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Quantum-mechanical path integrals with Wiener measure for all polynomial Hamiltonians. II

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The coherent-state representation of quantum-mechanical propagators as well-defined phase-space path integrals involving Wiener measure on continuous phase-space paths in the limit that the diffusion constant diverges is formulated and proved. This construction covers a wide class of self-adjoint Hamiltonians, including all those which are polynomials in the Heisenberg operators; in fact, this method also applies to maximal symmetric Hamiltonians that do not possess a self-adjoint extension. This construction also leads to a natural covariance of the path integral under canonical transformations. An entirely parallel discussion for spin variables leads to the representation of the propagator for an arbitrary spin-operator Hamiltonian as well-defined path integrals involving Wiener measure on the unit sphere, again in the limit that the diffusion constant diverges.

I. INTRODUCTION

For quantum systems the problem of providing a well-defined meaning for the heuristic and formal path-integral expressions for the propagator has attracted the attention of a number of workers.[1] The most commonly used prescription involves the continuum limit of a time-slicing formulation which, although perfectly correct,[2] is sometimes criticized as being far removed from the idealized desired goal of an integration over a space of paths defined for a continuous-time parameter. Unfortunately, in such quantum formulations, and unlike the Feynman–Kac formula, the orders of integration and the continuum limit cannot be interchanged to yield a formulation on continuous-time path spaces. Not only does this procedure fail for configuration-space path integrals, but seemingly even more so for the far more widely applicable phase-space path integrals.[3]

In this paper we propose an alternative to the time-slicing and continuum-limit procedure to define path integrals that leads to the quantum-mechanical propagator being given by well-defined path integrals involving Wiener measure on continuous phase-space paths in the limit that the diffusion constant diverges.[4] We are able to prove the existence of this formulation for a wide class of quantum Hamiltonians (described below) which includes all those that are polynomials in (Cartesian) P's and Q's. Indeed, our construction leads to a natural definition for the propagator even in cases where the Hamiltonian operator is maximal symmetric and admits *no* self-adjoint extension. Moreover, a formulation in terms of continuous phase-space paths permits one to make a transformation of integration variables, such as that involved in canonical transformations, with much greater care than usual (see the end of this section). We feel this possibility is just one of several advantages offered by our approach.

A. Motivation, summary of principal results, and outline of the paper

We begin by giving a heuristic overview of our formulation of quantum-mechanical phase-space path integrals. In terms of the canonical coherent states, defined in Dirac notation for all $(p,q)\in\mathbb{R}^2$ as

$$|p,q\rangle \equiv e^{i(pQ-qP)}|0\rangle,$$

where $|0\rangle$ is the normalized ground state of $(P^2+Q^2)/2$, one can write the following formal expression for the coherent state matrix elements of $\exp(-iTH)$ (see Ref. 5):

$$\langle p'',q''|e^{-iTH}|p',q'\rangle = \mathscr{N}^{-1}\int \exp\left[i\frac{1}{2}\int(p\dot q - q\dot p)dt - i\int H(p,q)dt\right]\prod_t dp(t)dq(t). \tag{1.1}$$

This is only a formal expression because there is no well-defined measure underlying this "integral"; $\mathscr{N}$ stands for a formal (actually infinite) "normalization constant." The function $H(p,q)$ was defined in Ref. 5 as the diagonal coherent state matrix element of H

$$H(p,q) = \langle p,q|H|p,q\rangle,$$

which, in the terminology of pseudodifferential operators, is equivalent with the "ordered symbol" corresponding to the operator H.

It is possible to give meaning to the formal expression (1.1) by inserting an extra factor

$$\exp\left[-\frac{1}{2\nu}\int(\dot p^2+\dot q^2)dt\right] \tag{1.2}$$

into the integrand, and redefining $\mathscr{N}$ in such a way that

$$\mathscr{N}^{-1}\exp\left[-\frac{1}{2\nu}\int(\dot p^2+\dot q^2)dt\right]\prod_t dp(t)dq(t)$$

can be interpreted as a Wiener measure with diffusion constant ν. The measure is pinned at p',q' at the initial time and at p'',q'' at the final time, a conditioning made possible by the use of the overcomplete coherent states. Since $\int(p\,dq - q\,dp)$ is a well-defined stochastic integral for this Wiener measure (and in fact the Itô and Stratonovich rules give the same result), then the function

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$$\exp\left[i\frac{1}{2}\int(p\,dq-q\,dp)-i\int H(p,q)dt\right]$$

is integrable with respect to the Wiener measure, and the resulting expression is a well-defined path integral.

In the limit $\nu\to\infty$ the extra regularizing factor (1.2) formally tends to unity and the ν-dependent path integrals revert to the original formal expression. This entirely formal argument suggests that the coherent state matrix element $\langle p'',q''|\exp(-iTH)|p',q'\rangle$ might be considered as the limit, as the diffusion constant ν tends to ∞, of well-defined phase-space path integrals with Wiener measure.

Our main result is that this heuristic argument indeed contains some truth. More precisely, we will show that

$$\langle p'',q''|\exp(-iTH)|p',q'\rangle$$
$$=\lim_{\nu\to\infty}2\pi e^{\nu T/2}\int\exp\left[i\frac{1}{2}\int(p\,dq-q\,dp)\right.$$
$$\left.-i\int h(p,q)dt\right]d\mu_W^\nu(p,q), \tag{1.3}$$

where μ_W^ν is the product of two independent Wiener measures (one in p, one in q) with diffusion constant ν, pinned at p',q' for $t=0$, and at p'',q'' for $t=T$. The normalization of the measure is given by

$$\int d\mu_W^\nu(p,q)$$
$$=[2\pi\nu T]^{-1}\exp\left\{-\frac{(p''-p')^2+(q''-q')^2}{2\nu T}\right\}. \tag{1.4}$$

Its connected covariance is (x is either p or q) ($t_1\leqslant t_2$)

$$\langle x(t_1)x(t_2)\rangle^c=\langle x(t_1)x(t_2)\rangle-\langle x(t_1)\rangle\langle x(t_2)\rangle$$
$$=\nu t_1(1-t_2/T), \tag{1.5}$$

where $\langle(\cdot)\rangle\equiv\int(\cdot)d\mu_W^\nu/\int d\mu_W^\nu$. The formula is valid for all self-adjoint Hamiltonians for which the finite linear span D_c of the harmonic oscillator eigenstates is a core, and which can be written as

$$H=\int\frac{dp\,dq}{2\pi}h(p,q)|p,q\rangle\langle p,q|. \tag{1.6}$$

The function $h(p,q)$ must satisfy, for all $\alpha>0$, the bound

$$\int dp\,dq|h(p,q)|^2\exp[-\alpha(p^2+q^2)]<\infty.$$

The class of Hamiltonians satisyfing these conditions contains all Hamiltonians polynomial in P and Q.

The function $h(p,q)$ used in the integrand in (1.3) is defined by (1.6). The relation between $h(p,q)$ and the diagonal matrix element $H(p,q)$ is given by

$$h(p,q)=\exp[-\tfrac{1}{2}(\partial_p^2+\partial_q^2)]H(p,q). \tag{1.7}$$

From (1.7) one sees that generally $h(p,q)\neq H(p,q)$; equality only holds when $H(p,q)$ is linear in p and q. In pseudodifferential operator terminology, $h(p,q)$ is equivalent to the "antiordered" symbol. From the difference between h and H one sees that (1.3) is more than just a "regularization" of (1.1) by (1.2). We shall return later (at the end of Sec. II) to the role played by $h(p,q)$.

As a matter of fact, our approach can also handle symmetric operators which are *not* self-adjoint. Formula (1.3) still holds if the closure of $H|_{D_c}$ is maximal symmetric, where we then have to write either $\exp(-iHT)$ or $\exp(-iH^*T)$ in the matrix element on the left-hand side, according to which deficiency index of $H|_{D_c}$ is zero (see Theorem 2.4 in Sec. II C). Here H is again defined by (1.6), and the growth restriction on h ensures that H is well defined on D_c.

Note also that the regularization procedure which consists of inserting terms of type (1.2) into (1.1) in order to obtain (1.3) cannot work for the ordinary configuration-space path integral (whereas we assert here that it *does* work for the coherent-state, phase-space path integral). The reason for this is that the configuration-space path integral contains (formally) factors of the type $\exp(i\frac{1}{2}\int\dot{q}^2\,dt)$ in the integrand. This cannot be regularized by inserting an extra factor $\exp(-\frac{1}{2}\nu^{-1}\int\dot{q}^2\,dt)$; an old argument[6] shows that it is impossible to define the Brownian measure with a nonreal diffusion constant [or, alternatively, $\exp(i\frac{1}{2}\int\dot{q}^2\,dt)$ is not a measurable function with respect to a Wiener measure]. One could imagine inserting $\exp(-\frac{1}{2}\nu^{-1}\int\ddot{q}^2\,dt)$; however, the additional data needed at the initial and final times are outside the scope of the configuration-space approach (it is more nearly like the coherent-state approach; compare, however, Itô, Ref. 1).

For the proof of (1.3) we shall first show that the path integral in the right-hand side of (1.3) can be considered, for finite ν, as the integral kernel of a contraction operator on $L^2(\mathbb{R}^2)$, the set of square-integrable functions on phase space. This will be done in Sec. II B, after we have defined all the necessary machinery in Sec. II A. In Sec. II C we take the limit $\nu\to\infty$, and prove (1.3) (Theorem 2.4). For reasons of simplicity we will restrict ourselves to the case of one degree of freedom, i.e., to a two-dimensional phase space. Everything we do can be trivially extended to any finite number of degrees of freedom.

In Sec. III we discuss path integrals for Hamiltonians containing spin operators. Again we consider path-integral expressions for coherent-state matrix elements of the evolution operators corresponding to these Hamiltonians. The coherent states used here are associated with SU(2) rather than with the Heisenberg group, and are labeled by elements of S^2 rather than of $\mathbb{R}^2$. In our construction we shall be able to treat an *arbitrary* Hamiltonian written, analogously to (1.6), as a superposition of diagonal dyadic operators in the spin coherent states (this representation has been studied and used before; its first use in the construction of path integrals for spin systems was by Lieb[7]). Once the appropriate definitions are formulated (Sec. III A), the analysis of Sec. II carries over to the spin case without any problem, and we therefore shall only state the result, without detailed proofs (Secs. III B and III C).

We have already announced our principal results in Ref. 4, in a slightly weaker version. The proofs outlined in Ref. 4 are, however, different from the ones we give here, though there is some connection. In the Appendix we compare the two versions, and show how our previous approach fits into the present framework.

B. Canonical transformations

As an illustration of our path-integral formalism we conclude this Introduction with a few remarks about how time-independent canonical transformations appear in our approach. For this purpose it is useful to interpret all stochastic integrals and stochastic differential equations in the sense of Stratonovich,[8] and this we shall do in this subsection. We introduce new canonical coordinates $\bar{p} = \bar{p}(p,q)$ and $\bar{q} = \bar{q}(p,q)$, which are classically connected, for example, by the relation

$$p\,dq - q\,dp = \bar{p}\,d\bar{q} - \bar{q}\,d\bar{p} + 2\,dF(\bar{p},\bar{q};p,q).$$

The stochastic variables $\bar{p}$ and $\bar{q}$ satisfy the stochastic differential equations given by

$$d\bar{p} = \frac{\partial\bar{p}}{\partial p}dp + \frac{\partial\bar{p}}{\partial q}dq, \quad d\bar{q} = \frac{\partial\bar{q}}{\partial p}dp + \frac{\partial\bar{q}}{\partial q}dq,$$

and their solution determines a new, generally non-Gaussian measure $\bar{\mu}^\nu(\bar{p},\bar{q})$ according to $d\bar{\mu}^\nu(\bar{p},\bar{q}) = d\mu_W^\nu(p,q)$. In the new canonical coordinates (1.3) becomes

$$\langle \bar{p}'',\bar{q}''|e^{-iTH}|\bar{p}',\bar{q}'\rangle = \lim_{\nu\to\infty} 2\pi e^{\nu T/2}\int \exp\Big[i\frac{1}{2}\int(\bar{p}\,d\bar{q} - \bar{q}\,d\bar{p}) - i\int \bar{h}(\bar{p},\bar{q})dt\Big]\,d\bar{\mu}^\nu(\bar{p},\bar{q}), \tag{1.8}$$

where $\bar{h}(\bar{p},\bar{q}) \equiv h(p(\bar{p},\bar{q}),q(\bar{p},\bar{q}))$, and where we have incorporated the effects of F by defining the states

$$|\bar{p},\bar{q}\rangle \equiv \exp[iF(\bar{p},\bar{q};p(\bar{p},\bar{q}),q(\bar{p},\bar{q}))]\,|p(\bar{p},\bar{q}),q(\bar{p},\bar{q})\rangle.$$

With this phase convection, (1.8) is canonically equivalent to (1.3); the phase is still given by the classical action for stochastic phase-space paths; what is different is the weighting of those paths by the integration measure. Note that the measures $\bar{\mu}^\nu$ and μ_W^ν are typically mutually singular, as is already the case if $\bar{p} = ap$, $\bar{q} = q/a$, for $a > 0$, $a \neq 1$.

It is straightforward to extend the foregoing discussion to time-dependent canonical transformations.

