{1}{2}\right)^2\right]B \ge 2\beta - 2\beta^2 + \frac{1}{2},$$

which is impossible since $2\beta - 2\beta^2 + \frac{1}{2}$ and $1 - \left(\beta - \frac{1}{2}\right)^2$ are both strictly positive for $\frac{1}{2} < \beta < 1$. Consequently $B > 0$, and (2.19) then implies $A \le 2\beta + 1$. Substituting this into (2.20) gives $B \ge 2\beta - 2\beta^2 + \frac{1}{2}$. □

This proof has implicitly proved the following corollary which we shall use in its own right:

Corollary 2.3. *For all* $x \in [0,1]$, $0 \le \varphi(x) \le (2\beta+1)\beta$.

Next, we use the upper bound from Lemma 2.2 in a bootstrapping argument to derive a tighter upper bound.

Lemma 2.4. *For all* $x \in [0,1]$,

$$\varphi(x) \le 2\beta \ . \tag{2.21}$$

Proof. (i) It is sufficient to prove (2.20) for $\frac{1}{2} \le x \le 1$, since $\varphi\left(\frac{x}{2}\right) = \beta\varphi(x)$.
(ii) Take $x = \frac{1}{2} + \frac{1}{4}y \in \left[\frac{1}{2}, \frac{3}{4}\right]$, i.e. $y \in [0,1]$. Then (2.17) implies

$$\begin{aligned}\varphi(x) &= \beta\left(\frac{1}{2}-\beta\right)\varphi(y) + \beta\left(2\beta + \frac{1}{2}y\right)\\ &\le \beta^2\left(\frac{1}{2}-\beta\right)(2\beta+1) + \beta\left(2\beta+\frac{1}{2}\right) \ ,\end{aligned} \tag{2.22}$$

where we have used the upper bound in Lemma 2.2. But (2.22) $=$ $\beta\left[-2\beta^3 + \frac{5}{2}\beta + \frac{1}{2}\right]$ and $-2\beta^3 + \frac{5}{2}\beta + \frac{1}{2} < 2$ for $\beta \in \left[\frac{1}{2}, 1\right]$. So (2.21) follows if $x \in \left[\frac{1}{2}, \frac{3}{4}\right]$.
(iii) Since $\varphi(1) = 2\beta$, we have therefore

$$\max_{x\in[0,1]} \varphi(x) = \max_{x\in[\frac{3}{4},1]} \varphi(x) = A \, .$$

For $x = \frac{3}{4} + \frac{1}{4}y \in [\frac{3}{4}, 1]$, i.e. $y \in [0,1]$, (2.14) implies

$$\begin{aligned}\varphi(x) &= \left(\frac{1}{2}-\beta\right)^2 \varphi(y) + \beta\left(2\beta + \frac{1}{2} + \frac{1}{2}y\right) + \beta\left(\frac{1}{2}-\beta\right)(2\beta+y)\\ &\le \left(\frac{1}{2}-\beta\right)^2 A + \beta(2\beta+1) + \beta\left(\frac{1}{2}-\beta\right)(2\beta+1) \ .\end{aligned}$$

Consequently

$$A\left[1 - \left(\frac{1}{2}-\beta\right)^2\right] \le (2\beta+1)\beta\left(\frac{3}{2}-\beta\right) \ ,$$

which gives $A \le 2\beta$. Consequently $\varphi(x) \le 2\beta$ for all $x \in [0,1]$. □

This upper bound implies a lower bound:

Lemma 2.5. *For all* $x \in [0,1]$, $\varphi(x) \ge 2\beta x$.

Proof. (i) We first prove this for $x \in [\frac{1}{2}, 1]$. Take $x = \frac{1}{2} + \frac{1}{2}z$, $z \in [0,1]$. Then

$$\begin{aligned}\varphi(x) - 2\beta x &= \left(\frac{1}{2} - \beta\right)\varphi(z) + \beta(2\beta + z) - \beta(1+z) \\ &= \left(\beta - \frac{1}{2}\right)[2\beta - \varphi(z)] \geq 0 .\end{aligned}$$

(ii) Take now $x \in [0, \frac{1}{2}]$. There exists L so that $2^{-L-1} \leq x \leq 2^{-L}$. Then

$$\begin{aligned}\varphi(x) &= \beta^L \varphi(2^L x) \geq \beta^L\, 2\beta\, 2^L x \\ &= (2\beta)^L\, 2\beta x \geq 2\beta x .\end{aligned}$$

□

And this in turn gives rise to an even more precise upper bound,

Lemma 2.6. *For $2^{-L-1} \leq x \leq 2^{-L}$, $L \in \mathbb{N}$, we have*

$$\varphi(x) \leq 2(2\beta)^{L+1}(1-\beta)x + 2\beta^{L+1}(2\beta - 1) .$$

Proof. (i) We first prove this for $L = 0$. Then $x \in [\frac{1}{2}, 1]$, i.e. $x = \frac{1}{2} + \frac{1}{2}y$ with $y \in [0,1]$, and

$$\begin{aligned}\varphi(x) &= \left(\frac{1}{2} - \beta\right)\varphi(y) + \beta(2\beta + y) \\ &\leq \left(\frac{1}{2} - \beta\right)2\beta y + \beta(2\beta + y) \\ &= 2\beta y(1-\beta) + 2\beta^2 \\ &= 4\beta(1-\beta)x + 2\beta(2\beta - 1) .\end{aligned}$$

(ii) For $2^{-L-1} \leq x \leq 2^{-L}$ with $L \geq 1$, we have

$$\begin{aligned}\varphi(x) &= \beta^L \varphi(2^L x) \\ &\leq \beta^L \left[4\beta(1-\beta)2^L x + 2\beta(2\beta - 1)\right] \\ &\leq 2(2\beta)^{L+1}(1-\beta)x + 2\beta^{L+1}(2\beta - 1) .\end{aligned}$$

□

We can now use all these results to derive our final lemma in this section, on upper and lower bounds on $\varphi(x + \frac{1}{2}) - \varphi(x)$ if β is not too large.

Lemma 2.7. *Assume $\frac{1}{2} \leq \beta \leq \frac{3}{4}$. Then we have, for all $x \in [0, \frac{1}{2}]$,*

$$\frac{3}{8} \leq \frac{1}{2}\,\beta(1 + 6\beta - 8\beta^2) \leq \varphi\left(x + \frac{1}{2}\right) - \varphi(x) \leq 2\beta^2 \leq \frac{9}{8} .$$

Moreover, if $x \le 2^{-L}$ with $L \ge 2$, then the lower bound can be sharpened to

$$\varphi\left(x+\frac{1}{2}\right)-\varphi(x) \ge 2\beta^2+\beta\left[2^{-L}-(4\beta-1)\beta^L\right] .$$

Proof. (i) From (2.17) we have

$$\begin{aligned}\varphi\left(x+\frac{1}{2}\right)-\varphi(x) &= \left[\frac{\frac{1}{2}-\beta}{\beta}-1\right]\varphi(x)+2\beta(\beta+x)\\ &= \left(\frac{1}{2\beta}-2\right)\varphi(x)+2\beta(\beta+x)\\ &\le \left(\frac{1}{2\beta}-2\right)2\beta x+2\beta(\beta+x)\\ &= 2\beta^2+(1-2\beta)x \le 2\beta^2 .\end{aligned}$$

(ii) Take $L \ge 1$, $L \in \mathbb{N}$ so that $2^{-L-1} \le x \le 2^{-L}$. Then

$$\varphi\left(x+\frac{1}{2}\right)-\varphi(x) \ge \left(\frac{1}{2\beta}-2\right)\left[2(2\beta)^{L+1}(1-\beta)x+2\beta^{L+1}(2\beta-1)\right] + 2\beta(\beta+x) .$$

The right-hand side is a linear function of x, with slope $2\beta-2(1-\beta)(4\beta-1)(2\beta)^L$; for $\frac{1}{2} \le \beta \le \frac{3}{4}$, this is bounded above by $2\beta[1-(2\beta)^{L-1}] \le 0$, so that the highest value is attained at the lower edge of the interval, $x = 2^{-L-1}$. This leads to

$$\varphi\left(x+\frac{1}{2}\right)-\varphi(x) \ge 2\beta^2+\beta\left[2^{-L}-(4\beta-1)\beta^L\right] . \tag{2.23}$$

(iii) Now $F(\lambda) = 2^{-\lambda}-(4\beta-1)\beta^\lambda$ has a unique minimum for $\lambda \in [0,\infty)$. If $F'(1) \ge 0$, then this minimum lies to the left of 1, so that $\inf_{\lambda\in[1,\infty)} F(\lambda) = F(1)$ in this case. Now $F'(1) = -\frac{1}{2}\log 2-\beta(4\beta-1)\log\beta$ has a zero at $\beta = \frac{1}{2}$ and another at $\beta \simeq .839$; it is positive in between. Consequently, for $\frac{1}{2} \le \beta \le \frac{3}{4}$ we have

$$\begin{aligned}\varphi\left(x+\frac{1}{2}\right)-\varphi(x) &\ge 2\beta^2+\beta\left[\frac{1}{2}-(4\beta-1)\beta\right]\\ &= \frac{1}{2}\beta(1+6\beta-8\beta^2) \ge \frac{3}{8} .\end{aligned}$$

(iv) The sharper lower bound for $x \le 2^{-L}$, $L \in \mathbb{N}$ and $L \ge 2$ immediately follows from (2.23) and $F'(\lambda) \ge 0$ for $\lambda \ge 1$. □

To conclude this section, let us have a look at the graph of φ for one particular value of β, namely $\beta = .74$. Fig. 1a shows $\varphi(x)$ for $0 \le x \le 3$, Fig. 1b the restriction to $0 \le x \le 1$. The lower and upper bounds for Lemmas 2.5 and 2.6 are also graphed. To illustrate the fractal nature we have also graphed two successive blowups of the function near $x = \frac{9}{16} = .5625$, in Figs. 2a and 2b. At the scale of Fig. 2b the region to the immediate left of 9/16 seems uneventful, but with sufficient magnification similar phenomena will show up there as well.

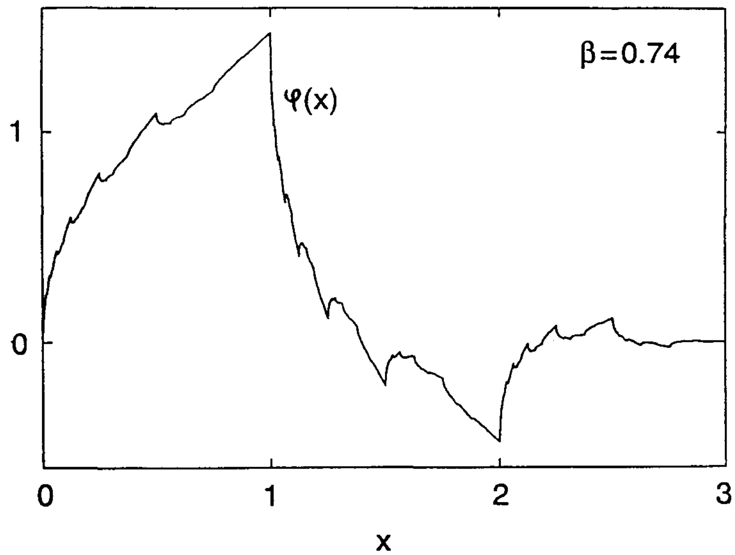

Fig. 1a. Graph of $\varphi(x)$ for $\beta = .74$.