II. THE CANONICAL CASE

A. Definitions and basic properties

We start by a review of the definition and some of the properties of the canonical coherent states. Let $\mathscr{H}$ be a separable complex Hilbert space carrying an irreducible, strongly continuous unitary representation $W(p,q)$ of the Weyl commutation relations

$$W(p',q')W(p'',q'') = \exp[i\tfrac{1}{2}(p'q'' - p''q')]W(p'+p'',q'+q'').$$

The position operator Q and the momentum operator P are the infinitesimal generators of the strongly continuous unitary groups $W(p,0)$, $W(0,-q)$, respectively; one has

$$W(p,q) = \exp[i(pQ - qP)] = \exp(-i\tfrac{1}{2}pq)\exp(ipQ)\exp(-iqP). \tag{2.1}$$

We define $\omega \in \mathscr{H}$ to be the normalized ground state of the harmonic oscillator Hamiltonian

$$\tfrac{1}{2}(P^2 + Q^2 - 1)\omega = 0.$$

The canonical coherent states (cs) are defined as

$$\omega^{p,q} = W(p,q)\omega.$$

They form an overcomplete set of vectors in $\mathscr{H}$ with "overlap function"

$$\langle\omega^{p'',q''},\omega^{p',q'}\rangle = \exp[i\tfrac{1}{2}(p'q'' - p''q') - \tfrac{1}{4}(p''-p')^2 - \tfrac{1}{4}(q''-q')^2]. \tag{2.2}$$

They also give rise to the following "resolution of unity":

$$\int \frac{dp\,dq}{2\pi}\langle\psi,\omega^{p,q}\rangle\langle\omega^{p,q},\phi\rangle = \langle\psi,\phi\rangle. \tag{2.3}$$

This can be viewed as a special case of

$$\int \frac{dp\,dq}{2\pi}\langle\psi_1,W(p,q)\psi_2\rangle\langle W(p,q)\phi_1,\phi_2\rangle = \langle\psi_1,\phi_2\rangle\langle\phi_1,\psi_2\rangle. \tag{2.4}$$

Note: In the usual Schrödinger representation, one has $\mathscr{H} = L^2(\mathbb{R})$. The $W(p,q)$ act then as follows:

$$[W(p,q)f](x) = \exp(-i\tfrac{1}{2}pq + ipx)f(x-q).$$

The vector $\omega^{p,q}$ is given by the familiar functions

$$\omega^{p,q}(x) = \pi^{-1/4}\exp[-i\tfrac{1}{2}pq + ipx - \tfrac{1}{2}(x-q)^2].$$

Setting $p = q = 0$ gives $\omega(x)$.

We shall also use the harmonic oscillator excited states ω_k, defined by

$$\tfrac{1}{2}(P^2 + Q^2 - 1)\omega_k = k\omega_k. \tag{2.5}$$

In analogy with the definition of the cs we define

$$\omega_k^{p,q} = W(p,q)\omega_k.$$

In order to alleviate many of the expressions in what follows, we shall often make use of Dirac's bra-ket notation in scalar products, matrix elements, and dyadic operators involving the coherent states. We shall write, e.g.,

$$\begin{aligned}
&\langle p,q|\phi\rangle \equiv \langle\omega^{p,q},\phi\rangle \quad (\phi\in\mathscr{H}),\\
&\langle k|\phi\rangle \equiv \langle\omega_k,\phi\rangle,\\
&\langle p,q;k|\phi\rangle \equiv \langle\omega_k^{p,q},\phi\rangle,\\
&\langle p'',q''|p',q'\rangle \equiv \langle\omega^{p'',q''},\omega^{p',q'}\rangle,\\
&|p,q\rangle\langle p,q| \equiv \omega^{p,q}\langle\omega^{p,q},\cdot\,\rangle,\\
&|p,q;k\rangle\langle p,q;l| \equiv \omega_k^{p,q}\langle\omega_l^{p,q},\cdot\,\rangle.
\end{aligned}$$

In these notations (2.3), e.g., can be written as

$$\int \frac{dp\,dq}{2\pi}|p,q\rangle\langle p,q| = \mathbf{1}_{\mathscr{H}}, \tag{2.6}$$

where the integral converges weakly, according to (2.3). As a matter of fact, (2.6) also converges strongly; see, e.g., the remark following Lemma 2.3 in Sec. II C. Equation (2.4) implies

$$\int \frac{dp\,dq}{2\pi}|p,q;k\rangle\langle p,q;l| = \delta_{kl}\mathbf{1}_{\mathscr{H}}. \tag{2.7}$$

For matrix elements $\langle\omega^{p'',q''},A\,\omega^{p',q'}\rangle$ we shall use the notation $\langle p'',q''|A\,p',q'\rangle$ rather than the more common bra-ket notation $\langle p'',q''|A|p',q'\rangle$ (i.e., we use a space instead of the second vertical bar) in order to avoid confusion in case A is not symmetric.

In the next section we shall interpret the right-hand side of (1.3) as the integral kernel of an operator on $L^2(\mathbb{R}^2)$; this operator can be constructed explicitly, and its limit for $\nu\to\infty$ can then be taken later. In order to do all this, we shall need the following definitions and constructions.

We shall use the notation $L^2(V)$ for the Hilbert space $L^2(\mathbb{R}^2)$ with the normalization

$$\|f\|^2=\int\frac{dp\,dq}{2\pi}\,|f(p,q)|^2.$$

For $\psi\in\mathscr{H}$, we shall denote by f_ψ the function

$$f_\psi(p,q)=\langle p,q|\psi\rangle.$$

It follows from (2.3) that the map $\psi\to f_\psi$ is isometric from $\mathscr{H}$ into $L^2(V)$; the image of $\mathscr{H}$ under this map is a closed subspace $\mathscr{H}_o$ of $L^2(V)$. The properties of $\mathscr{H}_o$ are well known[9]; its elements are products of analytic functions in $p+iq$ with the Gaussian $\exp[-\frac{1}{4}(p^2+q^2)]$. We shall denote the isomorphism between $\mathscr{H}$ and $\mathscr{H}_o$ by U

$$U:\mathscr{H}\to\mathscr{H}_o,\quad (U\psi)(p,q)=\langle p,q|\psi\rangle. \tag{2.8}$$

We shall also make use of the operator $\hat{U}:\mathscr{H}\to L^2(V)$, which is defined as $\hat{U}=I\circ U$, where I is the natural embedding of $\mathscr{H}_o$ into $L^2(V)$. The orthogonal projection operator in $L^2(V)$, onto $\mathscr{H}_o$, will be denoted by P_o.

Define also

$$h_{kl}(p,q)\equiv\langle W(p,q)\omega_k,\omega_l\rangle=\langle p,q;k\,|l\,\rangle. \tag{2.9}$$

These functions can be explicitly calculated; they are related to the generalized Laguerre functions, and can all be written as the product of a polynomial in p,q with $\exp[-(p^2+q^2)/4]$. One easily sees from (2.4) that the h_{kl} are orthonormal in $L^2(V)$; as a matter of fact, they form a complete orthonormal basis for $L^2(V)$ (see Ref. 10). From (2.8) one then sees that the h_{ol} are a complete orthonormal basis for $\mathscr{H}_o$. We shall use the notation D for the set of finite linear combinations of the h_{kl}. Note that for any $\psi\in\mathscr{H}$

$$\begin{aligned}\langle h_{kl},\hat{U}\psi\rangle&=\int\frac{dp\,dq}{2\pi}\,\langle l\,|p,q;k\,\rangle\langle p,q;0|\psi\rangle\\&=\delta_{ko}\langle l\,|\psi\rangle.\end{aligned}$$

Suppose that R is a (bounded) operator on $\mathscr{H}$. The unitary map U transports this operator to URU^{-1} on $\mathscr{H}_o$. A simple way to extend URU^{-1} to all of $L^2(V)$ is to "fill in zeros," i.e., we define $\hat{R}$ on $L^2(V)$ such that

$$\begin{aligned}&\hat{R}f=0,\quad \text{if } f\perp\mathscr{H}_o,\\&\hat{R}f=\hat{U}RU^{-1}f,\quad \text{if } f\in\mathscr{H}_o.\end{aligned} \tag{2.10}$$

It turns out that $\hat{R}$ is an integral operator on $L^2(V)$

$$\begin{aligned}&\langle h_{kl},\hat{R}h_{rs}\rangle\\&\quad=\delta_{ok}\delta_{or}\langle h_{ol},\hat{R}h_{os}\rangle\\&\quad=\delta_{ok}\delta_{or}\langle\omega_l,R\omega_s\rangle\\&\quad=\delta_{ok}\delta_{or}\int\frac{dp''\,dq''}{2\pi}\int\frac{dp'\,dq'}{2\pi}\\&\qquad\times\langle l\,|p'',q''\rangle\langle p'',q''|R\,p',q'\rangle\langle p',q'|s\rangle\\&\quad=\int\frac{dp''\,dq''}{2\pi}\int\frac{dp'\,dq'}{2\pi}\\&\qquad\times\langle l\,|p'',q'';k\,\rangle\langle p'',q'';0|R\,p',q',0\rangle\langle p',q';r|s\rangle\\&\quad=\int\frac{dp''\,dq''}{2\pi}\int\frac{dp'\,dq'}{2\pi}\\&\qquad\times\overline{h_{kl}(p'',q'')}\langle p'',q''|R\,p',q'\rangle h_{rs}(p',q').\end{aligned} \tag{2.11}$$

Hence $\hat{R}$ has integral kernel $\langle p'',q''|R\,p',q'\rangle$; note that this integral kernel (since it is a cs matrix element in $\mathscr{H}$) is a smooth function of p'',q'',p',q'.

For H a (possibly unbounded) self-adjoint operator on $\mathscr{H}$, we have

$$\widehat{(e^{itH})}=P_oe^{it\hat{H}}P_o \tag{2.12}$$

(here we have extended our construction of $\hat{H}$ to unbounded operators; for the Hamiltonians we shall consider, however, this is not a problem). It is necessary to introduce P_o in (2.12) because $\hat{H}(1-P_o)=0$, hence $[\exp(-it\hat{H})](1-P_o)=1-P_o$, whereas $\widehat{[\exp(-itH)]}(1-P_o)=0$ [see (2.10)].

For the special class of operators R and H, which can be written as

$$R=\int\frac{dp\,dq}{2\pi}\,r(p,q)|p,q\rangle\langle p,q|,$$

another natural extension, different from $\hat{R}$, is possible. Define the multiplication operator

$$R_V:L^2(V)\to L^2(V),\quad (R_Vf)(p,q)=r(p,q)f(p,q). \tag{2.13}$$

Then

$$\begin{aligned}\langle h_{ol},R_Vh_{os}\rangle&=\int\frac{dp\,dq}{2\pi}\,\langle l\,|p,q\rangle r(p,q)\langle p,q|s\rangle\\&=\langle l\,|R\,s\rangle,\end{aligned}$$

which shows that $P_oR_VP_o|_{\mathscr{H}_o}=URU^{-1}$, hence

$$P_oR_VP_o=\hat{R}. \tag{2.14}$$

We are now ready to tackle our path integral. In the next subsection we shall see that, for finite ν, the path integral in the right-hand side of (1.3) can be interpreted as the integral kernel of an operator on $L^2(V)$, which we can construct explicitly.

B. Interpretation of the path integral (for finite ν) as an integral kernel on $L^2(V)$

Let us introduce the symbol $\mathscr{P}_\nu(h)$ for the expression in the right-hand side of (1.3)

$$\begin{aligned}&\mathscr{P}_\nu(h;p'',q'',t'';p',q',t')\\&\quad=2\pi e^{\nu(t''-t')/2}\int\exp\Big[i\frac{1}{2}\int(p\,dq-q\,dp)\\&\qquad-i\int h(p,q)dt\Big]\,d\mu_W^\nu(p,q),\end{aligned} \tag{2.15}$$

where again the measure μ_W^ν is the product of two independent Wiener measures with diffusion constant ν, and pinned at p',q' for $t=t'$ and at p'',q'' for $t=t''$, respectively $(t''>t')$.

If we put $h=0$, $\mathscr{P}_\nu$ can be calculated explicitly; the result is (for the case $\nu=1$, this calculation was carried out in Ref. 11)

$$\mathscr{P}_\nu(h=0;p'',q'',t'',p',q',t')$$
$$=\frac{e^{\nu(t''-t')/2}}{2\sinh[\nu(t''-t')/2]}\exp\left\{\frac{i}{2}(p'q''-p''q')-\frac{1}{4}\coth\frac{\nu(t''-t')}{2}\times[(p''-p')^2+(q''-q')^2]\right\}. \tag{2.16}$$

By their definition, these $\mathscr{P}_\nu(h=0)$ have a semigroup property, as can also be checked by explicit calculation

$$\int\frac{dp'\,dq'}{2\pi}\mathscr{P}_\nu(h=0;p'',q'',t'';p',q',t')\times\mathscr{P}_\nu(h=0;p',q',t';p,q,t)=\mathscr{P}_\nu(h=0;p'',q'',t'',p,q,t). \tag{2.17}$$

From (2.16) one sees that

$$|\mathscr{P}_\nu(h=0;p'',q'',t'';p',q',t')|<e^{\nu(t''-t')/2}[\nu(t''-t')]^{-1}\times\exp\{-[(p''-p')^2+(q''-q')^2]/[2\nu(t''-t')]\}. \tag{2.18}$$

Since for all $\alpha>0$ the function $\rho_\alpha(p,q)=\exp[-\alpha(p^2+q^2)]$ is in $L^1(\mathbf{R}^2)$, the upper bound (2.18) implies that for $t''\neq t'$, $\mathscr{P}_\nu(h=0)$ is the integral kernel of a bounded operator $E^0(\nu;t'',t')$ on $L^2(V)$

$$\mathscr{P}_\nu(h=0;p'',q'',t'';p',q',t')=[E^0(\nu;t'',t')](p'',q'';p',q'). \tag{2.19}$$

The bound (2.18) implies $\|E^0(\nu;t'',t')\|<e^{\nu(t''-t')}$. From the semigroup relation (2.17) we see that

$$E^0(\nu;t'',t)E^0(\nu;t,t')=E^0(\nu;t'',t'). \tag{2.20}$$

It is also easy to check from (2.16) that

$$[E^0(\nu;t'',t')]^*=E^0(\nu;t'',t'). \tag{2.21}$$

As t'' tends to $t'(t''\to t')$, it is clear from (2.16) that $\mathscr{P}_\nu(h=0;p'',q'',t'';p',q',t')$ tends to $2\pi\delta(p''-p')\delta(q''-q')$ in the sense of distributions. Using (2.20), (2.21), and $\|E^0(\nu;t'',t')\|\leqslant e^{\nu(t''-t')}$, this implies s-$\lim_{t''\to t'}E^0(\nu;t'',t')=\mathbf{1}$ on $L^2(V)$. Moreover, one sees from the explicit expression (2.16) that the $\mathscr{P}_\nu(h=0)$ depend on the initial and final times t',t'' only through the difference $(\nu t''-\nu t')$. Putting all this together, we conclude that, for fixed ν, the $E^0(\nu,t)\equiv E^0(\nu;t,0)$ form a strongly continuous semigroup (in t) of bounded operators. Hence

$$E^0(\nu;t'',t')=E^0(\nu,t''-t')=\exp[-\nu A(t''-t')], \tag{2.22}$$

where A is a self-adjoint operator on $L^2(V)$. The bound $\|E^0(\nu,t)\|\leqslant e^{\nu t}$ implies $A\geqslant-1$. The operator A can be calculated explicitly from (2.16). One finds

$$A=\tfrac{1}{2}[-(\partial_p^2+\partial_q^2)+i(p\,\partial_q-q\,\partial_p)+\tfrac{1}{4}(p^2+q^2)-1]=\tfrac{1}{2}[(-i\,\partial_p+q/2)^2+(-i\,\partial_q-p/2)^2-1]. \tag{2.23}$$

It is particularly interesting to note, if A is interpreted as a Hamiltonian on $L^2(\mathbf{R}^2)$, that it describes a two-dimensional particle in the presence of a constant magnetic field orthogonal to the plane of motion. Indeed, exactly such a Hamiltonian arises in the two-dimensional quantized Hall effect, and good use has been made of coherent state techniques in the study of this problem.[12] Because of this magnetic field analogy, we know immediately that A has a purely discrete spectrum (the Landau levels for the corresponding magnetic field). As a matter of fact, we already have an expression for a set of eigenvalues and eigenvectors for A. Recalling the definition (2.9) of the h_{kl}, and using the definitions (2.5) and (2.1) of the ω_k and $W(p,q)$, respectively, one finds

$$Ah_{kl}=kh_{kl}. \tag{2.24}$$

This will be useful for our analysis of the $\nu\to\infty$ limit below. It follows immediately from (2.24) that D, the set of finite linear combinations of the h_{kl}, is a core for A. Note that (2.24) also implies $A\geqslant0$. We have therefore $\|E^0(\nu,t)\|=\|\exp(-\nu At)\|\leqslant1$, which means that the $E^0(\nu,t)$ are a strongly continuous *contraction* semigroup.

Let us now look at the case where h is not identically zero. We shall consider functions h satisfying, for all $\alpha>0$, the condition

$$\int dp\,dq|h(p,q)|^2\exp[-\alpha(p^2+q^2)]<\infty. \tag{2.25}$$

This is automatically fulfilled if, e.g.,

$$\int dp\,dq|h(p,q)|^2\exp[-\beta(p^2+q^2)^\gamma]<\infty,$$

for some $\beta>0$ and $0<\gamma<1$.

Condition (2.25) ensures that the path integral (2.15) is well defined. To see this, we only need to check that $|\int_0^T h(p(t),q(t))dt|$ is finite for almost all paths $(p(t),q(t))$ in the support of μ_W^ν. This is certainly true if

$$\int\left\{\int_0^T|h(p(t),q(t))|dt\right\}d\mu_W^\nu(p,q)<\infty.$$

Using the definition of μ_W^ν, we can rewrite this condition as

$$\int_0^T dt\int dp\,dq|h(p,q)|(2\pi\nu t)^{-1}\times\exp\left\{-\frac{(p-p')^2+(q-q')^2}{2\nu t}\right\}\times[2\pi\nu(T-t)]^{-1}\exp\{-[(p-p'')^2+(q-q'')^2]/[2\nu(T-t)]\}<\infty. \tag{2.26}$$

For h satisfying condition (2.25), we can use the Cauchy–Schwarz inequality to bound the left-hand side of (2.26) by

$$C\int_0^T t^{-1}(T-t)^{-1}\int dp\,dq\exp\left\{-\frac{(p-p')^2+(q-q')^2}{\nu t}-\frac{(p-p'')^2+(q-q'')^2}{\nu(T-t)}+\frac{(p^2+q^2)}{\nu T}\right\}$$
$$<C\int_0^T dt\,\nu T[T^2-t(T-t)]^{-1}\times\exp\left\{\frac{2T(|p'||p''|+|q'||q''|)}{\nu[T^2-t(T-t)]}\right\}. \tag{2.27}$$

Since, for $t \in [0,T]$, $T^2 - t(T-t) \geqslant 3T^2/4$, one immediately sees that expression (2.27) is finite, and hence that $\mathscr{P}_\nu(h)$ is well defined for all p',q',p'',q''. From (2.15) we see then that, for all h satisfying (2.25),

$$|\mathscr{P}_\nu(h;p'',q'',t'';p',q',t')|$$
$$<2\pi e^{\nu(t''-t')/2}\int d\mu_W^\nu(p,q)$$
$$<\frac{e^{\nu(t''-t')/2}}{\nu(t''-t')}\exp\left\{-\frac{(p''-p')^2+(q''-q')^2}{2\nu(t''-t')}\right\}. \tag{2.28}$$

Since h is time independent, $\mathscr{P}_\nu(h)$ will depend on t'',t' only through the difference $t''-t'$. Together with (2.28) this implies that $\mathscr{P}_\nu(h)$ is the integral kernel of a bounded operator $E(\nu,h;t''-t')$ on $L^2(V)$

$$\mathscr{P}_\nu(h;p'',q'',t'';p',q',t') = [E(\nu,h;t''-t')](p'',q'';p',q'), \tag{2.29}$$

with

$$\|E(\nu,h;t)\| < e^{\nu t/2}. \tag{2.30}$$

From the path integral definition (2.15) one immediately sees that the semigroup property (2.17) for the $\mathscr{P}_\nu(h)$ also holds for $h \neq 0$. This implies $[t_1,t_2 \geqslant 0; E(\nu,h;0) \equiv \mathbb{1}]$

$$E(\nu,h;t_1)E(\nu,h;t_2) = E(\nu,h;t_1+t_2), \tag{2.31}$$

i.e., the $E(\nu,h;t)$ form a semigroup.

We want to show that the $E(\nu,h;t)$ actually form a *strongly continuous* semigroup of *contractions* on $L^2(V)$ [which means we have to do better than (2.30)!]. In order to do this, we shall proceed in several steps. We shall first consider h in C_0^∞, the C^∞ functions vanishing at ∞. Then we shall extend our results to bounded h, and in a third step we generalize to all h satisfying condition (2.25).

For any h satisfying (2.25), we define H_V to be [as in (2.13)] the multiplication operator by $h(p,q)$

$$(H_V f)(p,q) = h(p,q)f(p,q). \tag{2.32}$$

We shall always assume that h is a real function, which implies that H_V is self-adjoint. If h is not bounded, the domain $D(H_V)$ of H_V consists of all $f \in L^2(V)$ for which hf is still square integrable. Because of the special form of the h_{kl} we have

$$|h_{kl}(p,q)| \leqslant C(1+p^2+q^2)^n \exp[-(p^2+q^2)/4]$$

(C and n depend on k and l); together with (2.25) this implies that $h_{kl} \in D(H_V)$ for all k,l.

Let us now consider $h \in C_0^\infty$. Then H_V is a bounded operator. On the domain $D(A)$ of A we can define $\nu A + iH_V$. This is a closed operator, which is the generator of a strongly continuous contraction semigroup (both νA and iH_V are generators, and H_V is A bounded with relative bound zero; see, e.g., Kato's book,[13] p. 499). The integral kernel of $\exp[-(\nu A + iH_V)T]$ is given by the Trotter product formula

$$\{\exp[-(\nu A + iH_V)T]\}(p'',q'';p',q')$$
$$= \lim_{N\to\infty}\int\cdots\int\prod_{j=0}^N\{[\exp(-\nu A\epsilon)](p_{j+1},q_{j+1};p_j,q_j)$$
$$\times\exp[-ih(p_j,q_j)\epsilon]\}\prod_{j=1}^N\left(\frac{dp_j\,dq_j}{2\pi}\right)$$
$$= \lim_{N\to\infty} 2\pi\int\cdots\int\prod_{j=0}^N\left(\frac{\exp(\nu\epsilon/2)}{4\pi\sinh(\nu\epsilon/2)}\right.$$
$$\times\exp\{(i/2)(p_jq_{j+1}-p_{j+1}q_j)$$
$$-\tfrac14\coth(\nu\epsilon/2)[(p_{j+1}-p_j)^2+(q_{j+1}-q_j)^2]\}$$
$$\left.\times\exp[-ih(p_j,q_j)\epsilon]\right)\prod_{j=1}^N(dp_j\,dq_j)$$

[use (2.16) and (2.17)].

Here we have used the notations $p_o = p'$, $q_o = q'$, $p_{N+1} = p''$, $q_{N+1} = q''$, and $\epsilon \equiv T/(N+1)$. In the limit for $N\to\infty$, we can replace $[\sinh(\nu\epsilon/2)]^{-1}$ and $\coth(\nu\epsilon/2)$ by their first-order approximation $2/\epsilon\nu$ (higher-order terms do not contribute in the limit). This leads to

$$\{\exp[-(\nu A + iH_V)T]\}(p'',q'';p',q')$$
$$= 2\pi e^{\nu T/2}\lim_{N\to\infty}\int\cdots\int\prod_{j=0}^N\left(\exp\left\{\frac12 i[p_j(q_{j+1}-q_j)-q_j(p_{j+1}-p_j)]\right\}\exp[-ih(p_j,q_j)\epsilon]\right)$$
$$\times\prod_{j=0}^N\left([2\pi\nu\epsilon]^{-1}\exp\left\{-\frac{(q_{j+1}-q_j)^2+(p_{j+1}-p_j)^2}{2\nu\epsilon}\right\}\right)\prod_{j=1}^N(dp_j\,dq_j)$$
$$= 2\pi e^{\nu T/2}\int\exp\left[i\frac12\int(p\,dq-q\,dp)-i\int h(p,q)dt\right]d\mu_W^\nu(p,q)$$
$$= \mathscr{P}_\nu(h;p'',q'',T;p',q',0). \tag{2.33}$$

Here we have used the continuity of h in the limit so that as $N\to\infty$,

$$\exp\left[-i\sum_{j=0}^N h(p_j,q_j)\epsilon\right]\to\exp\left[-i\int h(p,q)dt\right].$$

Comparing (2.33) with (2.29) we see that for $h \in C_0^\infty$,

$$E(\nu,h;t) = \exp[-(\nu A + iH_V)t]. \tag{2.34}$$

This implies that for h in C_0^∞ we have achieved our goal: the $E(\nu,h;t)$ form a strongly continuous contraction semigroup.

For h in L^∞, the operator H_V defined by (2.32) is still bounded. The operator $\nu A + iH_V$, defined on $D(A)$, is therefore still a generator of a strongly continuous contraction semigroup. We can find functions h_n in C_0^∞ such that $|h_n(p,q)| \leqslant \|h\|_\infty$ for all p,q and $h_n(p,q)\to h(p,q)$ almost everywhere (a.e.). By the dominated convergence theorem, one sees immediately from (2.15) that this implies $\mathscr{P}_\nu(h_n)$

$\to_{n\to\infty} \mathscr{P}_\nu(h)$ pointwise. On the other hand, $(\nu A + iH_{V,n})f \to (\nu A + iH_V)f$ for all $f\in D(A)$, and therefore $\nu A + iH_{V,n}$ converges to $\nu A + iH_V$ in the strong resolvent sense (see Ref. 13, Theorem VIII 1.5), where we have used the notation $H_{V,n}$ for the multiplication operator by h_n on $L^2(V)$. Hence

$$\text{s-}\lim_{n\to\infty} \exp[-(\nu A + iH_{V,n})t] = \exp[-(\nu A + iH_V)t].$$

Using (2.28) [note that this upper bound on $\mathscr{P}_\nu(h)$ is independent of h!] one can therefore apply the dominated convergence theorem to see that $[f, g \in L^2(V)]$

$$\begin{aligned}
&\langle f, \exp[-(\nu A + iH_V)t]\, g\rangle \\
&= \lim_{n\to\infty} \langle f, \exp[-(\nu A + iH_{V,n})t]\, g\rangle \\
&= \lim_{n\to\infty} \int \frac{dp''\, dq''}{2\pi} \int \frac{dp'\, dq'}{2\pi}\, \overline{f(p'',q'')} \\
&\quad \times \mathscr{P}_\nu(h_n; p'',q'',t; p',q',0)\, g(p',q') \\
&= \int \frac{dp''\, dq''}{2\pi} \int \frac{dp'\, dq'}{2\pi}\, \overline{f(p'',q'')} \\
&\quad \times \mathscr{P}_\nu(h; p'',q'',t; p',q',0)\, g(p',q') \\
&= \langle f, E(\nu,h;t) g\rangle.
\end{aligned}$$

Hence $E(\nu,h;t) = \exp[-(\nu A + iH_V)t]$ for $h\in L^\infty$, which implies again that the $E(\nu,h;t)$ form a strongly continuous contraction semigroup, now for all $h\in L^\infty$.