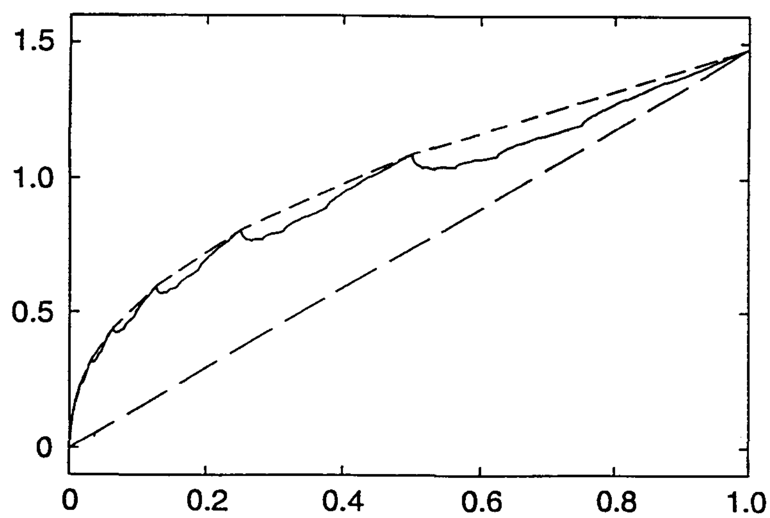

Fig. 1b. Blowup of the restriction of φ to $[0,1]$. The upper bound of Lemma 2.6 is graphed in short dashes; the lower bound of Lemma 2.5 in long dashes.

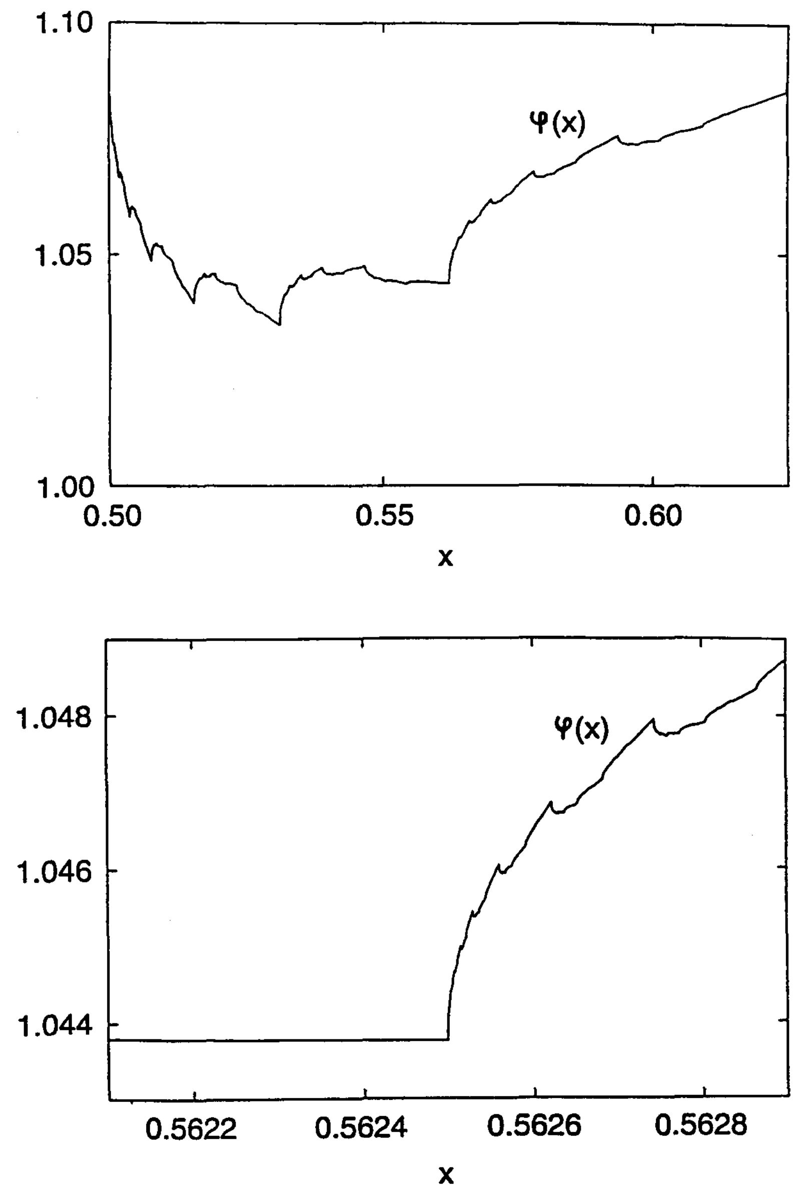

Fig. 2. Two successive blowups of $\varphi(x)$ near $x = \frac{9}{16} = .5625$, again for $\beta = .74$.

3. Computing $\alpha(x)$

Proposition 2.1 iii) already implies that $\alpha(x) \geq |\log_2 \beta| = h$ for all $x \in [0,3]$. This worst Hölder exponent $|\log_2 \beta|$ is achieved in e.g. $x = 0$, as shown by

$$|\varphi(2^{-n}) - \varphi(0)| = |\varphi(2^{-n})| = \beta^n |\varphi(1)| = C 2^{-n|\log_2 \beta|} ,$$

where $C > 0$. It then follows from (2.12) and (2.13) that $\alpha(1) = \alpha(2) = \alpha(0) = |\log_2 \beta|$ as well. The best possible Hölder exponent $\alpha(x)$ for $x \in [0,3]$ is

$|\log_2(\beta - \frac{1}{2})|$; this is achieved in $x = 3$; indeed, for $0 \le t \le 1$, we have

$$|\varphi(3 - 2^{-n}t) - \varphi(3)| = |\varphi(3 - 2^{-n}t)| = \left(\beta - \frac{1}{2}\right)^n |\varphi(3 - t)| \\ \le C\, 2^{-n|\log_2(\beta - \frac{1}{2})|} ,$$

where the inequality is an equality, with $C = |\varphi(2)| = 2\beta - 1$, if $t = 1$.

A continuum of intermediate values for $\alpha(x)$ between these upper and lower bounds can be attained; in this section we compute the exact value of $\alpha(x)$ in sufficiently many points x to allow us to derive $f(\alpha)$ in the next section. This constitutes a refinement of results in Daubechies and Lagarias (1992), where only a lower bound on $\alpha(x)$ was derived, for fewer points x.

The following proposition enables us to concentrate mostly on $]0,1[$.

Proposition 3.1. *Take $x \in\,]0,1[$. Then the following are true:*

(i) $\alpha(x) < 1 \Leftrightarrow \alpha(x+1) < 1 \Leftrightarrow \alpha(x+2) < 1$ *and moreover* $\alpha(x) = \alpha(x+1) = \alpha(x+2)$ *if any of the three is less than* 1.

(ii) $\alpha(x) \ge 1 \Leftrightarrow \alpha(x+1) \ge 1 \Leftrightarrow \alpha(x+2) \ge 1$

(iii) *if one of* $\alpha(x)$, $\alpha(x+1)$, $\alpha(x+2)$ *is strictly larger than* 1, *then the other two equal* 1.

Proof. These are immediate consequences of (2.12) and (2.13). □

We shall therefore compute $\alpha(x)$ only for $x \in\,]0,1[$. Our main tool for the computation of $\alpha(x)$ will be formula (2.15), which gives $\varphi(x+t) - \varphi(x)$ if x and $x+t$ have the same first n binary digits,

$$\varphi(x+t) - \varphi(x) = \mu(n;x)[\varphi(\sigma^n(x+t)) - \varphi(\sigma^n x)] + \beta t \sum_{k=1}^{n} d_k(x) 2^k \mu(k-1;x) . \quad (3.1)$$

Let us be naive, and assume that $|t| \simeq 2^{-n-1}$. We assume furthermore that for the purposes of estimating (3.1) we may take $|\mu(\ell;x)| = |\mu_0|^{\ell - s_\ell(x)} |\mu_1|^{s_\ell(x)} \simeq \left[\beta^{1-r(x)} \left(\beta - \frac{1}{2}\right)^{r(x)}\right]^\ell$, where $s_\ell(x) = \sum_{k=1}^{\ell} d_k(x)$ is the number of times 1 occurs in the first ℓ digits of x, and where $r(x) = \lim_{\ell\to\infty} \frac{1}{\ell} s_\ell(x)$ is presumed to exist. If $\beta^{1-r(x)} \left(\beta - \frac{1}{2}\right)^{r(x)} < \frac{1}{2}$, then the second term in (3.1), of order 2^{-n}, will dominate, and we will find $\alpha(x) = 1$. If $\beta^{1-r(x)} \left(\beta - \frac{1}{2}\right)^{r(x)} > \frac{1}{2}$, then we expect the first term to dominate, leading to the Hölder exponent

$$\alpha(x) = [1 - r(x)]\, |\log_2 \beta| + r(x)\, \left|\log_2 \left(\beta - \frac{1}{2}\right)\right| < 1 .$$

This intuition is essentially correct, but we will need some amount of work in order to bridge the gap between being naive and proving a theorem. What can go wrong? First of all, the binary expansion of x could contain very long stretches of ones or zeros, so that $|t|$ may have to be much smaller than 2^{-n} in order to ensure that x and $x+t$ have the same first n digits. Second, $\frac{1}{\ell}s_\ell(x)$ may fail to have a limit for $\ell \to \infty$. Finally, we have been very cavalier in our estimations: since $\mu_1 = \frac{1}{2} - \beta < 0$, (3.1) contains positive and negative terms. When we "expect" the first term to dominate, the second term in (3.1) is the partial sum of an infinite series which does not converge absolutely, so we have to be very careful in our estimates. In fact, the second term will be of the same order as the first one, and we will have to prove that no cancellations occur that could spoil our intuition.