Finally, let us take a general function h satisfying (2.25). Let $h_n(p,q)$ be defined as

$$\begin{aligned}
&h_n(p,q) = h(p,q), \quad \text{if } |h(p,q)| \leqslant n, \\
&h_n(p,q) = 0, \quad \text{otherwise.}
\end{aligned} \tag{2.35}$$

Clearly $\lim_{n\to\infty} h_n(p,q) = h(p,q)$ (h_n converges pointwise to h, a.e.), while $|h_n(p,q)| \leqslant |h(p,q)|$ for all p,q. By the dominated convergence theorem we have therefore, for every path $[p(t),q(t)]$ for which $\int_0^T |h(p,q)|dt$ is finite, that

$$\lim_{n\to\infty} \int_0^T h_n(p,q)dt = \int_0^T h(p,q)dt. \tag{2.36}$$

Since $\int_0^T |h(p,q)|dt$ is finite a.e. with respect to μ_W^ν (see above), (2.36) implies, again by the dominated convergence theorem, that for all p',q',p'',q''

$$\begin{aligned}
&\lim_{n\to\infty} \mathscr{P}_\nu(h_n; p'',q'',T; p',q',0) \\
&\quad = \mathscr{P}_\nu(h; p'',q'',T; p',q',0).
\end{aligned} \tag{2.37}$$

Take now any $f\in L^2(V)$. We have then

$$\begin{aligned}
&\|[E(\nu,h;t) - E(\nu,h_n;t)]f\|^2 \\
&= \int \frac{dp\, dq}{2\pi} \int \frac{dp_1\, dq_1}{2\pi} \int \frac{dp_2\, dq_2}{2\pi}\, \overline{f(p_1,q_1)} \\
&\quad \times \overline{[\mathscr{P}_\nu(h; p,q,t; p_1,q_1,0) - \mathscr{P}_\nu(h_n; p,q,t; p_1,q_1,0)]} \\
&\quad \times [\mathscr{P}_\nu(h; p,q,t; p_2,q_2,0) - \mathscr{P}_\nu(h_n; p,q,t; p_2,q_2,0)] \\
&\quad \times f(p_2,q_2).
\end{aligned}$$

Using the pointwise convergence (2.37) and the upper bound (2.28) one sees that this integral converges to zero for $n\to\infty$, by the dominated convergence theorem. Hence

$$\text{s-}\lim_{n\to\infty} E(\nu,h_n;t) = E(\nu,h;t). \tag{2.38}$$

Since $h_n \in L^\infty$ for all n, the $E(\nu,h_n;t)$ are contraction operators. Hence the strong convergence (2.38) implies

$$\|E(\nu,h;t)\| \leqslant 1. \tag{2.39}$$

Taking (2.31) into account, we therefore only need to prove still that

$$\text{s-}\lim_{t\to 0} E(\nu,h;t) = 1 \tag{2.40}$$

in order to conclude that the $E(\nu,h;t)$ form a strongly continuous contraction semigroup. Since the $E(\nu,h;t)$ are uniformly bounded, and since D, the set of finite linear combinations of the h_{kl}, is dense, it is sufficient to prove, for all k,l,

$$\lim_{t\to 0} \|E(\nu,h;t)h_{kl} - h_{kl}\| = 0. \tag{2.41}$$

To prove (2.41) we shall again use the L^∞ functions h_n defined by (2.35). Since $h_n \in L^\infty$, we know that $E(\nu,h_n;t) = \exp[-(\nu A + iH_{V,n})t]$. Fix k,l. Since $h_{kl} \in D(A) = D(\nu A + iH_{V,n})$, $G_n(t) \equiv E(\nu,h_n;t)h_{kl}$ is differentiable, and

$$\begin{aligned}
\left\|\frac{d}{dt} G_n(t)\right\| &= \|E(\nu,h_n;t)[\nu A + iH_{V,n}]h_{kl}\| \\
&\leqslant \nu k + \||h_n| h_{kl}\| \leqslant \nu k + \||h|\, h_{kl}\|,
\end{aligned} \tag{2.42}$$

where we have used $\|E(\nu,h_n;t)\| \leqslant 1$, $Ah_{kl} = kh_{kl}$, and $|h_n(p,q)| \leqslant |h(p,q)|$. The fact that the upper bound (2.42) on $\|(d/dt)G_n(t)\|$ is independent of n implies that the G_n form an equicontinuous family of vector-valued functions of t. Since $G_n(t)$ converges to $E(\nu,h;t)h_{kl}$ for every t, the equicontinuity of the G_n implies that $E(\nu,h;t)h_{kl}$ is continuous in t. We have thus proved (2.41), and hence (2.40).

We have now achieved our goal, i.e., we have shown that for all functions h satisfying (2.25), $\mathscr{P}_\nu(h; p'',q'',t; p',q',0)$ is the integral kernel for a strongly continuous contraction semigroup $E(\nu,h;t)$. This contraction semigroup can be considered as a "perturbation" of the semigroup $E^0(\nu,T) = \exp(-\nu At)$. If h is a bounded function, the multiplication operator H_ν is bounded, and one sees from (2.34) that $E(\nu,h;t)$ satisfies the integral equation

$$E(\nu,h;T) = E^0(\nu,T) - i\int_0^T ds\, E(\nu,h;T-s) H_V E^0(\nu,s). \tag{2.43}$$

For a general unbounded h, (2.43) cannot be written as an operator equation, because of domain problems. However, we shall see that (2.43) is still true in a "weak" sense for functions h satisfying (2.25).

Take any (real) function h satisfying (2.25). Let $[p(t),q(t)]$ be a path in the support of μ_W^ν for which $\int_0^T |h(p,q)|dt$ is finite. Then $F(s) = \exp[-i\int_s^T h(p,q)dt]$ is a function of bounded variation on the interval $[0,T]$. Hence, F is differentiable a.e. and by the fundamental theorem of calculus

$$F'(s) = ih(p(s),q(s))F(s), \quad \text{a.e.}$$

Hence

$$F(T) = F(0) + i\int_0^T ds\, h(p(s),q(s))F(s),$$

or

$$\exp\left[-i\int_0^T h(p,q)dt\right]$$
$$= 1 - i\int_0^T ds\, h(p(s),q(s))\exp\left[-i\int_s^T h(p,q)dt\right].$$

Since for h satisfying (2.25), $\int_0^T |h(p,q)|dt$ is finite a.e. with respect to μ_W^ν, we can insert the above expression into the definition (2.15) of $\mathscr{P}_\nu(h)$, which gives

$$\mathscr{P}_\nu(h;p'',q'',T;p',q',0)$$
$$= \mathscr{P}_\nu(h=0;p'',q'',T;p',q',0) - i2\pi e^{\nu T/2}$$
$$\times \int \left\{ \int_0^T ds\, h(p(s),q(s)) \exp\left[i\frac{1}{2}\int (p\,dq - q\,dp)\right]\right.$$
$$\left.\times \exp\left[-i\int_s^T h(p,q)dt\right]\right\} d\mu_{W;p',q',0}^{\nu;p'',q'',T}(p,q). \qquad (2.44)$$

In order to avoid confusion we have, for this computation only, explicitly labeled the Brownian bridge measure μ_W^ν by its initial and final times, together with the pinned values of p,q at these times. For every s in $(0,T)$ we can write

$$d\mu_{W;p',q',0}^{\nu;p'',q'',T} = \int dp\,dq\, d\mu_{W;p,q,s}^{\nu;p'',q'',T} \otimes d\mu_{W;p',q',0}^{\nu;p,q,s}.$$

We insert this into (2.44). The multiple integral we thus obtain is absolutely convergent if h satisfies (2.25) [see (2.26)]. We are therefore allowed to change the order of the integrations, which yields

$$\mathscr{P}_\nu(h;p'',q'',T;p',q',0)$$
$$= \mathscr{P}_\nu(h=0;p'',q'',T;p',q',0)$$
$$- i\int_0^T ds \int \frac{dp\,dq}{2\pi}\,\mathscr{P}_\nu(h;p'',q'',T;p,q,s)$$
$$\times h(p,q)\mathscr{P}_\nu(h=0;p,q,s;p',q',0). \qquad (2.45)$$

For $f_1,f_2 \in D$, we multiply this expression by $\overline{f_1(p'',q'')}f_2(p',q')$, and integrate over p'',q'',p',q'. We can use the fact that for some C,n, $|f_j(p,q)| < C(1+p^2+q^2)^n \exp[-(p^2+q^2)/4]$ $(j=1,2)$ to show that the multiple integral converges absolutely; we may therefore again change the order of the integrations, which leads, for all $f_1,f_2 \in D$, to

$$\langle f_1, E(\nu,h;T)f_2\rangle = \langle f_1, E^o(\nu,T)f_2\rangle$$
$$- i\int_0^T ds \langle f_1, E(\nu,h;T-s)H_V E^o(\nu,s)f_2\rangle. \qquad (2.46)$$

Formula (2.46) holds for *all* functions h satisfying (2.25). Note that $E^o(\nu,t)$ leaves D invariant; since $D \subset D(H_V)$, all the terms in (2.46) are well defined.

Putting together all the preceding results, we see that we have proved the following proposition.

Proposition 2.1: Let h be a real function satisfying, for all $\alpha > 0$,

$$\int dp\,dq |h(p,q)|^2 \exp[-\alpha(p^2+q^2)] < \infty.$$

Then

$$\mathscr{P}_\nu(h;p'',q'',T;p',q',0)$$
$$= 2\pi e^{\nu T/2}\int \exp\left[i\frac{1}{2}\int (p\,dq - q\,dp)\right.$$
$$\left.- i\int h(p,q)dt\right] d\mu_W^\nu$$

is well defined. Here μ_W^ν is a Gaussian measure completely determined by its normalization (1.4) and its connected covariance (1.5). Moreover, there exists a strongly continuous contraction semigroup $E(\nu,h;t)$ on $L^2(V)$ such that we have the following.

(1) $E(\nu,h;t)$ is an integral operator, with kernel $[E(\nu,h;t)](p'',q'';p',q') = \mathscr{P}_\nu(h;p'',q'',t;p',q',0)$.

(2) For all $f,g \in D$,

$$\langle f, E(\nu,h;T)g\rangle = \langle f, E^o(\nu,T)g\rangle$$
$$- i\int_0^T dt \langle f, E(\nu,h;T-t)H_V E^o(\nu,t)g\rangle.$$

This proposition will enable us, in the next section, to study the limit for $\nu \to \infty$.

From the path integral definition (2.15) for $\mathscr{P}_\nu(h)$ one can easily check that

$$\mathscr{P}_\nu(-h;p'',q'',T;p',q',0) = \overline{\mathscr{P}_\nu(h;p',q',T;p'',q'',0)}. \qquad (2.47a)$$

This implies

$$E(\nu,-h;t) = E(\nu,h;t)^*. \qquad (2.47b)$$

C. Taking the limit $\nu \to \infty$

Let us again first consider the case $h = 0$. Since the h_{ol} span the subspace $\mathscr{H}_o$, we see from (2.24) that $e^{-\nu TA} \to P_o$ as $\nu \to \infty$. Hence, as $\nu \to \infty$,

$$\mathscr{P}_\nu(h=0;p'',q'',T;p',q',0)$$
$$= [\exp(-\nu AT)](p'',q'';p',q') \to P_o(p'',q'';p',q'). \qquad (2.48)$$

Since $P_o = (\mathbf{1}_{\mathscr{H}})\hat{}$ [using the definition (2.10) of $\hat{R}$ for $R \in \mathscr{B}(\mathscr{H})$], its integral kernel is given by [use (2.11)]

$$P_o(p'',q'';p',q') = \langle p'',q''|p',q'\rangle. \qquad (2.49)$$

Putting (2.48) and (2.49) together yields, as $\nu \to \infty$,

$$\mathscr{P}_\nu(h=0;p'',q'',T;p',q',0) \to \langle p'',q''|p',q'\rangle.$$

This is exactly statement (1.3), specialized to the case $h \equiv 0$.

For $h \neq 0$, the same will happen. If h is bounded, $\mathscr{P}_\nu(h)$ is the integral kernel of $\exp[-(\nu A + iH_V)T]$. Since $A(1-P_o) \geqslant 1 - P_o$, the effect of the $-\nu TA$ term in the exponent, in the limit $\nu \to \infty$, is that everything happening outside $\mathscr{H}_o = P_o L^2(V)$ gets damped out. An analogous phenomenon takes place for unbounded h. This is the content of the following proposition.

Proposition 2.2: Let h be a real function on $\mathbb{R}^2$ satisfying (2.25). Let H_V be the operator on $L^2(V)$ defined by

$$(H_V f)(p,q) = h(p,q) f(p,q).$$

Let $E(\nu,h;t)$ be the contraction semigroup given by Proposition 2.1. Define the operator $P_o H_V P_o$ on the domain $\{f; P_o f \in D(H_V)\}$. Obviously $D \subset D(P_o H_V P_o)$. Assume that $P_o H_V P_o$ is essentially self-adjoint on D. Then, for all $T > 0$,

$$\operatorname*{s-lim}_{\nu\to\infty} E(\nu,h;T) = P_o \exp(-iP_o H_V P_o T) P_o, \tag{2.50}$$

where, with a slight abuse of notation, we write $\exp(-iP_o H_V P_o T)$ for $\exp(-i\,\overline{P_o H_V P_o}\,T)$.