We shall start our computation of $\alpha(x)$ by the computation of a lower bound, where the cancellations mentioned above don't matter (they can only decrease $|\varphi(x+t) - \varphi(x)|$, thereby increasing the Hölder exponent). The first two difficulties are then addressed by introducing the following appropriate definitions. Given $x \in [0,1]$, with binary digits $d_k(x)$, and given $N \in \mathbb{N}$, we define

$$\ell_N^1(x) = \begin{cases} 0 & \text{if } d_N(x) = 0 \\ k & \text{if } d_N(x) = 1 = \cdots = d_{N-k+1}(x) \\ & \text{and } d_{N-k}(x) = 0 \ . \end{cases}$$

Clearly $\ell_N^1(x)$ is the length of the uninterrupted stretch of ones preceding (and including) $d_N(x)$. We define $\ell_N^0(x)$ analogously as the length of the stretch of zeros preceding $d_N(x)$. Define now $a_N(x)$ by

$$\begin{aligned} 2^{-a_N(x)N} &= |\mu(N - \ell_N^1(x); x)| \ \mu_0^{\ell_N^1(x)} \\ &= \beta^N \left[\frac{\beta - 1/2}{\beta}\right]^{s_N(x) - \ell_N^1(x)} , \end{aligned} \tag{3.2}$$

i.e.

$$a_N(x) = |\log_2 \beta| + \frac{s_N(x) - \ell_N^1(x)}{N} \left|\log_2 \frac{\beta - 1/2}{\beta}\right| .$$

Finally, define $a(x)$ by

$$a(x) = \liminf_{N\to\infty} a_N(x) .$$

Remark. Note that we don't even have to worry about which binary expansion to take if x is a dyadic rational; for sufficiently large N we have

$$2^{-a_N^+(x)N} = \frac{\beta - 1/2}{\beta} 2^{-a_N^-(x)N} ,$$

so that $\liminf_{N\to\infty} a_N^+(x) = \liminf_{N\to\infty} a_N^-(x)$; both are equal to $|\log_2 \beta|$.

Lemma 3.2. *If $a(x) = \alpha \leq 1$, then $\alpha(x) \geq a(x)$.*

Proof. (i) There exists N_0 so that $a_N(x) > \alpha - \epsilon$ for all $N \geq N_0$, implying

$$|\mu(N - \ell_N(x); x)| \leq 2^{-N(\alpha - \epsilon)} ,$$

and a fortiori

$$| \mu(N;x) | \leq 2^{-N(\alpha-\epsilon)}$$

(since $| \mu(N;x) | = \left(\frac{\beta-1/2}{\beta}\right)^{\ell_N(x)} | \mu(N-\ell_N(x);x) |$).

(ii) Choose now $0 \leq t < 2^{-N_0}$. There exists $N \geq N_0$ so that $2^{-N-1} \leq t < 2^{-N}$. If $t < 2^{-N}(1-\sigma^N x)$, then x and $x+t$ have the same first N digits, and because φ is bounded, (3.1) leads to

$$\begin{aligned} | \varphi(x+t) - \varphi(x) | &\leq C\, 2^{-Na_N(x)} + C\, 2^{-N} \sum_{k=1}^{N} 2^k \,| \mu(k-1;x) | \\ &\leq C\, 2^{-N(\alpha-\epsilon)} + C\, 2^{-N} \sum_{k=1}^{N_0} 2^k \,| \mu(k-1;x) | \\ &\quad + C\, 2^{-N} \sum_{k=N_0+1}^{N} 2^k 2^{-k(\alpha-\epsilon)} \\ &\leq C\, 2^{-N(\alpha-\epsilon)} + \tilde{C}\, 2^{-N} + C\, 2^{-N}\, 2^{N(1-\alpha+\epsilon)+1} \\ &\leq C''\, 2^{-N(\alpha-\epsilon)}\,, \end{aligned}$$

where C'' depends on x and on ϵ, but not on N.

(iii) If $t \geq 2^{-N}(1-\sigma^N x)$, then only the first $N-\ell_N(x)-1$ digits of x and $x+t$ are the same. Then

$$\begin{aligned} | \varphi(x+t) - \varphi(x) | \leq C\, | \mu(N-\ell_N(x)-1;x) | \; \mu_0^{\ell_N(x)+1} \\ + C\, 2^{-N} \sum_{k=1}^{N-\ell_N(x)-1} 2^k \,| \mu(k-1;x) | \,, \end{aligned} \tag{3.3}$$

where we have used that φ is Hölder continuous with exponent $| \log_2 \mu_0 |$, so that

$$\begin{aligned} | \varphi(\sigma^{N-\ell_N(x)-1}x + 2^{N-\ell_N(x)-1}t) - \varphi(\sigma^{N-\ell_N(x)-1}x) | \\ \leq C\, | 2^{N-\ell_N(x)-1}t |^{|\log_2 \mu_0|} \leq C \mu_0^{\ell_N(x)+1}\,. \end{aligned}$$

The first term in (3.3) is bounded by $C'\, 2^{-Na_N(x)}$; the remainder of the estimate proceeds as in (ii).

(iv) If $t \leq 0$, the discussion is even easier. We choose again $|t| < 2^{-N_0}$, and we identify N so that $2^{-N-1} \leq |t| < 2^{-N}$. We now have again two cases to distinguish: $|t| \leq \sigma^N x$ or $|t| > \sigma^N x$. In the first case the estimates are exactly the same as in (ii) above. In the second case $d_N(x) = 0$, and it may be preceded by a stretch of zeros, $d_N(x) = 0 = \cdots = d_{N-L+1}(x)$, with $d_{N-L}(x) = 1$. Then only the first $N-L-1$ digits of x and $x+t$ are the same, and

$$\begin{aligned} | \varphi(x+t) - \varphi(x) | \leq C\, | \mu(N-L-1;x) | \; \mu_0^{L+1} \\ + C\, 2^{-N} \sum_{k=1}^{N-L-1} 2^k \,| \mu(k-1;x) | \,, \end{aligned}$$

as in (iii). But now $|\mu(N;x)| = |\mu_1|\,|\mu_0|^L |\mu(N-L-1;x)|$, so that we still have

$$|\varphi(x+t)-\varphi(x)| \le C' \;|\mu(N;x)|\; + C\,2^{-N}\sum_{k=1}^{N} 2^k\,|\mu(k-1;x)|\,,$$

and the estimate of (ii) still works. □

Next we derive a lower bound on $\alpha(x)$ if $a(x) > 1$.

Lemma 3.3. *If $a(x) > 1$, then $\varphi \in C^1(x)$.*

Proof. (i) There exists N_0 so that $a_N(x) > \alpha = \frac{1+a(x)}{2} > 1$ for all $N \ge N_0$. It follows that, for $N \ge N_0$,

$$|\mu(N;x)| \le |\mu(N-\ell_N(x);x)| \le 2^{-N\alpha}\,.$$

(ii) Choose $|t| \le 2^{-N_0}$. Find N so that $2^{-N-1} \le |t| < 2^{-N}$. By the same arguments as in the proof of Lemma 3.2 we can then bound the first term in (3.1) by $C\,2^{-N\alpha}$, regardless of whether $x+t$ and t have the same first N digits or not. Consequently

$$\begin{aligned} |\varphi(x+t)-\varphi(x)| &\le C\,2^{-N\alpha} + C\,2^{-N}\sum_{k=1}^{N_0} 2^k\,|\mu(k-1;x)| \\ &\quad + C\,2^{-N}\sum_{k=N_0+1}^{N} 2^k 2^{-k\alpha} \\ &\le C'\,2^{-N} \le C''\,|t|\,. \end{aligned}$$

It follows that $\varphi \in C^1(x)$. □

Together, Lemmas 3.2 and 3.3 imply the following lower bound on $\alpha(x)$,

$$\alpha(x) \ge \min[1, a(x)]\,.$$

In order to prove an upper bound on $\alpha(x)$, we need to address the problem of cancellations in the right-hand side of (3.1). Note that there is never a problem if x is a dyadic rational: in that case, $\sigma^K x = 0$ for some K, so that if we take $t = 2^{-N}$ with $N > K$, then

$$\begin{aligned} \varphi(x+2^{-N}) - \varphi(x) &= \mu^+(N-1;x)[\varphi(1)-\varphi(0)] \\ &\quad + \beta 2^{-N}\sum_{k=1}^{K} d_k^+(x)\,2^k \mu^+(k-1;x)\,. \end{aligned}$$

It follows that

$$\begin{aligned}
&| \varphi(x+2^{-N}) - \varphi(x) | \\
&\geq | \mu^+(K;x) | \ \beta^{N-K-1} \ | \varphi(1) - \varphi(0) | \ -\beta 2^{-N} \sum_{k=1}^{K} 2^k \ | \mu^+(k-1;x) | \\
&\geq 2\beta^{N-K} \ | \mu^+(K;x) | \ -2^{-N} 2\beta^2 \ \frac{(2\beta)^K - 1}{2\beta - 1} \\
&\geq \beta^N \left[2\beta^{-K} \left(\beta - \frac{1}{2}\right)^K - \frac{2\beta^2}{2\beta-1} \ (2\beta)^{K-N} \right] \geq C\beta^N
\end{aligned}$$

with $C > 0$, because $\beta > \frac{1}{2}$. Consequently φ has Hölder exponent at most $| \log_2 \beta |$ in any dyadic rational $x \in [0,1[$, which is exactly $a(x)$.

In other points x where very long but finite stretches of zeros and ones occur things are less simple. We shall restrict ourselves below to the set $\mathcal{R}$ defined by

$$\mathcal{R} = \left\{ x \in [0,1]; \ \lim_{N\to\infty} N^{-1} \ell_N^1(x) = 0 = \lim_{N\to\infty} N^{-1} \ell_N^0(x) \right\} \ .$$

We shall see in Sec. 4 that restricting ourselves to only those x that belong to $\mathcal{R}$ does not affect the computation of $f(\alpha)$. For $x \in \mathcal{R}$, we have $a(x) = \liminf_{N\to\infty} \tilde{a}_N(x)$, with

$$\tilde{a}_N(x) = | \log_2 \beta | \ + \frac{s_N(x)}{N} \left| \log_2 \frac{\beta - \frac{1}{2}}{\beta} \right| \ .$$

Note that $N^{-1} s_N(x)$ may still fail to tend to a limit for general $x \in \mathcal{R}$. Using the detailed estimates from Sec. 2, we can now prove

Lemma 3.4. *Assume $\frac{1}{2} < \beta < \frac{3}{4}$. If $x \in \mathcal{R}$ and $a(x) \leq 1$, then $\alpha(x) = a(x)$.*

Proof. (i) In view of Lemma 3.2, it is sufficient to show that for arbitrarily small $\delta > 0$ we can find a constant $C > 0$ and a sequence of t_n with $|t_n| \downarrow 0$ so that

$$| \varphi(x + t_n) - \varphi(x) | \geq C |t_n|^{\alpha + \delta} \ , \tag{3.4}$$

where $\alpha = a(x) \leq 1$.