Remark: Note that the condition that $\overline{P_o H_V P_o|_D}$ be self-adjoint is an extra condition on h, which is not fulfilled by all h satisfying (2.25), not even if H_V is essentially self-adjoint on D, and $P_o D \subset D$ notwithstanding. It may happen that $P_o H_V P_o|_D$ has more than one self-adjoint extension [e.g., $h(p,q) = p^2 + (1-3\lambda)q^2 - \lambda q^4, \lambda > 0$] or none at all [e.g., $h(p,q) = pq^3 + \frac{3}{2}pq$]. The condition that $\overline{P_o H_V P_o|_D}$ be self-adjoint ensures that $\exp[-i\,\overline{P_o H_V P_o|_D}\,T]$ is well defined and unitary. This is needed in point (7) of the proof (see below). The condition on $P_o H_V P_o|_D$ may be weakened, however (see the remark following the proof of Proposition 2.1); it is sufficient to require that $\overline{P_o H_V P_o|_D}$ be maximal symmetric. In that case a slightly weaker conclusion than (2.50) holds: in some cases one has to substitute "weak limit" for "strong limit." Note that if $\overline{P_o H_V P_o|_D}$ is maximal symmetric (this includes the self-adjoint case), we automatically have $\overline{P_o H_V P_o|_D} = \overline{P_o H_V P_o}$.

Before starting on the proof of Proposition 2.2, we state the following lemma which we shall need. Since it is easy to prove, we omit the proof here (see also Ref. 13, Lemma V 1.2).

Lemma 2.3: Let $\mathscr{H}$ be any separable complex Hilbert space. Let B_n be a sequence of bounded operators on $\mathscr{H}$, with $\text{w-lim}_{n\to\infty} B_n = B$. suppose that

$$\|B_n\| \leqslant 1, \quad \text{all } n,$$

$$\|B\psi\| = \|\psi\|, \quad \text{all } \psi \in \mathscr{H}.$$

Then the B_n converge strongly to B.

Remark: A corollary to this lemma is that the integral in (2.6) actually converges in the strong sense; that is, for any increasing sequence of compact sets K_n ("increasing" means $K_n \subset K_{n+1}$ for all n) such that $\cup_n K_n = \mathbb{R}^2$, one has

$$\operatorname*{s-lim}_{n\to\infty} \int_{K_n} \frac{dp\,dq}{2\pi} |p,q\rangle\langle p,q| = \mathbf{1}$$

[take $B_n = \int_{K_n}(dp\,dq/2\pi)|p,q\rangle\langle p,q|, B = \mathbf{1}$. Since $\|B_n\| \leqslant 1$, all the conditions for the lemma are fulfilled, and the above statement follows].

We now proceed to prove proposition 2.2.

Proof of Proposition 2.2: (1) Since we shall work with one fixed h, we shall drop this label in our notation for $E(\nu,h;t)$:

$$E(\nu,t) = E(\nu,h;t), \quad E^o(\nu,t) = E(\nu,0;t) = \exp(-\nu t A).$$

Since the $E(\nu,t)$ are contractions, $\|E(\nu,t)\| \leqslant 1$, it suffices to prove the strong convergence on D. For $f, g \in D$, we have [see (2.46)], for all $t \geqslant 0$,

$$\langle f, E(\nu,t) g\rangle = \langle f, E^o(\nu,t) g\rangle - i\int_0^t ds \langle f, E(\nu,t-s) H_V E^o(\nu,s) g\rangle. \tag{2.51}$$

Hence, for $k > 0$,

$$|\langle f, E(\nu,t) h_{kl}\rangle| \leqslant |\langle f, E^o(\nu,t) h_{kl}\rangle| + \int_0^t ds |\langle f, E(\nu,t-s) H_V E^o(\nu,s) h_{kl}\rangle|$$
$$\leqslant e^{-\nu k t}\|f\| + (\nu^{-1}k^{-1})\|f\|\,\|H_V h_{kl}\|.$$

Since D is dense, this implies

$$\|E(\nu,t) h_{kl}\| \leqslant e^{-\nu k t} + (\nu^{-1}k^{-1})\|H_V h_{kl}\|. \tag{2.52}$$

Hence, for $k > 0$: $\|E(\nu,T) h_{kl}\| \to 0$ as $\nu \to \infty$ (since $T > 0$), which proves

$$\operatorname*{s-lim}_{\nu\to\infty} [E(\nu,T)(1-P_o)] = 0. \tag{2.53}$$

(2) We can use (2.52) to prove an estimate that will be useful below. From (2.52) we see that for any $f \in D$ there exists a constant C_f (depending on f) such that, for all $t \geqslant 0$,

$$\|E(\nu,t)(1-P_o) f\| \leqslant C_f(e^{-\nu t} + \nu^{-1}).$$

Take $g_1 \in L^2(V)$. For arbitrary $\epsilon > 0$, we can find $f \in D$ such that $\|f - g_1\| \leqslant \epsilon$. Hence

$$\|E(\nu,t)(1-P_o) g_1\| \leqslant \epsilon + C_f(e^{-\nu t} + \nu^{-1}).$$

This implies, for $g_2 \in L^2(V)$, that

$$\int_0^T dt\, |\langle g_2, E(\nu,T-t)(1-P_o) g_1\rangle|$$
$$\leqslant \epsilon T\|g_2\| + C_f\|g_2\|[\nu^{-1}T + \nu^{-1}(1-e^{-\nu T})].$$

It is always possible to choose ν_o such that for $\nu \geqslant \nu_o$ the second term in the right-hand side of this inequality becomes smaller than ϵ. Since ϵ was arbitrary to start with, we have therefore proved for all $g_1, g_2 \in L^2(V)$ that

$$\lim_{\nu\to\infty} \int_0^T dt\, \langle g_2, E(\nu,T-t)(1-P_o) g_1\rangle = 0. \tag{2.54}$$

(3) We now concentrate on $P_o E(\nu,T) P_o$. Take any strictly increasing sequence $(\nu_n)_n, \nu_n \geqslant 0$, with $\lim_{n\to\infty} \nu_n = \infty$. Fix $t \geqslant 0$. Since $\|P_o E(\nu_n,t) P_o\| \leqslant 1$, there exists a subsequence $(\nu_{n(k)})_k$ of $(\nu_n)_n$ such that $P_o E(\nu_{n(k)},t) P_o$ converges weakly. By the standard diagonalization trick, one finds there exists a subsequence $(\hat\nu_k)_k$ of $(\nu_n)_n$ such that the $P_o E(\hat\nu_k,t) P_o$ converge weakly for all rational values of t in $\mathbb{R}_+$.

(4) For $f, g \in D_o = P_o D$ (= finite linear span of the h_{ol}), and any $t \in \mathbf{R}_+$, we see from (2.51) that [use $E^o(\nu,t)g = g$]

$$\left| \frac{d}{dt} \langle f, E(\hat{\nu}_k, t) g\rangle \right| = | -i \langle f, E(\hat{\nu}_k,t) H_V\, g\rangle |$$
$$\leq \|f\|\, \|H_V\, g\|.$$

Since this bound is independent of k (and of t) this implies that for fixed $f, g \in D_o$, the functions $F_{k,f,g}(t) = \langle f, E(\hat{\nu}_k,t)g\rangle$ form a (uniformly) equicontinuous family of functions $\mathbf{R}_+ \to \mathbb{C}$. Since the F_k converge for $t \in \mathbb{Q}_+$, a dense set in $\mathbf{R}_+$, this implies that they converge for all $t \in \mathbf{R}_+$; namely, for all $t > 0$, $\langle f, E(\hat{\nu}_n,t)g\rangle \to F_{\infty,f,g}(t)$ as $\nu \to \infty$.

(5) for any t, $F_{\infty,f,g}$ is obviously sesquilinear in f, g. Moreover, since the $E(\nu,t)$ are all contractions, we have $|F_{\infty,f,g}(t)| \leq \|f\|\, \|g\|$. By Riesz' lemma this implies that there exist operators $L_t \in \mathscr{B}(\mathscr{H}_o)$ such that, for all $f, g \in \mathscr{H}_o$ and all $t \in \mathbf{R}_+$,

$$\langle f, L_t\, g\rangle = F_{\infty,f,g}(t) = \lim_{k\to\infty} \langle f, E(\hat{\nu}_k,t)g\rangle. \tag{2.55}$$

(6) Putting together (2.51) with some estimates, we can find an explicit form for the operators L_t. Let f, g be arbitrary elements of D_o. Then

$$\langle f, L_t\, g\rangle$$
$$= \lim_{k\to\infty} \langle f, E(\hat{\nu}_k,t)g\rangle \quad [\text{see } (2.55)]$$
$$= \lim_{k\to\infty} \left[\langle f,g\rangle - i \int_0^T dt\, \langle f, E(\hat{\nu}_k, T-t) H_V\, g\rangle \right]$$
$$[\text{use } (2.51), \text{ together with } E^o(\nu,t)g = g]$$
$$= \langle f, g\rangle - i \lim_{k\to\infty} \int_0^T dt\, \langle f, E(\hat{\nu}_k, T-t)(1-P_o) H_V\, g\rangle$$
$$- i \lim_{k\to\infty} \int_0^T dt\, \langle f, E(\hat{\nu}_k, T-t) P_o H_V\, g\rangle.$$

The second term is zero by (2.54); in the third term we can interchange the limit and the integration because of the dominated convergence theorem, which gives

$$\langle f, L_T\, g\rangle = \langle f, g\rangle - i \int_0^T dt\, \langle f, L_{T-t} P_o H_V\, g\rangle. \tag{2.56}$$

Equation (2.56) holds for $f, g \in D_o$. Introducing the operator $\hat{L}_t$ on $L^2(V)$, defined as the trivial extension of L_t

$$\hat{L}_t f = 0, \quad \text{if } f \perp \mathscr{H}_o,$$
$$\hat{L}_t f = L_t f, \quad \text{if } f \in \mathscr{H}_o,$$

we can rewrite (2.56), for all $f, g \in D$ and all $T > 0$, as

$$\langle f, \hat{L}_T\, g\rangle = \langle f, P_o\, g\rangle - i \int_0^T dt\, \langle f, \hat{L}_{T-t}\, P_o H_V P_o\, g\rangle. \tag{2.57}$$

Since $P_o H_V P_o$ is essentially self-adjoint on D, (2.57) implies for all $T > 0$ that

$$\hat{L}_T = P_o \exp[-iP_o H_V P_o T] P_o. \tag{2.58}$$

From (2.55) and the definition of $\hat{L}_T$, one sees that this implies

$$\operatorname*{w-lim}_{k\to\infty} P_o E(\hat{\nu}_k, T) P_o = P_o \exp[-iP_o H_V P_o T] P_o. \tag{2.59}$$

Since $(\hat{\nu}_k)_k$ was a well-chosen subsequence of an *arbitrary* increasing sequence $(\nu_n)_n$, with $\lim_{n\to\infty} \nu_n = \infty$, (2.59) implies

$$\operatorname*{w-lim}_{\nu\to\infty} P_o E(\nu, T) P_o = P_o \exp[-iP_o H_V P_o T] P_o. \tag{2.60}$$

(7) Equation (2.60) is still not quite what we want, since it only gives weak convergence, while we are interested in strong convergence. However, we shall see that we can, by applying Lemma 2.3, convert this weak into strong convergence. Let us restrict ourselves to $\mathscr{H}_o$. If we define $\tilde{E}(\nu,t)$, $\tilde{H}_V$ to be the obvious restrictions to $\mathscr{H}_o$ of $P_o E(\nu,t) P_o$, $P_o H_V P_o$, respectively, (2.60) can be rewritten as

$$\operatorname*{w-lim}_{\nu\to\infty} \tilde{E}(\nu, T) = \exp(-i\tilde{H}_V T).$$

Since $\exp(-i\tilde{H}_V T)$ is a unitary operator on $\mathscr{H}_o$, and $\|\tilde{E}(\nu,T)\| \leq 1$, we can apply Lemma 2.3, and conclude

$$\operatorname*{s-lim}_{\nu\to\infty} \tilde{E}(\nu, T) = \exp(-i\tilde{H}_V T).$$

This then implies, on $L^2(V)$,

$$\operatorname*{s-lim}_{\nu\to\infty} P_o E(\nu, T) P_o = P_o \exp[-iP_o H_V P_o T] P_o. \tag{2.61}$$

(8) Comparing (2.61) and (2.53) with (2.50) one sees that we only need to prove still that $(1 - P_o) E(\nu,t) P_o$ converges to zero. This is an easy consequence of the fact that the $E(\nu,t)$ are contractions, while $\exp(-iP_o H_V P_o T)$ is unitary. Take $f \in L^2(V)$. Let ϵ be arbitrary, with $\epsilon > 0$. Because of (2.61) we know that there exists a ν_o such that

$$\|P_o [E(\nu,t) - \exp(-iP_o H_V P_o t)] P_o f\| \leq \epsilon, \quad \text{for all } \nu \geq \nu_o.$$

Since $P_o \exp(-iP_o H_V P_o t) P_o = \exp(-iP_o H_V P_o t) P_o$, this implies

$$\|e^{-iP_o H_V P_o t} P_o f\|^2 - \|P_o E(\nu,t) P_o f\|^2 \leq 2\epsilon \|P_o f\|.$$

Hence, for $\nu \geq \nu_o$,

$$\|(1 - P_o) E(\nu,t) P_o f\|^2$$
$$= \|E(\nu,t) P_o f\|^2 - \|P_o E(\nu,t) P_o f\|^2$$
$$\leq \|P_o f\|^2 - \|e^{-iP_o H_V P_o t} P_o f\|^2 + 2\epsilon \|P_o f\|$$
$$= 2\epsilon \|P_o f\|.$$

Since ϵ and f were arbitrary, this proves

$$\operatorname*{s-lim}_{\nu\to\infty} (1 - P_o) E(\nu,t) P_o = 0, \tag{2.62}$$

and (2.50) now follows from (2.53), (2.61), and (2.62).

Remark: We already noted above that the condition on $P_o H_V P_o$ may be weakened. We only required that $P_o H_V P_o|_D$ be essentially self-adjoint in order to be allowed to make the transition from the integral equation (2.57) to the "integrated form" (2.58) for $\hat{L}_T$. There are, however, more general conditions under which this transition is still permitted.

We first make some general remarks. Let T be a closed symmetric operator on a complex Hilbert space $\mathscr{H}$. We define its deficiency indices $n_\pm$ as $n_\pm = \dim \operatorname{Ker}(T^* \pm i1)$. Let ξ be any strictly positive real number. One checks easily that $iT + \xi$ is closed, and for all $\phi \in D(T)$,

$$\|(iT+\xi)\phi\| \geq \xi \|\phi\|.$$

This implies that $\operatorname{Ran}(iT+\xi)$ is closed. If $\operatorname{Ran}(iT+\xi) = \mathscr{H}$ then $(iT+\xi)^{-1}$ exists, and $\|(iT+\xi)^{-1}\| \leq \xi^{-1}$. This is a necessary and sufficient condition for iT to be the generator of a strongly continuous contraction semigroup (this is the Hille–Yosida theorem; see, e.g., Kato[13]); we denote this semigroup by $\exp(-iTt)$. But $[\operatorname{Ran}(iT+\xi)]^\perp = \operatorname{Ker}(-iT^* + \xi)$, hence $\operatorname{Ran}(iT+\xi) = \mathscr{H}$ if and only if $n_+ = 0$. Therefore, iT generates a strongly continuous contraction semigroup if and only if $n_+ = 0$. It turns out, due to the fact that T is symmetric, that this semigroup consists entirely of isometries [for $\phi \in D(T)$, one checks that $(d/dt)\|\exp(-iTt)\phi\|^2 = 0$, hence $\|\exp(-iTt)\phi\| = \|\phi\|$. Since $D(T)$ is dense and $\exp(-iTt)$ a contraction, this extends to all of $\mathscr{H}$]. If $n_- = 0$, the same analysis as above holds for $-T$ instead of T. We have then that all strictly positive real numbers lie in the resolvent set of iT, and $\|(-iT+\xi)^{-1}\| \leq \xi^{-1}$ for $\xi > 0$. This implies $\|(iT^* + \xi)^{-1}\| \leq \xi^{-1}$, i.e., iT^* is a generator of a strongly continuous contraction semigroup $\exp(-iT^*t)$. In fact, $\exp(-iT^*t) = [\exp(iTt)]^*$.

Let us now specialize this to the case at hand. It is clear that we have to assume at least that $\overline{P_o H_V P_o|_D}$ is maximal symmetric. Hence either $n_+ = 0$ or $n_- = 0$. If $n_+ = 0$, $\overline{P_o H_V P_o|_D}$ is the generator of a contraction semigroup, and we are allowed to conclude (2.58) from (2.57). This then leads to the weak limit statement (2.59). Since $\exp(-iP_o H_V P_o T)$ is still an isometry, Lemma 2.3 can still be applied, and the arguments in points (7) and (8) of the proof of Proposition 2.2 still carry through.

If $n_- = 0$ we can apply the above to $-h$. We have thus

$$\operatorname*{s-lim}_{\nu\to\infty} E(\nu,-h;T) = P_o \exp[iP_o H_V P_o T] P_o.$$

Taking adjoints, we see that this implies [use (2.47)]

$$\operatorname*{w-lim}_{\nu\to\infty} E(\nu,h;T) = P_o \exp[-i(P_o H_V P_o)^* T] P_o,$$

where we have used $(P_o H_V P_o)^* = (P_o H_V P_o|_D)^*$ since $\overline{P_o H_V P_o|_D} = \overline{P_o H_V P_o}$. Note that we only can conclude weak convergence in this case (due to the taking of adjoints).

Summarizing, we see that if $\overline{P_o H_V P_o|_D}$ is maximal symmetric, then

$$\begin{aligned} &\operatorname*{s-lim}_{\nu\to\infty} P_o E(\nu,h;T) P_o = P_o \exp[-iP_o H_V P_o T] P_o, \\ &\qquad \text{if } n_+(P_o H_V P_o) = 0, \\ &\operatorname*{w-lim}_{\nu\to\infty} P_o E(\nu,h;T) P_o = P_o \exp[-i(P_o H_V P_o)^* T] P_o, \\ &\qquad \text{if } n_-(P_o H_V P_o) = 0. \end{aligned} \tag{2.63}$$

Ultimately we are interested in convergence of the $\mathscr{P}_\nu(h)$, the integral kernels of the operators $E(\nu,h;t)$, rather than in convergence of the operators themselves. As a consequence of the constructions we made in Sec. II A, we can easily conclude that the $\mathscr{P}_\nu(h)$ converge in the sense of the Schwartz distributions.

To see this, let us first define the operator H on $\mathscr{H}$ by

$$H = \int \frac{dp\,dq}{2\pi} |p,q\rangle h(p,q) \langle p,q|.$$

This definition is consistent with our earlier notations; the operator H_V on $L^2(V)$ associated to H by means of (2.13) coincides with the multiplication operator H_V defined by (2.32). We have, therefore, according to (2.14),

$$P_o H_V P_o = \hat{H}, \tag{2.64}$$

where $\hat{H}$ is defined by (2.10). This implies [use (2.12)]

$$P_o \exp(-iP_o H_V P_o T) P_o = [\exp(-iTH)]\hat{}.$$

According to (2.11), $[\exp-(iTH)]\hat{}$ is an integral operator, and its integral kernel is given by the cs matrix elements of $\exp(-iTH)$. Hence

$$\begin{aligned} &[P_o \exp(-iP_o H_V P_o T) P_o](p'',q'';p',q') \\ &\quad = \langle p'',q''|\exp(-iHT)|p',q'\rangle. \end{aligned}$$

We find therefore that both the $E(\nu,h;T)$ *and* their limiting operator $P_o \exp(-iP_o H_V P_o T) P_o$ are integral operators on $L^2(V)$. It is then easy to show, using the fact that the Schwartz test functions on $\mathbb{R}^2$ are elements of $L^2(V)$, that the convergence proved in Proposition 2.2 implies convergence, in the sense of the distributions, of the integral kernels $\mathscr{P}_\nu(h)$, for all $T > 0$, to the integral kernel of the limiting operator. We have thus (d-lim = limit in the sense of the Schwartz distributions)

$$\begin{aligned} &\operatorname*{d-lim}_{\nu\to\infty} \mathscr{P}_\nu(h;p'',q'',T;p',q',0) \\ &\quad = \langle p'',q''|\exp(-iHT)|p',q'\rangle. \end{aligned} \tag{2.65}$$

In the final theorem of this section we shall see that we can do better than this, i.e., that we can prove in addition *pointwise* convergence of the $\mathscr{P}_\nu(h)$, provided the function h satisfies a condition slightly stronger than (2.25). To prove this theorem we shall make use of formula (2.45) and of Proposition 2.2. Actually we shall only need weak convergence of the operators $E(\nu,h;t)$; this enables us to consider also operators H which are not self-adjoint, but only maximal symmetric [see (2.63)].

Theorem 2.4: Let H be a maximal symmetric operator on $\mathscr{H}$, which can be written as

$$H = \int \frac{dp\,dq}{2\pi} |p,q\rangle h(p,q) \langle p,q|.$$

Assume that D_c, the finite linear span of the harmonic oscillator eigenstates ω_k, is a core for H. Suppose that the function h satisfies the following.

(C1) For every $\alpha > 0$

$$\int dp\,dq |h(p,q)|^2 \exp[-\alpha(p^2+q^2)] < \infty.$$

(C2) For some $0 \leq \beta < 1$

$$\int dp\,dq |h(p,q)|^4 \exp\left[-\frac{\beta(p^2+q^2)}{2}\right] = C_\beta < \infty.$$

Then, for all p'',q'',p',q' in $\mathbb{R}$, and all $t'' > t'$,

$$\lim_{\nu\to\infty} 2\pi e^{\nu(t''-t')/2}\int \exp\left[\frac{i}{2}\int (p\,dq - q\,dp) - i\int h(p,q)dt\right] d\mu_W^\nu(p,q)$$

$$= \begin{cases} \langle p'',q''|\exp[-i(t''-t')H]\,p',q'\rangle, \\ \quad \text{if } n_+(H)=0, \\ \langle p'',q''|\exp[-i(t''-t')H^*]\,p',q'\rangle, \\ \quad \text{if } n_-(H)=0. \end{cases} \tag{2.66}$$

Here μ_W^ν is the product of two independent Wiener measures (one in p, one in q), pinned at p',q' for $t=t'$, and at p'',q'' for $t=t''$. The normalization of μ_W^ν is given by

$$\int d\mu_W^\nu(p,q) = [2\pi\nu(t''-t')]^{-1} \times \exp\left\{-\frac{(p''-p')^2+(q''-q')^2}{2\nu(t''-t')}\right\}$$

and the connected covariance is (x either p or q; $t_1<t_2$)

$$\langle x(t_1)x(t_2)\rangle^c \equiv \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle = \nu t_1[1-t_2/(t''-t')],$$

where $\langle f\rangle = (\int d\mu_W^\nu f)/(\int d\mu_W^\nu)$. If the limit is taken in the sense of the Schwartz distributions, then (2.66) already holds if only (C1) is satisfied.

Proof: (1) We take, without loss of generality, $t'=0$ and $t''=T>0$.

(2) We shall use (2.45), relating $\mathscr{P}_\nu(h)$ with $\mathscr{P}_\nu(h=0)$. If we write (2.45) also for $\mathscr{P}_\nu(-h)$, take the complex conjugate, and apply (2.47a), we find another such integral equation for $\mathscr{P}_\nu(h)$. Combining this with (2.45) leads to

$$\begin{aligned} &\mathscr{P}_\nu(h;p'',q'',T;p',q',0) \\ &= \mathscr{P}_\nu(0;p'',q'',T;p',q',0) - i\int_0^T dt \int \frac{dp\,dq}{2\pi} \\ &\quad\times \mathscr{P}_\nu(0;p'',q'',T;p,q,t)h(p,q)\mathscr{P}_\nu(0;p,q,t;p',q',0) \\ &\quad - \int_0^T dt_1 \int_0^{t_1} dt_2 \int \frac{dp_1\,dq_1}{2\pi}\int \frac{dp_2\,dq_2}{2\pi} \\ &\quad\times \mathscr{P}_\nu(0;p'',q'',T;p_1,q_1,t_1)h(p_1,q_1) \\ &\quad\times \mathscr{P}_\nu(h;p_1,q_1,t_1;p_2,q_2,t_2)h(p_2,q_2) \\ &\quad\times \mathscr{P}_\nu(0;p_2,q_2,t_2;p',q',0). \end{aligned} \tag{2.67}$$

A calculation analogous to what was done above [see (2.26)] shows that all the integrals in (2.67) converge absolutely (for fixed ν).

(3) Let us introduce a new notation. For $p,q\in\mathbb{R},\nu,t>0$, we define $\phi_{p,q,\nu,t}\in L^2(V)$ by

$$\begin{aligned} \phi_{p,q,\nu,t}(p_1,q_1) &= \mathscr{P}_\nu(0;p_1,q_1,t;p,q,0) \\ &= (e^{\nu t/2}/(2\sinh[\nu t/2]))\exp\{(i/2)(pq_1-p_1q) \\ &\quad - \tfrac14\coth(\nu t/2)[(p-p_1)^2+(q-q_1)^2]\}. \end{aligned}$$

One easily calculates

$$\|\phi_{p,q,\nu,t}\| = (1-e^{-2\nu t})^{-1/2}.$$

Using (C1), one can check that $\phi_{p,q,\nu,t}\in D(H_V)$. As ν tends to ∞ (the other parameters remaining fixed), $\phi_{p,q,\nu,t}(p_1,q_1)$ converges pointwise to a familiar expression

$$\begin{aligned} &\phi_{p,q,\nu,t}(p_1,q_1) \\ &\quad\to \exp\{(i/2)(pq_1-p_1q) - \tfrac14[(p-p_1)^2+(q-q_1)^2]\} \\ &\quad= \langle p_1,q_1|p,q\rangle = (\hat U\omega^{p,q})(p_1,q_1), \end{aligned}$$

where we have used the notation of Sec. II A. An easy calculation shows that this convergence also holds in $L^2(V)$:

$$\|\phi_{p,q,\nu,t} - \hat U\omega^{p,q}\| = (e^{2\nu t}-1)^{-1/2}. \tag{2.68}$$

(4) With this new notation we can rewrite (2.67) as

$$\begin{aligned} &\mathscr{P}_\nu(h;p'',q'',T;p',q',0) \\ &= \mathscr{P}_\nu(0;p'',q'',T;p',q',0) \\ &\quad - i\int_0^T dt\,\langle \phi_{p'',q'',\nu,T-t}, H_V\phi_{p',q',\nu,t}\rangle \\ &\quad - \int_0^T dt_1\int_0^{t_1} dt_2\langle H_V\phi_{p'',q'',\nu,T-t_1}, E(\nu,h;t_1-t_2) \\ &\qquad\times H_V\phi_{p',q',\nu,t_2}\rangle \end{aligned} \tag{2.69}$$

(5) One can derive a similar integral equation for $\langle p'',q''|\exp(-iTH)\,p',q'\rangle$ [we assume $n_+(H)=0$; a similar derivation can be made if $n_-(H)=0$]. Since (C1) ensures that the $\omega^{p,q}$ lie in the domain of H, one can write

$$\begin{aligned} &\langle p'',q''|\exp(-iTH)p',q'\rangle \\ &= \langle p'',q''|p',q'\rangle - i\int_0^T dt\,\langle p'',q''|\exp(-itH)H\,p',q'\rangle \\ &= \langle p'',q''|p',q'\rangle - i\int_0^T dt\,\langle p'',q''|H\,p',q'\rangle \\ &\quad - \int_0^T dt_1\int_0^{t_1} dt_2\langle p'',q''|H\exp[-i(t_1-t_2)H]\,H\,p',q'\rangle. \end{aligned} \tag{2.70}$$

Transposing the inner products in (2.70) to $L^2(V)$ by means of $\hat U$, and subtracting the resulting equation from (2.69), we obtain

$$\begin{aligned} &\mathscr{P}_\nu(h;p'',q'',T;p',q',0) - \langle p'',q''|\exp(-iTH)\,p',q'\rangle \\ &= [\mathscr{P}_\nu(h=0;p'',q'',T;p',q',0) - \langle p'',q''|p',q'\rangle] \\ &\quad - i\int_0^T dt\,\langle \phi_{p'',q'',\nu,T-t}, H_V(\phi_{p',q',\nu,t} - \hat U\omega^{p',q'})\rangle \\ &\quad - i\int_0^T dt\,\langle \phi_{p'',q'',\nu,T-t} - \hat U\omega^{p'',q''}, H_V\hat U\omega^{p',q'}\rangle \\ &\quad - i\int_0^T dt_1\int_0^{t_1} dt_2\langle H_V\phi_{p'',q'',\nu,T-t_1}, \\ &\qquad E(\nu,h;t_1-t_2)H_V(\phi_{p',q',\nu,t_2} - \hat U\omega^{p',q'})\rangle \\ &\quad - i\int_0^T dt_1\int_0^{t_1} dt_2\langle H_V(\phi_{p'',q'',\nu,T-t_1} - \hat U\omega^{p'',q''}), \end{aligned}$$

$$E(\nu,h;t_1-t_2)H_V\hat{U}\omega^{p',q'}\rangle$$

$$-\int_0^T dt_1\int_0^{t_1} dt_2\langle H_V\hat{U}\omega^{p'',q''},\{E(\nu,h;t_1-t_2)$$

$$-P_o\exp[-iP_oH_VP_o(t_1-t_2)]P_o\}H_V\hat{U}\omega^{p',q'}\rangle.$$

Let us denote these six terms by $\Delta_1,\dots,\Delta_6$ (in the above order). We shall see that each $\Delta_j\to 0$ as $\nu\to\infty$, which proves the theorem.