(ii) Fix $\epsilon > 0$. Since $x \in \mathcal{R}$, there exists N_0 so that, for $N \geq N_0$,

$$\frac{1}{N} \ell_N^0(x), \quad \frac{1}{N} \ell_N^1(x) \leq \epsilon \ .$$

On the other hand, for all N there exists $\tilde{N} \geq N$ such that

$$\tilde{a}_{\tilde{N}}(x) < \alpha + \epsilon \quad \text{or} \quad | \mu(\tilde{N}, x) | > 2^{-\tilde{N}(\alpha+\epsilon)} \ .$$

Define now $N' = \tilde{N} - \ell_{\tilde{N}}^1(x)$, $\bar{N} = N' - \ell_{N'}^0(x)$. Then $d_{\bar{N}}(x) = 1$, $d_{\bar{N}+1}(x) = 0$. Moreover, if we restrict ourselves to $N \geq N_1 = (1-\epsilon)^{-1} N_0$, then $\tilde{N} \geq N \geq N_1 \geq N_0$ and $N' = \tilde{N} - \ell_{\tilde{N}}^1(x) \geq \tilde{N}(1-\epsilon) \geq N_0$, so that $\ell_{N'}^0(x) \leq \epsilon N'$. It follows that

$$\bar{N} \geq \tilde{N} - \epsilon \tilde{N} - \epsilon N' \geq (1 - 2\epsilon) \tilde{N} \ ,$$

so that

$$\begin{aligned} |\, \mu(\bar{N};x) \,| &\geq |\, \mu(\tilde{N};x) \,| \\ &\geq 2^{-\tilde{N}(\alpha+\epsilon)} \geq 2^{-(\alpha+\epsilon)(1-2\epsilon)^{-1}\bar{N}} . \end{aligned}$$

(iii) We shall choose two possibilities for t_N and then take the best one. Define $u_N = 2^{-\bar{N}-2}(-1)^{d_{\bar{N}+2}(x)}$, where $\bar{N} = \bar{N}(N)$ is as in (ii) above. The first $\bar{N}+1$ binary digits of x and $x+u_N$ are the same, so that

$$\begin{aligned} &(-1)^{d_{\bar{N}+2}(x)}[\varphi(x+u_N) - \varphi(x)] \\ &\quad = \mu(\bar{N}+1,x)\left[\varphi\left(\frac{1}{2}\,\sigma^{\bar{N}+2}x + \frac{1}{2}\right) - \varphi\left(\frac{1}{2}\,\sigma^{\bar{N}+2}x\right)\right] \\ &\qquad + \beta 2^{-\bar{N}-2}\sum_{k=1}^{\bar{N}+1} d_k(x)2^k\mu(k-1;x) \\ &\quad = \mu(\bar{N}+1,x)\left[\varphi\left(y_{\bar{N}+1} + \frac{1}{2}\right) - \varphi(y_{\bar{N}+1}) + \beta 2^{-\bar{N}-2}\frac{\Sigma(\bar{N}+1,x)}{\mu(\bar{N}+1;x)}\right] \end{aligned}$$

where we have introduced the notations

$$\begin{aligned} y_M &= \frac{1}{2}\,\sigma^{M+1}x \\ \Sigma(M,x) &= \sum_{k=1}^{M} d_k(x)2^k\mu(k-1;x) . \end{aligned} \tag{3.5}$$

Similarly, if we define $v_N = -2^{-\bar{N}}$, then

$$\begin{aligned} &-[\varphi(x+v_N) - \varphi(x)] \\ &\quad = \mu(\bar{N}-1;x)\left[\varphi\left(y_{\bar{N}-1} + \frac{1}{2}\right) - \varphi(y_{\bar{N}-1}) + \beta 2^{-\bar{N}}\frac{\Sigma(\bar{N}-1,x)}{\mu(\bar{N}-1;x)}\right] . \end{aligned}$$

Note that $y_{\bar{N}-1} = \frac{1}{2}\sigma^{\bar{N}}x \leq \frac{1}{4}$, since the first binary digit of $\sigma^{\bar{N}}x$ is $d_{\bar{N}+1}(x) = 0$.

(iv) Define $\Gamma(M,x) = \varphi\left(y_M + \frac{1}{2}\right) - \varphi(y_M) + \beta 2^{-M-1}\frac{\Sigma(M,x)}{\mu(M;x)}$. If we can show that

$$\max\left[|\,\Gamma(\bar{N}-1,x)\,|,\ |\,\Gamma(\bar{N}+1,x)\,|\right] = C > 0 , \tag{3.6}$$

with C independent of $\bar{N}$, then we can choose $t_N = u_N$ if $|\,\Gamma(\bar{N}+1)\,|$ is largest, $t_N = v_N$ otherwise, and it will follow that

$$\begin{aligned} |\,\varphi(x+t_N) - \varphi(x)\,| &\geq \frac{1}{|\,\beta - \frac{1}{2}\,|}\,C\,|\,\mu(\bar{N},x)\,| \\ &\geq C'\,2^{-\bar{N}(\alpha+\epsilon)(1-2\epsilon)^{-1}} \geq C'\,2^{-\bar{N}(\alpha+\epsilon)(1+4\epsilon)} \end{aligned}$$

if $\epsilon \leq \frac{1}{2}$. Since $|t_N| \downarrow 0$, this implies (3.4), with $\delta = \epsilon(1+4\alpha+4\epsilon)$ arbitrarily small.

(v) It remains therefore to establish (3.6).

First note that $\mu(\bar{N}+1;x) = \beta\left(\frac{1}{2}-\beta\right)\mu(\bar{N}-1;x)$, $\Sigma(\bar{N}+1,x) = \Sigma(\bar{N}-1,x) + 2^{\bar{N}}\mu(\bar{N}-1,x)$, so that

$$2^{-\bar{N}-2}\frac{\Sigma(\bar{N}+1)}{\mu(\bar{N}+1)} = -\frac{1}{2\beta(2\beta-1)}\left[2^{-\bar{N}}\frac{\Sigma(\bar{N}-1,x)}{\mu(\bar{N}-1,x)} + 1\right] .$$

On the other hand, we know from Lemma 2.7 that

$$\varphi\left(y_{\bar N+1}+\frac{1}{2}\right)-\varphi\left(y_{\bar N+1}\right)\geq \frac{1}{2}\beta(1+6\beta-8\beta^2)$$

and, since $y_{\bar N-1}\leq \frac{1}{4}$,

$$\varphi\left(y_{\bar N-1}+\frac{1}{2}\right)-\varphi\left(y_{\bar N-1}\right)\geq 2\beta^2+\beta\left[\frac{1}{4}-\beta^2(4\beta-1)\right] .$$

Define now $\rho = 2^{-\bar N}\frac{\Sigma(\bar N-1,x)}{\mu(\bar N-1;x)}$. Then

$$\Gamma(\bar N-1,x)\geq 2\beta^2+\beta\left[\frac{1}{4}-\beta^2(4\beta-1)\right]+\beta\rho \tag{3.7a}$$

and

$$\Gamma(\bar N+1,x)\geq \frac{1}{2}\beta(1+6\beta-8\beta^2)-\frac{1}{2(2\beta-1)}(\rho+1) . \tag{3.7b}$$

The right-hand sides of (3.7a) and (3.7b) are increasing and decreasing linear functions of ρ, respectively. Their intersection is given by equating (3.7a) = (3.7b) which gives

$$\rho_{\text{int}}=\left(32\beta^5-56\beta^4+28\beta^3-2\beta^2-\beta-2\right)\left(8\beta^2-4\beta+2\right)^{-1} ;$$

substituting this into the right-hand side of (3.7) gives

$$\frac{\beta(2\beta-1)(3-4\beta)(8\beta^2+2\beta+1)}{4(4\beta^2-2\beta+1)} , \tag{3.8}$$

which is >0 for $\beta\in\,]\frac{1}{2},\frac{3}{4}[$. Consequently

$$\max\left(\Gamma(\bar N-1,x),\ \Gamma(\bar N+1,x)\right)\geq\ (3.8)>0 ,$$

which proves (3.6) and thereby the whole lemma. □

Lemma 3.4 settled the case if $x\in\mathcal{R}$ and $a(x)\leq 1$. If $a(x)>1$, then we have

Lemma 3.5. *Suppose $a(x)>1$. Then $\Sigma(x)=\sum_{k=1}^{\infty}d_k(x)2^k\mu(k-1;x)$ converges absolutely. If $\Sigma(x)\neq 0$, then $\alpha(x)=1$.*

Proof. (i) Again, we only have to find $t_n\to 0$ and $C>0$ so that

$$|\,\varphi(x+t_n)-\varphi(x)\,|\geq C|t_n| .$$

(ii) Since $a(x)>1$, it follows that there exists N_0 so that for $N\geq N_0$

$$\tilde a_N(x)\geq a_N(x)>\frac{1+a(x)}{2}=\gamma>1, \quad \text{or} \quad |\,\mu(N;x)\,|\leq 2^{-\gamma N} .$$

This already proves the absolute convergence of the series in $\Sigma(x)$.

(iii) Take now $t_N = (-1)^{d_{N+1}(x)}2^{-N-1}$. Then

$$\begin{aligned}[\varphi(x+t_N)-\varphi(x)](-1)^{d_{N+1}(x)} \\ = \mu(N;x)\left[\varphi\left(\frac{1}{2}\,\sigma^{N+1}x+\frac{1}{2}\right)-\varphi\left(\frac{1}{2}\right)\right] \\ + 2^{-N-1}\beta\sum_{k=1}^{N} d_k(x)2^k\mu(k-1;x)\,,\end{aligned}$$

so that

$$|\,\varphi(x+t_N)-\varphi(x)\,| \geq \beta 2^{-N-1}\ \ |\,\Sigma(N,x)\,|\ \ -2\beta^2 2^{-\gamma N}\ ,$$

with $\Sigma(N,x)$ as defined by (3.5). Since $\gamma > 1$, it therefore suffices to prove a lower bound on $|\,\Sigma(N,x)\,|$,

$$|\,\Sigma(N,x)\,| \geq C > 0\ , \tag{3.9}$$

with $C = C(x)$ independent of N. But $|\,\Sigma(N,x)\,| \longrightarrow |\,\Sigma(x)\,| > 0$, which implies $|\,\Sigma(N,x)\,| \geq \frac{1}{2}\ \ |\,\Sigma(x)\,| > 0$ for sufficiently large N. □

If x is such that $a(x) > 1$ and $\Sigma(x) = 0$, then $\alpha(x)$ may be larger than 1. An example is given by $\beta = .5898\ldots$, a root of $1+(4\beta)^2(1-2\beta) = 0$, and by $x = \frac{9}{15}$ which has binary expansion .10011001100110011... . One easily checks that

$$\Sigma(4,x) = 1+(4\beta)^2(1-2\beta) = 0\ ,$$

and similarly

$$\Sigma(4k,x) = \Sigma(4(k-1),x) = \cdots = 0\ .$$

Consequently, for $\ell = 0$, 1, 2 or 3

$$|\,\Sigma(4k+\ell,x)\,| \leq C\ |\,\mu(4k-1;x)\,|\ 2^{4k}\ . \tag{3.10}$$

In this case $a(x) = \frac{1}{2}\ \ |\log_2\beta\,|\ \ +\frac{1}{2}\ \ |\log_2\left(\beta-\frac{1}{2}\right)| \simeq 2.12 > 1$. If x and $x+t$ have the same first N binary digits (which happens whenever $|t| < 2^{-N-2}$), then (3.10) leads to

$$|\,\varphi(x+t)-\varphi(x)\,| \leq C\ 2^{-a(x)N}\ ,$$

which means that for all $|t| \leq 1/4$,

$$|\,\varphi(x+t)-\varphi(x)\,| \leq C'\ |t|^{-a(x)}\ .$$

This construction can easily be adapted to yield x with other values for $\alpha(x)$ between 1 and $|\log_2(\beta-\frac{1}{2})\,|$. However, for most $x \in\,]0,1[$ with $a(x) > 1$, we have $\alpha(x) = 1$. This is shown by the following lemma