(6) Using the explicit expression (2.16) for $\mathscr{P}_\nu(h=0)$, one easily finds

$$|\Delta_1|<(e^{\nu t}-1)^{-1}[1+\tfrac{1}{2}(p''-p')^2+\tfrac{1}{2}(q''-q')^2]$$
$$\times\exp[-(p''-p')^2/4-(q''-q')^2/4],$$

hence $\Delta_1\to 0$.

(7) For Δ_3 we can use (2.68) and Cauchy–Schwarz, which leads to

$$|\Delta_3|<\int_0^T dt\,(e^{2\nu t}-1)^{-1/2}\|H_V\hat{U}\omega^{p,q}\|$$

$$<\nu^{-1}\|H_V\hat{U}\omega^{p,q}\|\int_0^\infty ds(e^s-1)^{-1/2}.$$

This implies $\Delta_3\to 0$.

(8) Let us now consider Δ_6. This term is the integral, on a bounded domain, of a function uniformly bounded by $2\|H_V\hat{U}\omega^{p'',q''}\|\cdot\|H_V\hat{U}\omega^{p',q'}\|$, and converging pointwise a.e. to zero for ν tending to ∞ [by (2.63)]. Hence, by the dominated convergence theorem, $\Delta_6\to 0$.

(9) For the remaining three terms Δ_2, Δ_4, and Δ_5 we need estimates of $\|H_V\phi_{p,q,\nu,t}\|$ and $\|H_V(\phi_{p,q,\nu,t}-\hat{U}\omega^{p,q})\|$, which we shall compute using (C2). We have

$$\|H_V\phi_{p',q',\nu,t}\|^2$$
$$=\int\frac{dp\,dq}{2\pi}|h(p,q)|^2\frac{e^{\nu t}}{4\sinh^2(\nu t/2)}$$
$$\times\exp\{-\tfrac{1}{2}\coth(\nu t/2)[(p-p')+(q-q')^2]\}$$
$$<\left\{\int\frac{dp\,dq}{2\pi}|h(p,q)|^4\exp\left[-\frac{\beta(p^2+q^2)}{2}\right]\right\}^{1/2}$$
$$\times e^{\nu t}\left[4\sinh^2\frac{\nu t}{2}\right]^{-1}\exp\left[\frac{\beta(p'^2+q'^2)}{2}\right]$$
$$\times\left\{\int\frac{dp\,dq}{2\pi}\exp\left[-\left(\coth\frac{\nu t}{2}-\beta\right)(p^2+q^2)\right]\right\}^{1/2}$$
$$=\frac{1}{2}\left(\frac{C_\beta}{\pi}\right)^{1/2}e^{\nu t}\left[4\sinh^2\frac{\nu t}{2}\right]^{-1}\left(\coth\frac{\nu t}{2}-\beta\right)^{-1/2}$$
$$\times\exp[\beta(p'^2+q'^2)/2]$$
$$<K\exp[\beta(p'^2+q'^2)/2](1-e^{-\nu t})^{-3/2}.$$

Hence

$$\|H_V\phi_{p',q',\nu,t}\|<K_1\exp[\beta(p'^2+q'^2)/4](1-e^{-\nu t})^{-3/4}.$$

An analogous calculation can be made for $\|H_V\phi_{p',q',\nu,t}-\hat{U}\omega^{p',q'})\|$. Putting $y=e^{\nu t}-1$, $a=2(1-\beta)<2$, one finds

$$\|H_V(\phi_{p',q',\nu,t}-\hat{U}\omega^{p',q'})\|^2<\tfrac{1}{2}(C_\beta/\pi)^{1/2}\exp[\beta(p'^2+q'^2)/2]\{y^{-3}a^{-1}(1+ay)^{-1}$$
$$\times(2+ay)^{-1}(3+ay)^{-1}(4+ay)^{-1}[(a^4-4a^3+12a^2-24a+24)y^3$$
$$+6a(a^2-2a+2)y^2+6a(11a-8)y+6a]\}^{1/2}$$
$$<K'\exp[\beta(p'^2+q'^2)/2]y^{-3/2}(1+y)^{-1/2}.$$

Hence

$$\|H_V(\phi_{p',q',\nu,t}-\hat{U}\omega^{p',q'})\|<K_2\exp[\beta(p'^2+q'^2)/4](e^{\nu t}-1)^{-3/4}.$$

(10) With the help of these two estimates we can now discuss Δ_2, Δ_4, and Δ_5. We give here the explicit estimate for Δ_4

$$|\Delta_4|<K_1K_2\exp\left[\frac{\beta(p'^2+p''^2+q'^2+q''^2)}{4}\right]\int_0^T dt_1\int_0^{t_1}dt_2[1-e^{-\nu(T-t_1)}]^{-3/4}[e^{\nu t_2}-1]^{-3/4}$$

$$<C\int_0^T ds(1-e^{-\nu s})^{-3/4}\nu^{-1}\int_0^\infty ds(e^x-1)^{-3/4}\leqslant C'\nu^{-1}\left\{\nu^{-1}\int_0^2 ds\left[s\left(1-\frac{s}{2}\right)\right]^{-3/4}+(1-e^{-2})^{-3/4}\left(T-\frac{2}{\nu}\right)\right\}.$$

Since C' does not depend on ν, we clearly have $\Delta_4\to 0$. Estimates for Δ_2,Δ_5 can be computed analogously; one also finds $\Delta_2\to 0$, $\Delta_5\to 0$.

(11) Since $\Delta_j\to 0$, $j=1,\dots,6$, we have shown that

$$|\mathscr{P}_\nu(h,p'',q'',T;p',q',0)-\langle p'',q''|\exp(-iTH)|p',q'\rangle|\to 0.$$

This proves the main statement of the theorem.

(12) The fact that convergence in the sense of distributions follows already if only (C1) is satisfied was proved by the argument preceding the theorem. Note that there, too, only weak convergence was needed for the operators $E(\nu,h;T)$; the argument therefore extends trivially to maximal symmetric H. ■

Remarks: (1) In our formulation of Theorem 2.4, we have used initial and final times t' and t'', respectively, while in all our preceding analyses we took $t'=0$ (and $t''=T$). Since $h(p,q)$ and therefore also H are time independent, this simply amounts to a relabeling of t. It is certainly plausible that all the above also holds for time-dependent Hamiltonians, where the evolution operators are then taken to be time-ordered products. For quadratic Hamiltonians, where everything can be calculated explicitly, this is indeed the case.

(2) Strictly speaking, the pointwise limit proved in Theorem 2.4 is *not* stronger than the limit in the sense of the Schwartz distributions proved before. A close inspection of the proof shows indeed that our estimates of the difference functions Δ_j contain factors of the form $\exp[\frac{1}{4}\beta(p''^2+p'^2+q''^2+q'^2)]$, which are not tempered distributions. If h is polynomially bounded, better estimates can be made for the Δ_j, containing only polynomials in the p',q',p'',q''. These estimates then automatically imply convergence in the sense of the Schwartz distributions.

(3) Note that we have always restricted ourselves to strictly positive time intervals: $T>0$ in Proposition 2.2, and $t''>t'$ in Theorem 2.4. For $T=0$ or $t'=t''$ there is no hope of proving convergence, since

$$E(\nu,h;0)=\mathbf{1},$$

$$\mathscr{P}_\nu(h;p'',q'',0;p',q',0)=\delta(p''-p')\delta(q''-q')$$
$$\neq\langle p'',q''|p',q'\rangle.$$

(4) The construction above shows how the "antiordered symbol" $h(p,q)$ comes into play, rather than the more expected (and much more well-behaved) "ordered symbol" $H(p,q)$ (as defined in the Introduction). Note that for quadratic Hamiltonians a result similar to (1.3), but where the function $H(p,q)$ is used in the path integral instead of $h(p,q)$, also holds.[14] The price to pay for this change is that the measure has then to be replaced by a Wiener measure with drift terms (depending on H; see Ref. 14). This suggests that (1.3) might be one element of a family of related results, each with slightly different Hamiltonian functions in the action, and accordingly different measures.

III. THE SPIN CASE

The spin case can be treated completely analogously to the canonical case, modulo a change in the basic setting of course. In Sec. III A we define our notation, and in Sec. III B we reinterpret the spin path integral for finite ν as the integral kernel of an operator on $L^2(S^2)$. We state our final result (limit for $\nu\to\infty$) in Sec. III C, without proof since the proofs are the same as in Sec. II.

A. Notations and definitions

At the end of Sec. III C we shall see, analogously to (2.66) in the canonical case, that the matrix element between spin coherent states of the unitary evolution operator associated to a spin Hamiltonian for spin s can be written as the limit, for diverging diffusion constant, of path integrals on S^2 involving Wiener measure on the sphere. In all this the spin value s is fixed; s occurs also as a parameter in the path integral. In order to prove this relation we shall, however, also need matrix elements relating to other spin values than s (this is similar to the use of the $|p,q;k\rangle$ in the arguments in Sec. II, even though the final result involved only the $|p,q\rangle$). In order to make this distinction clear, we shall use the symbol j for any arbitrary integer or half-integer value, while s will be used only for the particular spin value (which can also be any integer or half-integer) for which we wish to construct a path integral.

Let $\mathscr{H}_j$ be a $(2j+1)$-dimensional complex Hilbert space ($j=\frac{1}{2},1,\frac{3}{2},\ldots$) carrying an irreducible representation of the Lie group SU(2). We denote the generators of the corresponding Lie algebra by S_k, $k=1,2,3$; one has

$$[S_1,S_2]=iS_3 \quad \text{(plus cyclic permutations).} \tag{3.1}$$

It follows that

$$\sum_{k=1}^{3} S_k^2=j(j+1)\mathbf{1}_j, \tag{3.2}$$

where $\mathbf{1}_j$ is the unit operator in $\mathscr{H}_j$. Let $|j,m\rangle$ be a normalized vector in $\mathscr{H}_j$ such that

$$S_3|j,m\rangle=m|j,m\rangle, \quad m=-j,-j+1,\ldots,j.$$

(We use here directly Dirac's bra-ket notation.) Let θ,ϕ denote the usual coordinates on the unit sphere, where $0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$. We define unitary operators on $\mathscr{H}_j$ by

$$U(\Omega)\equiv U(\theta,\phi)=\exp(-i\phi S_3)\exp(-i\theta S_2).$$

The spin coherent states (for state $|j,m\rangle$) are then given by[15]

$$|\Omega;j,m\rangle\equiv|\theta,\phi;j,m\rangle=U(\theta,\phi)|j,m\rangle.$$

We shall be more specifically interested in the $|\Omega;j,j\rangle$ ($m=j$). We therefore also introduce the notation

$$|\Omega;j\rangle\equiv|\theta,\phi,j\rangle=U(\theta,\phi)|j,j\rangle.$$

For a given value of s, the states $|\Omega;s\rangle$ will be the analog of the states $|p,q\rangle$ in the canonical case, while the $|\Omega;j,m\rangle$ will play a role analogous to the $|p,q;n\rangle$. As in the canonical case, the $|\Omega;s\rangle$ form an overcomplete set in $\mathscr{H}_s$; their "overlap function" is given by

$$\langle\theta'',\phi'',s|\theta',\phi';s\rangle$$
$$=\{\cos[(\theta''-\theta')/2]\cos[(\phi''-\phi')/2]$$
$$+i\cos[(\theta''+\theta')/2]\sin[(\phi''-\phi')/2]\}^{2s}.$$

As in the canonical case, the spin coherent states $|\Omega;j,m\rangle$ give rise to a resolution of the identity in $\mathscr{H}_j$ (for any m value)

$$N_j\int d\Omega\,|\Omega;j,m\rangle\langle\Omega;j,m|=\mathbf{1}_j, \tag{3.3}$$

where $d\Omega\equiv\sin\theta\,d\theta\,d\phi$, $N_j\equiv(2j+1)/4\pi$. One can also prove [from the orthogonality relations for the representations of SU(2)] that for $|j-j'|>0$ and integer,

$$\psi\in\mathscr{H}_j,\ \chi\in\mathscr{H}_{j'}\Rightarrow\int d\Omega\,\langle\psi|\Omega;j,m\rangle\langle\Omega;j',m|\chi\rangle=0. \tag{3.4}$$

Note well the same value of m!