Lemma 3.6. *For $x \in\,]0,1[$ with $a(x) > 1$ and $\Sigma(x) = 0$, we have:*

(i) *if $x > \frac{1}{2}$, then $\Sigma(\sigma x) \neq 0$ and $a(\sigma x) = a(x) > 1$, hence $\alpha(\sigma x) = 1$;*
(ii) *if $x < \frac{1}{2}$, then $\Sigma(x+\frac{1}{2}) \neq 0$, and $a(x+\frac{1}{2}) = a(x) > 1$, hence $\alpha(x+\frac{1}{2}) = 1$.*

Proof. In the first case, $\Sigma(x) = (2\beta-1)\,\Sigma(\sigma x)+2$; in the second case $\Sigma(x+\frac{1}{2}) = 2 + \left(\frac{1}{2}-\beta\right)^{-1}\beta\Sigma(x)$. It follows that $\Sigma(\sigma x) \neq 0$ for $x > \frac{1}{2}$, $\Sigma(x+\frac{1}{2}) \neq 0$ for $x < \frac{1}{2}$. It is easy to see that the shift σ or adding $\frac{1}{2}$ does not affect $a(x)$. The remainder then follows from Lemma 3.5. □

This settles our discussion for $x \in\,]0,1[$. By Proposition 3.1, most of this can be carried over by simple translation to $x \in\,]1,2[$ or $x \in\,]2,3[$. There is one exception: the "anomalous" $x \in\,]1,2[$ or $\in\,]2,3[$ where $a(x) > 1$ and $\alpha(x) > 1$ are now given by $\Sigma(x-1) = \frac{1}{2}$ and $\Sigma(x-2) = 1$, respectively.

The following theorem summarizes all the findings of this section.

Theorem 3.7. *Assume $\frac{1}{2} < \beta < \frac{3}{4}$, and let φ be as defined in Sec. 2. Choose $y \in [0,3[$, $y = n+x$, with $n = 0$, 1 or 2, and $x \in [0,1[$ the decimal part of y. Define*

$$a(y) = a(x) = \liminf_{N\to\infty} \left[|\log_2\beta| + \frac{s_N(x) - \ell_N^1(x)}{N} \left|\log_2 \frac{\beta - 1/2}{\beta}\right| \right] , \quad (3.11)$$

where

$$s_N(x) = \sum_{k=1}^{N} d_k(x) \quad \text{and} \quad \ell_N^1(x) = \min\{k \in \mathbb{N};\ d_{N-k}(x) = 0\} .$$

Then $\alpha(y) = \liminf_{|t|\to 0} [\log|\varphi(y+t) - \varphi(y)|/\log|t|]$ satisfies

$$\alpha(y) \geq \min[1, a(y)] .$$

Moreover, if $x \in \mathcal{R} = \{z \in [0,1[;\ \lim_{N\to\infty} \frac{1}{N}\ell_N^1(z) = 0 = \lim_{N\to\infty} \frac{1}{N}\ell_n^0(z)\}$, where $\ell_N^0(z) = \min\{k \in \mathbb{N};\ d_{N-k}(z) = 1\}$, and if $a(y) = a(x) \leq 1$, then $\alpha(y) = a(y)$. If $a(y) = a(x) > 1$, then $\alpha(y)$ may be larger than 1, but at least two of the three values $\alpha(x)$, $\alpha(x+1)$, $\alpha(x+2)$ equal 1.

Proof. For $x \in\,]0,1[$ this follows from our lemmas. For $y = 0$, 1 or 2 this follows from the discussion at the start at the section and $a(y) = |\log_2\beta|$. □

4. The Singularity Spectrum $f(\alpha)$ and its Legendre Transform

We shall split our computation of $f(\alpha)$ into two parts: $\alpha < 1$ and $\alpha \geq 1$. In the latter part we shall concentrate mostly on $\alpha = 1$, but we shall come back to this below. For $\alpha < 1$, we shall prove

Theorem 4.1. *Let $\varphi(x)$, $\alpha(x)$ be as in Theorem 3.7. Define S_α by*

$$S_\alpha = \{x \in [0,3];\ \alpha(x) = \alpha\} . \quad (4.1)$$

Then, for $\alpha < 1$, the Hausdorff dimension of S_α is given by

$$\dim_H S_\alpha = \rho\,|\log_2\rho| + (1-\rho)\ |\log_2(1-\rho)| \quad (4.2)$$

where ρ is determined by $|\log_2\beta| + \rho\left|\log_2 \frac{\beta-1/2}{\beta}\right| = \alpha$.

The proof will follow from several observations and lemmas. First of all, note that for $\alpha < 1$, it follows from Theorem 3.7 that

$$\{x \in [0,3],\ x - \lfloor x \rfloor \in \mathcal{R} \text{ and } a(x) = \alpha\} \subset S_\alpha \subset \{x \in [0,3];\ a(x) \le \alpha\}\ , \tag{4.3}$$

where $a(x)$ is as in (3.11). Let us start by stripping this of all the extraneous factors $|\log_2 \beta|$ and $|\log_2(\beta - \frac{1}{2})|$. Define $b(x) = \liminf_{N\to\infty} N^{-1}[s_N(x) - \ell_N^1(x)]$. Then (4.3) can be rewritten as

$$\{x \in [0,3],\ x - \lfloor x \rfloor \in \mathcal{R} \text{ and } b(x) = \rho\} \subset S_\alpha \subset \{x \in [0,3];\ b(x) \le \rho\}\ . \tag{4.4}$$

In Besicovitch (1934) and Eggleston (1949) the following theorem is proved:

Lemma 4.2. *Define* $R_\gamma = \{x \in [0,1];\ \lim_{N\to\infty} N^{-1} s_N(x) = \gamma\}$. *Then*

$$\dim_H R_\gamma = \gamma\,|\log_2 \gamma| + (1-\gamma)\ |\log_2(1-\gamma)|\ .$$

Note that the existence of a limit for $N^{-1}s_N(x)$ necessarily implies that $N^{-1}\ell_N^0(x)$ and $N^{-1}\ell_N^1(x)$ both tend to 0, i.e. that $x \in \mathcal{R}$. We therefore have

Corollary 4.3. *For* $\alpha < 1$, $\dim_H S_\alpha \ge h(\rho)$, *with*

$$h(\rho) = \rho\ |\log_2 \rho| + (1-\rho)\ |\log_2(1-\rho)|\ .$$

Proof. It follows from (4.4) that $S_\alpha \supset R_\rho$. □

It remains to prove that $\dim_H S_\alpha \le h(\rho)$. Note that because $\frac{1}{2} < \beta < \frac{3}{4}$, $|\log_2(\beta - \frac{1}{2})| > 2$, implying that $\rho = [\alpha - |\log_2 \beta|]/[|\log_2(\beta - \frac{1}{2})| - |\log_2 \beta|] < 1/2$. We now have

Lemma 4.4. *If* $\rho < \frac{1}{2}$, *then the set* $W_\rho = \{x \in [0,1];\ b(x) \le \rho\}$ *satisfies* $\dim_H W_\rho = h(\rho)$.

Proof. (i) It suffices to show for any $\eta > h(\rho)$ and any $\delta, \epsilon > 0$ that there is a δ-cover $\underline{\mathcal{U}} = \{U_i\}$ of W_ρ with intervals U_i such that

$$\sum_{U_i \in \underline{\mathcal{U}}} |U_i|^\eta < \epsilon\ . \tag{4.5}$$

(ii) Write $\eta = h(\rho) + \lambda$ and pick n_0 large enough so that $\delta > 2^{-n_0}$ and also $\sum_{j=n_0}^{\infty} j\ e^{-\frac{1}{2}j\lambda} < \epsilon$. Now define λ' by $h(\rho + \lambda') = \eta + \frac{1}{2}\lambda$; this is always possible for small enough λ because $\rho < 1/2$ and h is strictly increasing on $[0, \frac{1}{2}]$. For each $x \in W_\rho$, choose the smallest $N = N(x) \ge n_0$ such that

$$\frac{1}{N}[s_N(x) - \ell'_N(x)] < \rho + \lambda'\ ; \tag{4.6}$$

such N always exists because $b(x) \leq \rho$. Define $\tilde{x}$ to be the truncation of x after N binary digits, $\tilde{x} = x - 2^{-N}\sigma^N x = .d_1 \ldots d_N\, 0\, 0 \ldots$, and assign to x the dyadic interval $U = [\tilde{x}, \tilde{x} + 2^{-N}]$ of length 2^{-N}. Let $\mathcal{U}$ denote the collection of all such intervals.

(iii) Next we want to count how many intervals of length 2^{-N} occur in $\mathcal{U}$. We first map any $\tilde{x}$ to the number $\tilde{\tilde{x}}$ obtained by setting the final string of consecutive 1's in the digit pattern of $\tilde{x}$ to consecutive 0's, i.e.

$$\tilde{x} = .d_1 d_2 \ldots d_j\, 0\, 1\, 1 \ldots 1 \Rightarrow \tilde{\tilde{x}} = .d_1 d_2 \ldots d_j\, 0\, 0\, 0 \ldots 0 \ .$$

By (4.6) any such $\tilde{\tilde{x}}$ has at most $N(\rho + \lambda')$ digits equal to 1. Moreover any $\tilde{\tilde{x}}$ has $\ell_N^1(x) \leq N$ pre-images $\tilde{x}$ under this map. It follows that for any N, the number of different $\tilde{x}$ for which $N(x) = N$ is bounded by

$$N \sum_{j=1}^{N(\rho+\lambda')} \binom{N}{j} \leq N\ 2^{h(\rho+\lambda')N} ,$$

where we have used $\rho + \lambda' \leq 1/2$ together with Lemma 4.7.2 from Ash (1965).

(iv) It follows that

$$\sum_{U \in \mathcal{U}} |U|^\eta \leq \sum_{N=n_0}^{\infty} [N\ 2^{h(\rho+\lambda')N}]\ 2^{-N\eta}$$
$$\leq \sum_{N=n_0}^{\infty} N\ 2^{-\frac{1}{2}\lambda N} < \epsilon \ . \qquad \square$$

Combining the observations $\rho < 1/2$ and (4.4) with Corollary 4.3 and Lemma 4.4, we see that Theorem 4.1 is proved completely. This gives us an explicit formula for $f(\alpha)$ if $\alpha < 1$.

The case $\alpha = 1$ is handled by

Lemma 4.5. *Let $\varphi(x)$, $\alpha(x)$, S_α be defined as above, under the assumption $\frac{1}{2} < \beta < \frac{3}{4}$. Then $\dim_H S_1 = 1$.*

Proof. (i) Define, for $j = 1$, 2 or 3, $A_j = \{x \in [j-1, j];\ \alpha(x) > 1\}$. Then $\tilde{S} = \bigcup_{\alpha \geq 1} S_\alpha = S_1 \cup A_1 \cup A_2 \cup A_3$. From Theorem 3.7 we have $A_1 + 1$, $A_1 + 2$, $A_2 - 1$, $A_2 + 1$, $A_3 - 1$, $A_3 + 1 \subset S_1$. It follows that $\dim_H\, S_1 \geq \max\{\dim_H\, A_j;\ j = 1, 2, 3\}$. Consequently $\dim_H\, S_1 = \dim_H\, \tilde{S}$.

(ii) On the other hand, $[0,3] \backslash \tilde{S} = \bigcup_{\alpha < 1} S_\alpha \subset \{x \in [0,3];\ a(x) < 1\}$. It follows that $\tilde{S}$ contains all the normal points in $[0,3]$, because $a(x) = \frac{1}{2} \mid \log_2 \beta \mid + \frac{1}{2} \mid \log_2(\beta - \frac{1}{2}) \mid > 1$ if x is normal. Since the normal points constitute a full set, and full sets necessarily have Hausdorff dimension 1, we have therefore $\dim_H\, \tilde{S} = 1$. Hence $\dim_H\, S_1 = 1$. $\square$

The theorems above give us the value of $f(\alpha)$ for all $\alpha \leq 1$. We need not compute $f(\alpha)$ for $\alpha > 1$: because $f(1)$ equals 1, the maximum possible value of f, the values of $f(\alpha)$ for $\alpha > 1$ do not affect the Legendre transform of $1-f$ for $q \geq 0$.

Let us now compute the Legendre transform of $1 - f(\alpha)$,

$$S(q) = \inf_{\alpha} \; [q\alpha + 1 - f(\alpha)] \; .$$

Then $1-S(q)$ can be interpreted as the largest possible intercept with the y-axis for a line with slope q which has at least one point in common with the graph of f. Fig. 3 shows the graph of $f(\alpha)$ for $\alpha \leq 1$ and $\beta = .74$. It clearly shows the jump at $\alpha = 1$. For $\alpha > 1$, $f(\alpha) \leq 1$ (although it is probably much smaller than 1; we suspect $f(\alpha) = 0$ for $\alpha > 1$). For large positive q, $1 - S(q)$ will simply be the intersection with the y-axis of the tangent to $f(\alpha)$ with slope q. As q decreases, it reaches a critical value q_c for which the tangent with slope q_c also goes through $(1,1)$; this tangent is drawn in dashed lines on Fig. 3. For $q < q_c$, the straight line with slope q which has the largest intercept with the y-axis does not even touch the curved part of the graph: it is simply the straight line $y = q(\alpha - 1) + 1$. It follows that $S(q) = q$ for $q \leq q_c$. It remains to determine q_c and to compute $S(q)$ for $q > q_c$.

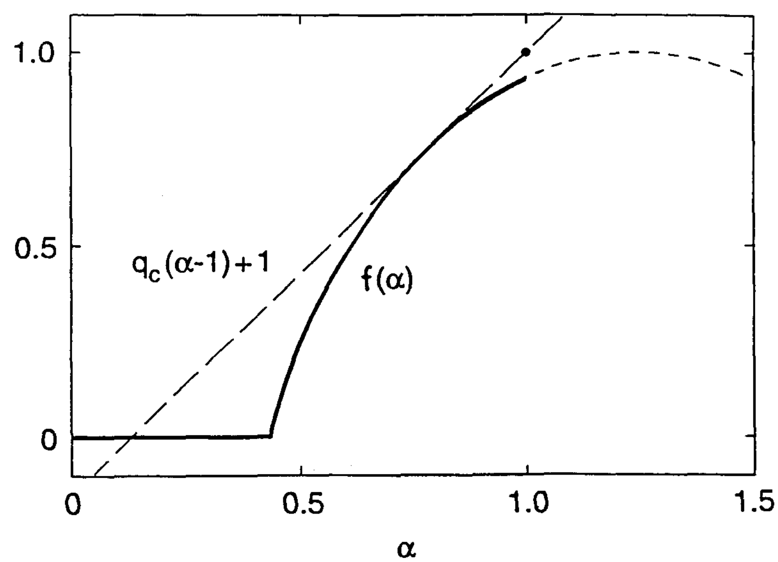

Fig. 3. The graph of $f(\alpha)$ for $\alpha \leq 1$ (fat curve + the point $(1,1)$) in the case $\beta = .74$. We have not plotted $f(\alpha)$ for $\alpha > 1$. The graph also shows the tangent (to the smooth part of $f(\alpha)$) that goes through $(1,1)$.

Lemma 4.6. *For $\frac{1}{2} < \beta < \frac{3}{4}$ the Legendre transform $S(q)$ of $1 - f(\alpha)$, for $q \geq 0$, is given by*

$$S(q) = q \qquad \text{if } q \leq q_c$$

$$S(q) = 1 - \log_2 \left[\beta^q + \left(\beta - \frac{1}{2}\right)^q\right] \qquad \text{if } q \geq q_c$$

where $q_c > 0$ is determined by

$$(2\beta)^{q_c} + (2\beta - 1)^{q_c} = 2 .$$

Proof. (i) Let us, for this proof only, define the function $g(\alpha)$ as follows:

$$\begin{aligned} g(\alpha) &= 0 \qquad \text{if} \quad \alpha \leq \alpha_0 = |\log_2 \beta| \quad \text{or} \quad \alpha \geq \alpha_1 = \left|\log_2\left(\beta - \frac{1}{2}\right)\right| \\ g(\alpha) &= \rho(\alpha)\,|\log_2 \rho(\alpha)| \; + [1-\rho(\alpha)]\,|\log_2[1-\rho(\alpha)]| \\ &\qquad \text{if} \quad \alpha_0 \leq \alpha \leq \alpha_1 , \end{aligned} \tag{4.7}$$

where

$$\rho(\alpha) = \frac{\alpha - \alpha_0}{\alpha_1 - \alpha_0} .$$

The restriction of $g(\alpha)$ to $[\alpha_0, \alpha_1]$ is C^∞, concave and symmetric around $\alpha_s = \frac{1}{2}(\alpha_0 + \alpha_s)$, where it attains its maximum value $g(\alpha_s) = 1$; it is strictly increasing on $[\alpha_0, \alpha_s]$ and strictly decreasing on $[\alpha_s, \alpha_1]$. Because $\beta(\beta - \frac{1}{2}) < \frac{1}{4}$ for $\frac{1}{2} < \beta < \frac{3}{4}$, we have $\alpha_s > 1$. Since $f(\alpha) \equiv g(\alpha)$ for $0 \leq \alpha < 1$, and $f(1) = 1$, the graph of $f(\alpha)$ therefore looks like Fig. 3 for all β between $\frac{1}{2}$ and $\frac{1}{4}$, with a jump at $\alpha = 1$. Define $G(q)$ to be the Legendre transform of $1 - g\,|_{[\alpha_0,\alpha_1]}$:

$$G(q) = \inf_{\alpha_0 \leq \alpha \leq \alpha_1} [q\alpha + 1 - g(\alpha)] .$$

Then, for any q, the straight line $y = q\alpha + 1 - G(q)$ is tangent to $y = g(\alpha)$, which means that $G(q)$ is determined by the equations

$$\begin{aligned} g'(\alpha) &= q \\ G(q) &= q\alpha + 1 - g(\alpha) . \end{aligned}$$

Moreover, if the intercept of the line $q\alpha + 1 - G(q)$ with the $\alpha = 1$ vertical exceeds 1, i.e. if $q \geq G(q)$, then $S(q) = G(q)$. If $q < G(q)$, then $S(q) = q$ (see above). The critical value q_c is therefore determined by $q_c = G(q_c)$.

(ii) From (4.7) we get

$$g'(\alpha) = \frac{1}{\alpha_1 - \alpha_0} \log_2 \frac{1 - \rho(\alpha)}{\rho(\alpha)} ,$$

so that $g'(\alpha) = q$ leads to

$$\rho(\alpha) = \left[1 + \left(\frac{\beta}{\beta - \frac{1}{2}}\right)^q\right]^{-1}$$

or

$$\alpha = \left[\log_2 \frac{\beta}{\beta - \frac{1}{2}}\right]\left[1 + \left(\frac{\beta}{\beta - \frac{1}{2}}\right)^q\right]^{-1} + |\log_2 \beta| .$$

Substituting this into $G(q) = q\alpha + 1 - g(\alpha)$ leads to

$$G(q) = 1 - \log_2\left[\beta^q + \left(\beta - \frac{1}{2}\right)^q\right] .$$

(iii) The critical value q_c is now determined by $G(q_c) = q_c$, or $\beta^{q_c} + (\beta - \frac{1}{2})^{q_c} = 2^{1-q_c}$, which can be rewritten as $(2\beta)^{q_c} + (2\beta - 1)^{q_c} = 2$. □

Figure 4 shows a graph of $S(q)$ for $\beta = .74$. One finds $q_c \simeq 1.151323$ in this case.

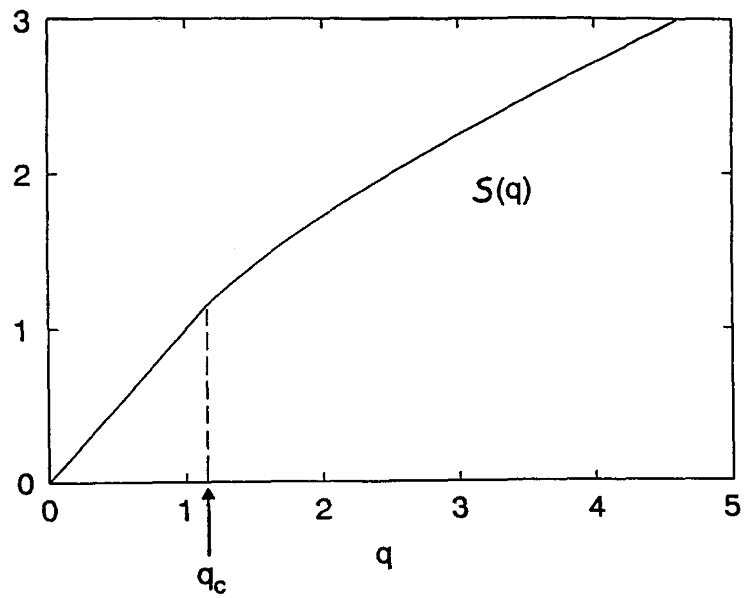

Fig. 4. Graph of $S(q)$ in the case $\beta = .74$; there are clearly two regimes separated by q_c. Asymptotically, for $q \to \infty$, $S(q) \sim 1 + q|\log_2 \beta|$.

5. Computing $\tau(q)$

In order to compute $\tau(q) = \liminf_{|t|\to 0} \frac{\log I(t,q)}{\log|t|}$, we need to find good upper and lower bounds for $I(t,q) = \int |\varphi(x+t) - \varphi(x)|^q \, dx$. This is the purpose of the next few lemmas.

Lemma 5.1. *There exist $C_1(q) > 0$ and $\lambda(q) > 0$ so that, for all $|t| < \frac{1}{2}$,*

$$I(t,q) \le C_1(q) \ \ |\log|t|\,|^{\lambda(q)} \left[|t|^q + |t|^{\xi_q}\right] ,$$

where

$$\xi_q = \left|\log_2 \left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]\right| .$$

Proof. (i) Since $I(t,q)$ is even in t, we only need to discuss $t \ge 0$.

(ii) For any such fixed $t < \frac{1}{2}$, we can find n so that $2^{-n-1} \le t < 2^{-n}$. Define now $E_n = [0, 1-2^{-n}] \cup [1, 2-2^{-n}] \cup [2, 3-2^{-n}]$, $F_n = [-2^{-n}, 0] \cup [1-2^{-n}, 1] \cup [2-2^{-n}, 2] \cup [3-2^{-n}, 3]$. Since support $\varphi = [0,3]$, we have

$$I(t,q) = \int_{E_n} |\varphi(x+t) - \varphi(x)|^q \, dx + \int_{F_n} |\varphi(x+t) - \varphi(x)|^q \, dx . \qquad (5.1)$$

(iii) The measure of F_n is 4×2^{-n}; together with the uniformly valid Hölder exponent $h = |\log_2 \beta|$ this gives the bound

$$\int_{F_n} |\varphi(x+t) - \varphi(x)|^q \, dx \leq C\, 2^{-n(1+qh)} . \tag{5.2}$$

(iv) To estimate the integral over E_n, we use again the vector notation introduced in Sec. 2:

$$\int_{E_n} |\varphi(x+t) - \varphi(x)|^q \, dx = \int_0^{1-2^{-n}} \sum_{i=1}^{3} |v_i(x+t) - v_i(x)|^q \, dx , \tag{5.3}$$

where v_i denotes the i-th component of the 3-vector v. Since all the norms on $\mathbb{R}^3$ are equivalent, and the three vectors $e_1 = (1,1,1)$, $e_2 = (2-2\beta, 1-2\beta, -2\beta)$ and $e_3 = (1,0,0)$ are independent, there exist (q-dependent) constants $C_1, C_2 > 0$ so that

$$\begin{aligned} C_1 \int_0^{1-2^{-n}} \sum_{j=1}^{3} |e_j \cdot [v(x+t) - v(x)]|^q &\leq (5.3) \\ \leq C_2 \int_0^{1-2^{-n}} \sum_{j=1}^{3} |e_j \cdot [v(x+t) - v(x)]|^q \, dx &. \end{aligned} \tag{5.4}$$

(v) By Proposition 2.1, it follows from (5.3), (5.4) that

$$\begin{aligned} &\int_{E_n} |\varphi(x+t) - \varphi(x)|^q \, dx \\ &\leq C \left\{ |t|^q (1 - 2^{-n}) + \int_0^{1-2^{-n}} |e_3 \cdot [v(x+t) - v(x)]|^q \, dx \right\} . \end{aligned} \tag{5.5}$$

The set $[0, 1-2^{-n}[$ can also be characterized as the set of all $x \in [0,1]$ which have at least one $d_k(x)$, $k = 1, \ldots, n$, equal to zero. Consequently $[\,0, 1-2^{-n}[= \bigcup_{k=1}^{n} A_{n,k}$, with $A_{n,k} = \{x \in [0,1],\ \ell_n^1(x) = n-k\}$. For $x \in A_{n,k}$, $x+t$ and x have the same first $k-1$ entries in their binary expansions (because $t < 2^{-n}$). Formula (2.8), Proposition 2.1 and $|\varphi(y+z) - \varphi(y)| \leq C|z|^h$ then lead to

$$\begin{aligned} &\int_0^{1-2^{-n}} |e_3 \cdot [v(x+t) - v(x)]|^q \, dx \\ &= \sum_{k=1}^{n} \int_{A_{n,k}} |e_3 \cdot [v(x+t) - v(x)]|^q \, dx \\ &= \sum_{k=1}^{n} \int_{A_{n,k}} \Bigg| \mu(k-1;x) e_3 \cdot [v(\sigma^{k-1}x + 2^{k-1}t) - v(\sigma^{k-1}x)] \\ &\quad + \beta t \sum_{\ell=1}^{k-1} d_\ell(x) 2^\ell \mu(\ell-1;x) \Bigg|^q \, dx \\ &\leq C \left(\sum_{k=1}^{n} I_{n,k} + |t|^q \sum_{k=1}^{n} J_{n,k} \right) , \end{aligned} \tag{5.6}$$

where

$$I_{n,k} = \int_{A_{n,k}} |\mu(k-1;x)|^q \; 2^{-(n-k+1)hq} \, dx$$

and

$$J_{n,k} = \int_{A_{n,k}} \left| \sum_{\ell=1}^{k-1} d_\ell(x) 2^\ell \, \mu(\ell-1;x) \right|^q dx \, .$$

(vi) The first $k-1$ binary digits of $x \in A_{n,k}$ are unconstrained, and are therefore equal to 0 or 1 with probability 1/2 each, all independently of each other. Therefore

$$\begin{aligned}\int_{A_{n,k}} |\mu(k-1;x)|^q \, dx &= \left(\frac{1}{2} \; |\mu_0|^q + \frac{1}{2} \; |\mu_1|^q\right)^{k-1} \int_{\sigma^{k-1}A_{n,k}} dx \\ &= \left(\frac{1}{2} \; |\mu_0|^q + \frac{1}{2} \; |\mu_1|^q\right)^{k-1} 2^{-n+k-1}\end{aligned}$$

so that

$$\begin{aligned}\sum_{k=1}^{n} I_{n,k} &= 2^{-n(1+hq)} \sum_{k=1}^{n} \left\{ \left(\frac{1}{2} \; |\mu_0|^q + \frac{1}{2} \; |\mu_1|^q\right) 2^{1+hq} \right\}^{k-1} \\ &= 2^{-n(1+hq)} \sum_{k=1}^{n} \left[1 + \frac{|\mu_1|^q}{|\mu_0|^q}\right]^{k-1} \\ &\le 2^{-n(1+hq)} \left(1 + \frac{|\mu_1|^q}{|\mu_0|^q}\right)^n \frac{|\mu_0|^q}{|\mu_1|^q} \\ &= \left(\frac{\beta}{\beta - 1/2}\right)^q \left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]^n \, . \end{aligned} \tag{5.7}$$

(vii) For $k = 1$, our summation conventions (see Sec. 2) give $J_{n,k} = 0$. For $k \geq 2$, a probabilistic argument similar to that in (vi) gives

$$\begin{aligned}J_{n,k} &= \sum_{d_1,\ldots,d_{k-1}=0 \text{ or } 1} 2^{-k+1} \left| \sum_{\ell=1}^{k-1} d_\ell 2^\ell \prod_{j=1}^{\ell-1} \mu_{d_j} \right|^q 2^{-n+k-1} \\ &= 2^{-n} \sum_{d_1,\ldots,d_{k-1}=0 \text{ or } 1} \left| \sum_{\ell=1}^{k-1} d_\ell 2^\ell \prod_{j=1}^{\ell-1} \mu_{d_j} \right|^q = 2^{-n} B_{k-1} \, .\end{aligned}$$

We shall see below that, for some C and $\lambda > 0$,

$$B_m \leq C m^\lambda \; \{2^m + (|2\beta|^q + |2\beta - 1|^q)^m\} \; , \tag{5.8}$$

implying

$$\begin{aligned}\sum_{k=1}^{n} J_{n,k} &\leq C n^\lambda \; 2^{-n} \{2^n + (|2\beta|^q + |2\beta - 1|^q)^n\} \\ &\leq C n^\lambda \left\{1 + \left(\frac{1}{2} \; |2\beta|^q + \frac{1}{2} \; |2\beta - 1|^q\right)^n\right\} \, . \end{aligned} \tag{5.9}$$

Substituting (5.7) and (5.9) into (5.6) then leads to

$$\int_{E_n} |\varphi(x+t)-\varphi(x)|^q \leq C\, n^\lambda \left\{2^{-nq}+\left(\frac{1}{2}\beta^q+\frac{1}{2}\left(\beta-\frac{1}{2}\right)^q\right)^n\right\} ,$$

where we have used $n \geq 1$. Together with (5.2) and $2^{-n} \leq 2t$ this implies

$$\int |\varphi(x+t)-\varphi(x)|^q \leq C\, |\log_2 |t||^\lambda \left\{|t|^q + |t|^{\xi_q}\right\}$$

where $\xi = \left|\log_2\left[\frac{1}{2}\beta^q+\frac{1}{2}\left(\beta-\frac{1}{2}\right)^q\right]\right|$. Modulo the proof of (5.8) (see Lemma 5.2 below), this proves the lemma. □

Lemma 5.2. *Define, for $m \geq 1$, $B_m = \sum_{d_1,\ldots,d_m=0}^1 \left|\sum_{\ell=1}^m d_\ell 2^\ell \mu(\ell-1;d)\right|^q$, where $\mu(k;d)=\prod_{j=1}^k \mu_{d_j}$, with the convention $\mu(0;d)=1$. Then there exist C and $\lambda > 0$ so that*

$$B_m \leq Cm^\lambda \left[2^m + (|\,2\beta\,|^q + |\,2\beta-1\,|^q)^m\right] . \tag{5.10}$$

Proof. (i) There exists L so that $2^{L-1} < m \leq 2^L$. Our proof shall work by induction on L. For $L=0$ we have only $m=1$, and $B_1 = \sum_{d=0}^1 |2d|^q = 2^q$. Hence (5.10) holds with $C \geq \beta^{-q}$.

(ii) Assume that (5.10) holds for all $m \leq 2^L$. Take now m such that $2^L < m \leq 2^{L+1}$. Then $m = n_1+n_2$, where $n_2 = n_1 = m/2$ if m is even, $n_2 = n_1+1 = \lfloor m/2 \rfloor + 1$ if m is odd. Clearly $n_1, n_2 \leq 2^L$.

(iii) Using that $|a+b|^q \leq c_q\,(|a|^q+|b|^q)$, with $c_q = 1$ if $q \leq 1$, $c_q = 2^{q-1}$ if $q \geq 1$, we have

$$\left|\sum_{\ell=1}^m d_\ell 2^\ell \mu(\ell-1;d)\right|^q$$
$$\leq c_q \left[\left|\sum_{\ell=1}^{n_1} d_\ell 2^\ell \mu(\ell-1;d)\right|^q + |2^{n_1}\mu(n_1;d)|^q \left|\sum_{r=1}^{n_2} d_{n_1+r} 2^r \mu(r-1;\sigma^{n_1}d)\right|^q\right] .$$

This implies

$$B_m \leq c_q \left[2^{n_2} B_{n_1} + \gamma^{n_1} B_{n_2}\right] , \tag{5.11}$$

where $\gamma = |\,2\mu_0\,|^q + |\,2\mu_1\,|^q = (2\beta)^q + (2\beta-1)^q \geq 1$ since $q \geq 0$ and $\beta > 1/2$. By the induction hypothesis, (5.11) implies

$$\begin{aligned} B_m &\leq c_q\, C n_2^\lambda \left[2^{n_2}(2^{n_1}+\gamma^{n_1}) + \gamma^{n_1}(2^{n_2}+\gamma^{n_2})\right] \\ &\leq 3c_q C \left(\frac{m+1}{2}\right)^\lambda (2^m+\gamma^m) . \end{aligned} \tag{5.12}$$

If λ is large enough so that $\left(\frac{4}{3}\right)^\lambda \geq 3c_q$, then $3c_q\left(\frac{m+1}{2m}\right)^\lambda \leq 1$ for all $m \geq 2$, and (5.12) implies (5.10). □

Lemma 5.1 gave us an upper bound on $I(t,q)$; the following lemma gives a lower bound for particular t_n.

Lemma 5.3. *Assume that* $\frac{1}{2} < \beta < \frac{3}{4}$. *Then there exist* $C_2(q) > 0$ *and a sequence of* $t_n \neq 0$ *with* $\lim_{n\to\infty} t_n = 0$ *so that*

$$I(t_n, q) \geq C_2(q)\left[|t_n|^q + |t_n|^{\xi_q}\right] .$$

Proof. (i) We shall choose $t_n = t_n^+ = -2^{-n}$ or $t_n = t_n^- = 2^{-n-2}$, according to a procedure to be explained below; we assume $n > 1$. Then we can copy the argument in (ii) of the proof of Lemma 5.1, and split the integration domain for $I(t,q)$ into $\tilde{E}_n = [2^{-n}, 1-2^{-n}] \cup [1+2^{-n}, 2-2^{-n}] \cup [2+2^{-n}, 3-2^{-n}]$ and $\tilde{F}_n = [-2^{-n}, 2^n] \cup [1-2^{-n}, 1+2^{-n}] \cup [2-2^{-n}, 2+2^{-n}] \cup [3-2^{-n}, 3+2^{-n}]$. With the obvious minor changes, the arguments in (iii) and (iv) of the proof of Lemma 5.1 then still hold, so that

$$I(t_n, q) \geq C\left[I_1(t_n, q) + I_2(t_n, q) - I_3(t_n, q)\right] \tag{5.13}$$

with

$$\begin{aligned} I_1(t_n, q) &= \int_{2^{-n}}^{1-2^{-n}} |e_2 \cdot [v(x+t_n) - v(x)]|^q \, dx \\ I_2(t_n, q) &= \int_{2^{-n}}^{1-2^{-n}} |e_3 \cdot [v(x+t_n) - v(x)]|^q \, dx \\ I_3(t_n, q) &= 2^{-n(1+qh)} . \end{aligned}$$

(ii) Since $|t_n| \geq 2^{-n-2}$, it follows that for $n \geq 1$,

$$I_1(t_n, q) = |t_n|^q (1 - 2^{-n+1}) \geq \frac{1}{2}\, 2^{-nq} . \tag{5.14}$$

(iii) We shall bound $I_2(t_n, q)$ below by restricting the integral domain to $D_n = \{x \in [0,1]; d_{n-1}(x) = d_{n+1}(x) = d_{n+2}(x) = 0, d_n(x) = 1\}$, a subset of $[2^{-n}, 1-2^{-n}]$. For $x \in D_n$, let us denote $y = \frac{1}{2}\,\sigma^{n+2}x$; y can take any value in $[0, \frac{1}{2}]$. Then $\sigma^{n-1}x = \frac{1}{2} + \frac{1}{4}y$, $\sigma^{n-1}(x + t_n^+) = \frac{1}{4}y$, $\sigma^{n+1}x = y$, $\sigma^{n+1}(x + t_n^-) = \frac{1}{2} + y$. Since $x + t_n^+$ and x have the same first $n-1$ entries, it follows that

$$\begin{aligned} -e_3 \cdot [v(x+t_n^+) - v(x)] &= -\varphi(x + t_n^+) + \varphi(x) \\ &= \mu(n-1; x)\left[\varphi\left(\frac{1}{2} + \frac{1}{4}y\right) - \varphi\left(\frac{1}{4}y\right)\right] + \beta\Sigma(n-1; x) , \end{aligned}$$

with $\Sigma(m;x)$ as defined by (3.5). Similarly $x + t_n^-$ and x have the same first $n+1$ entries, so that

$$e_3 \cdot [v(x+t_n^-) - v(x)] = \mu(n+1; x)\left[\varphi\left(\frac{1}{2} + y\right) - \varphi(y)\right] + \beta\Sigma(n+1; x) .$$

One can now copy all the arguments of (v) in the proof of Lemma 3.4 (this is where we need $\beta < \frac{3}{4}$), leading to

$$\max\left\{\left|e_3 \cdot [v(x+t_n^+) - v(x)]\right|, \quad \left|e_3 \cdot [v(x+t_n^-) - v(x)]\right|\right\} \geq C\,|\,\mu(n-1; x)\,| .$$

(iv) With the notation $g^{\pm}(x) = |e_3 \cdot [v(x+t_n^{\pm}) - v(x)]|^q$, it follows that

$$\begin{aligned}\int_{D_n} \max[g^+(x), g^-(x)] &\geq C \int_{D_n} \mid \mu(n-1;x) \mid^q \\ &\geq C\beta^q \frac{1}{4} 2^{-n+2} \sum_{d_1,\dots,d_{n-2}=0}^{1} \prod_{\ell=1}^{n-2} \mid \mu_{d_\ell} \mid^q \\ &\geq C' \left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]^n \\ &= C' \omega^n .\end{aligned}$$

Consequently

$$\begin{aligned}&\max\left(\int_{D_n} g^+(x)\,dx, \int_{D_n} g^-(x)\,dx\right) \\ &\geq \frac{1}{2}\int_{D_n} [g^+(x) + g^-(x)]\,dx \geq \frac{1}{2}\int_{D_n} \max[g^+(x), g^-(x)]\,dx \\ &\geq \frac{1}{2} C' \omega^n .\end{aligned}$$

Take now $t_n = t_n^+$ if $\int_{D_n} g^+(x)\,dx \geq \int_{D_n} g^-(x)\,dx$, $t_n = t_n^-$ otherwise. Then, since in either case $|t_n| \geq 2^{-n-2}$,

$$\begin{aligned}I_2(t_n, q) &\geq \int_{D_n} g^{\pm}(x)\,dx \geq C'' \left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]^n \\ &\geq C''' |t_n|^{\xi_q} .\end{aligned} \tag{5.15}$$

(v) Next we show that

$$I_3(t_n, q) \leq \frac{1}{2} I_2(t_n, q) \tag{5.16}$$

for n sufficiently large. By (5.15) and $\beta = 2^{-h}$ we have

$$I_2(t_n, q) \geq C\, 2^{-n(1+qh)} \left[1 + \left(\frac{\beta - 1/2}{\beta}\right)^q\right]^n .$$

This implies (5.16) if $n \geq \frac{\log 2 - \log C}{\log\left[1 + \left(\frac{\beta - 1/2}{\beta}\right)^q\right]}$.

(vi) Putting together (5.14), (5.15) and (5.16) we find from (5.13) that

$$I(t_n, q) \geq C\left(|t_n|^q + |t_n|^{\xi_q}\right) . \qquad \square$$

The lower and upper bounds on $I(t,q)$ allow us to prove

Theorem 5.4. *Assume that* $\frac{1}{2} < \beta < \frac{3}{4}$. *Take any* $q \geq 0$. *Then*

$$\begin{aligned}\tau(q) &:= \liminf_{|t| \to 0} \log\left[\int |\varphi(x+t) - \varphi(x)|^q\,dx\right] / \log|t| \\ &= \min\left(q, \left|\log_2\left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]\right|\right) .\end{aligned}$$

Proof. (i) Define, for the time being, $\zeta(q) = \min\left(q, \left|\log_2\left[\frac{1}{2}\beta^q + \frac{1}{2}\left(\beta - \frac{1}{2}\right)^q\right]\right|\right)$. If $|t| < \frac{1}{2}$, then Lemma 5.1 implies

$$I(t,q) \le 2C_1(q) \ \ |\log|t|\,|^{\lambda(q)} \ \ |t|^{\zeta(q)} \ ,$$

hence

$$\log I(t,q) \le \log[2C_1(q)] + \lambda(q)\log|\log|t|\,| + \zeta(q)\log|t| \ .$$

Since $\log|t| < 0$, this implies

$$\frac{\log I(t,q)}{\log|t|} \ge \zeta(q) + \lambda(q)\frac{\log|\log|t|\,|}{\log|t|} + \frac{\log[2C_1(q)]}{\log|t|} \ .$$

It follows that $\liminf\limits_{|t|\to 0} \frac{\log I(t,q)}{\log|t|} \ge \zeta(q)$.

(ii) On the other hand, Lemma 5.3 tells us that there exist $t_n \neq 0$, $|t_n| \downarrow 0$ so that

$$\log I(t_n,q) \ge \log[C_2(q)] + \zeta(q)\log t_n \ ,$$

hence

$$\frac{\log I(t_n,q)}{\log|t_n|} \le \zeta(q) + \frac{\log[C_2(q)]}{\log|t_n|} \ .$$

Together with (i), this proves $\tau(q) = \zeta(q)$. □ This then leads to the main result of this paper.

Theorem 5.5. *Choose $\beta \in \left]\frac{1}{2}, \frac{3}{4}\right[$, and let φ be the continuous L^1-function with support $[0,3]$ that solves the equation*

$$\varphi(x) = \beta\varphi(2x) + \left(\beta + \frac{1}{2}\right)\varphi(2x-1) + (1-\beta)\varphi(2x-2) + \left(\frac{1}{2} - \beta\right)\varphi(2x-3)$$

and that is normalized by $\int_0^3 \varphi(x)\,dx = 1$. Define, for α and $q \ge 0$,

$$f(\alpha) = \dim_H\left\{x \in [0,3];\ \ \liminf_{|t|\to 0}\ [\log|\varphi(x+t) - \varphi(x)|/\log|t|] = \alpha\right\}$$

$$\tau(q) = \liminf_{|t|\to 0}\ \left(\log\left[\int|\varphi(x+t) - \varphi(x)|^q\,dx\right]/\log|t|\right) \ .$$

Then $\tau(q)$ is the Legendre transform of $1 - f(\alpha)$, i.e.

$$\tau(q) = \inf_{\alpha\ge 0}\ [q\alpha + 1 - f(\alpha)] \,.$$

Proof. A simple comparison of the results in Sec. 4 and in Theorem 5.4. □

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Note Added in Proof

While this paper was in press we learned of the preprints of S. Jaffard "Multiresolution formalism for functions I. Results valid for all functions; II. Self-similar functions". It follows from (2.1), (2.12) and (2.13) that our functions are self-familiar in Jaffard's sense, so our Theorem 5.5 is included in Jaffard's results. For the specific examples here, our analysis provides more detailed information than Jaffard's.

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