One of the intermediate steps in Sec. II was the interpretation of the path integral as defining the integral kernel of an operator on the Hilbert space of square-integrable functions on the label space $\mathbb{R}^2$ for the $|p,q\rangle$. We shall need this here, too. We use the notation $L^2(S^2)$ for the square-integrable functions on S^2, with normalization

$$\|f\|^2=\int\frac{d\Omega}{4\pi}|f(\Omega)|^2, \quad d\Omega=\sin\theta\,d\theta\,d\phi.$$

For given spin value s (integer or half-integer, but fixed), we define functions h^s_{lm} by

$$h^s_{lm}(\theta,\phi) = \sqrt{2l+2s+1}\langle\Omega;l+s,s|l+s,m\rangle. \qquad (3.5)$$

Here l takes all non-negative integer values ($l = 0,1,2,...$); for each value of l, m takes any of the $2(l+s)+1$ values $-(l+s)$, $-(l+s)+1,...,(l+s)$. One checks from (3.3) and (3.4) that the h^s_{lm} form an orthonormal set in $L^2(S^2)$, i.e.,

$$\int \frac{d\Omega}{4\pi}\, \overline{h^s_{l'm'}(\Omega)}h^s_{lm}(\Omega) = \delta_{ll'}\delta_{mm'}. \qquad (3.6)$$

Note that for $s=0$, the $h^{s=0}_{lm}(\Omega) = (2l+1)^{1/2}\langle\Omega;l,0|l,m\rangle$ are exactly the familiar spherical harmonics

$$h^0_{lm} = Y_{lm}. \qquad (3.7)$$

Hence the h^0_{lm} are not only orthonormal, they also form a complete set for $L^2(S^2)$. We shall see that this is true for *any* value for s (integer or half-integer).

Explicit calculation (see, e.g., Ref. 16) shows that the h^s_{lm} can be written as

$$h^s_{lm}(\theta,\phi) = K_{lm}e^{im\phi}(1+\cos\theta)^{|s+m|/2}(1-\cos\theta)^{|s-m|/2} \times P^{(|s+m|,|s-m|)}_{l-\max(0,|m|-s)}(-\cos\theta), \qquad (3.8)$$

where the K_{lm} are constant factors (normalization plus phase), and where the $P^{(\alpha,\beta)}_k$ are the Jacobi polynomials. Using the well-known fact that the $P^{(\alpha,\beta)}_k(x)$, $k=0,1,...$, are a complete orthogonal set on $L^2([-1,1])$ with respect to the weight functions $(1-x)^\alpha(1+x)^\beta$, one sees again from (3.8) that the h^s_{lm} are orthogonal. Since for every value of m the associated allowed l values range from $\max(0,|m|-s)$ to ∞ [this follows from $-(l+s)\leqslant m \leqslant l+s$], the lower index of the Jacobi polynomial in the right-hand side (rhs) of (3.8) ranges from 0 to ∞. This ensures that for fixed m and ϕ, the $h^s_{lm}(\theta,\phi)$ form a complete set in the θ variable. Consequently, the h^s_{lm} are a complete set in $L^2(S^2)$. Taking into account (3.6) also, we conclude that the h^s_{lm}, $l=0,1,2,...,m = -(l+s), -(l+s)+1,...,l+s$, form a complete orthonormal basis in $L^2(S^2)$.

For $\psi \in \mathscr{H}_s$, we define by f_ψ the function

$$f_\psi(\Omega) = \sqrt{2s+1}\langle\Omega;s|\psi\rangle.$$

It follows from (3.3) that the map $\psi \to f_\psi$ is isometric from $\mathscr{H}_s$ into $L^2(S^2)$. The image of $\mathscr{H}_s$ under this map is a closed subspace of $L^2(S^2)$, which we shall denote by $\mathscr{H}^s_o$ [this is the analog of $\mathscr{H}_o$ in Sec. II; we have introduced an extra superscript s because different s values lead, of course, to different subspaces $\mathscr{H}^s_o$ of $L^2(S^2)$]. We shall denote the isomorphism between $\mathscr{H}_s$ and $\mathscr{H}^s_o$ by U_s:

$$U_s: \mathscr{H}_s \to \mathscr{H}^s_o, \quad (U\psi)(\Omega) = \sqrt{2s+1}\langle\Omega;s|\psi\rangle,$$

where the notation $\hat{U}_s$ stands for the operator $\mathscr{H}_s \to L^2(S^2)$ defined as $\hat{U}_s = I_s \circ U_s$, where I_s is the natural embedding $\mathscr{H}^s_o \to L^2(S^2)$. The orthogonal projection operator in $L^2(S^2)$, mapping $L^2(S^2)$ onto $\mathscr{H}^s_o$, will be denoted by P^s_o.

Again, we define possible ways of transporting and extending an operator on $\mathscr{H}_s$ to an operator on $L^2(S^2)$. These two constructions are completely analogous to what we did in Sec. II [cf. (2.10)–(2.14)].

(1) Given $R \in \mathscr{B}(\mathscr{H}_s)$, we define $\hat{R} \in \mathscr{B}[L^2(S^2)]$ by

$\hat{R}f = 0, \quad \text{if } f \perp \mathscr{H}^s_o,$

$\hat{R}f = \hat{U}_s R U_s^{-1} f, \quad \text{if } f \in \mathscr{H}^s_o.$

Then $\hat{R}$ is an integral operator on $L^2(S^2)$ with integral kernel $(2s+1)\langle\Omega'';s|R\,\Omega';s\rangle$

$$\langle f,\hat{R}g\rangle = (2s+1)\int\frac{d\Omega''}{4\pi}\int\frac{d\Omega'}{4\pi}\,\overline{f(\Omega'')} \times\langle\Omega'';s|R\,\Omega';s\rangle g(\Omega'). \qquad (3.9)$$

(2) Given $R \in \mathscr{B}(\mathscr{H}_s)$, with

$$R = (2s+1)\int\frac{d\Omega}{4\pi}\,|\Omega;s\rangle\langle\Omega;s|r(\Omega) \qquad (3.10)$$

[any operator in $\mathscr{B}(\mathscr{H}_s)$ can be written in this form, with r a smooth function], we define R_S on $L^2(S^2)$ by

$$(R_S f)(\Omega) = r(\Omega)f(\Omega). \qquad (3.11)$$

(The index S stands for sphere.)

One checks again that

$$P^s_o R_S P^s_o = \hat{R} \qquad (3.12)$$

and

$$[\exp(-itH)]\hat{} = P^s_o \exp(-it\hat{H})P^s_o. \qquad (3.13)$$

So much for our notation and definitions in the spin case. In the next subsection we introduce spin path integrals and show how they can be interpreted as integral kernels for operators in $L^2(S^2)$.

Note that since $\mathscr{H}_s$ is finite dimensional, we only have to deal with bounded operators this time (unlike the canonical case). Since the function $r(\Omega)$ in (3.10) can always be chosen as a smooth function, and since S^2 is compact, we also only have to consider bounded functions $r(\Omega)$. This simplifies the discussion considerably.

B. Definition of the path integral (for finite ν)

We define our ν-dependent spin path integral as

$$\mathscr{P}^s_{\nu,h}(\Omega'',t'';\Omega',t') = 4\pi e^{\nu s(t''-t')/2}\int\exp\left[is\int\cos\theta\,d\phi - i\int h(\theta,\phi)dt\right]d\mu^\nu_W(\theta,\phi),$$

where μ^ν_W is the Wiener measure on the sphere S^2, pinned at Ω'' for $t=t''$ and at Ω' for $t=t'$, and defined such that

$$\int d\mu^\nu_W(\theta,\phi) = (4\pi)^{-1}\left\{\exp\left[\frac{\nu(t''-t')\Delta}{2}\right]\right\}(\Omega'',\Omega') = \sum_{l=0}^{\infty}\exp\left[-\frac{\nu(t''-t')l(l+1)}{2}\right] \times\sum_{m=-l}^{l}Y_{lm}(\Omega'')Y^*_{lm}(\Omega'),$$

where the Y_{lm} are the standard spherical harmonic functions. For a restricted class of Hamiltonians, analogous path integrals for spin systems were already discussed in Ref. 15; the principal difference consists in the presence, in Ref. 15, of extra drift terms in the measure, which are absent here.

In the canonical case, in Sec. II, we could calculate $\mathscr{P}_\nu(h=0)$ explicitly, and thus identify the generator A of the semigroup with integral kernel $\mathscr{P}_\nu(h=0)$. In the present case, we have no explicit expression for $\mathscr{P}_\nu(h=0)$; we can, however, by standard techniques determine the partial differential equation associated with the above path integral. One finds thus that $\mathscr{P}_\nu(h;\Omega,t;\Omega',t')$ is a solution to the partial differential equation

$$\partial_t \mathscr{P}_\nu(h;\Omega,t) = -[\nu A_s + ih(\Omega)]\mathscr{P}_\nu(h;\Omega,t), \quad (3.14)$$

where

$$A_s = \frac{1}{2}\left[-\frac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta - \frac{1}{\sin^2\theta}(\partial_\phi - is\cos\theta)^2 - s\right]. \quad (3.15)$$

The kernel $\mathscr{P}_\nu(h;\Omega,t;\Omega',t')$ is completely determined by (3.14) and the initial condition

$$\mathscr{P}_\nu(h;\Omega,t';\Omega',t') = 4\pi\delta(\Omega-\Omega') \equiv 4\pi\delta(\phi-\phi')\delta(\cos\theta-\cos\theta'). \quad (3.16)$$

In order to define A_s, given by (3.15), as an operator on $L^2(S^2)$, we need also to specify the domain $D(A_s)$ of A_s. We define

$$D(A_s) = \{f; e^{is\phi/2}f \in D(-\Delta)\},$$

where $D(-\Delta)$ is the usual domain of the Laplacian on the sphere. Note that, as in the canonical case, A_s is "almost" equal to $(-\frac{1}{2})$ times the Laplacian. In the present case we even recover the Laplacian if we put $s=0$. For half-integer values of s, the functions in $D(A_s)$ are not continuous at $\phi=0$. This poses no problem, however; we define ∂_ϕ on these functions, at the points $\phi=0$ or $\phi=2\pi$, as the suitable right or left derivative.

It is easy to check that A_s, defined by (3.15), with domain $D(A_s)$, is a self-adjoint operator on $L^2(S^2)$. This operator will be the analog, for the spin case, of the operator A in the canonical case. The property of A which turned out to be crucial in the proof of Proposition 2.2 was (2.24); this showed that any vector in $\mathscr{H}_o$ was an eigenvector of A with eigenvalue 0, while on $\mathscr{H}_o^\perp$ the spectrum of $A|_{\mathscr{H}_o^\perp}$ was bounded below by a strictly positive number. In the limit $\nu\to\infty$, this made everything collapse onto $\mathscr{H}_o$. The same is true here. It is not difficult to check, using (3.5), (3.1), (3.2), and

$$sh^s_{lm}(\theta,\phi) = \langle U(\theta,\phi)S_3 \quad l+s,s|l+s,m\rangle,$$

that

$$A_s h^s_{lm} = [l(l+2s+1)/2]h^s_{lm}. \quad (3.17)$$

Since obviously $h^s_{lm}\in D(A_s)$ for all l,m, and since the h^s_{lm} are a complete orthonormal set of vectors in $L^2(S^2)$, (3.17) tells us that A_s has a purely discrete spectrum; its eigenvalues and eigenvectors are given by (3.17). We have, as in the canonical case, that $A_s|_{\mathscr{H}^s_o}=0$; moreover, on $\mathscr{H}^{s\perp}_o$, A_s is bounded below by $\frac{1}{2}>0$. For h a real smooth function on S^2, we define the operator H_S on $L^2(S^2)$ by [as in (3.11)]

$$(H_S f)(\Omega) = h(\Omega)f(\Omega).$$

Since h is a bounded function, H_S is a bounded operator, and $\nu A_s + iH_S$, defined on $D(A_s)$, is a closed operator generating a contraction semigroup on $L^2(S^2)$. From (3.14) and (3.16) one then sees that the integral kernel for this semigroup is given by $\mathscr{P}_\nu(h)$

$$\{\exp[-(\nu A_s + iH_S)(t''-t')]\}(\Omega'',\Omega') = \mathscr{P}_\nu(h;\Omega'',t'';\Omega',t'). \quad (3.18)$$

C. The limit for $\nu\to\infty$

Exactly the same arguments as in Sec. II show that

$$\operatorname*{s-lim}_{\nu\to\infty} \exp[-(\nu A_s + iH_S)T] = P^s_o \exp(-iP^s_o H_S P^s_o T)P^s_o. \quad (3.19)$$

Note that since H_S is always bounded, $\exp(-iP^s_o H_S P^s_o T)$ is always well defined and unitary (unlike the canonical case).

The integral kernel of the operator in the rhs is given by

$$\begin{aligned}[P^s_o \exp(-iP^s_o H_S P^s_o T)P^s_o](\Omega'',\Omega') &= [P^s_o \exp(-i\hat{H}T)P^s_o](\Omega'',\Omega') \quad [\text{use } (3.12)],\\ &= [\exp(-iHT)]\hat{\ }(\Omega'',\Omega') \quad [\text{use } (3.13)],\\ &= (2s+1)\langle\Omega'';s|\exp(-iHT)\,\Omega';s\rangle \quad [\text{use } (3.9)],\end{aligned}$$

where

$$H = (2s+1)\int \frac{d\Omega}{4\pi}|\Omega;s\rangle\langle\Omega;s|h(\Omega).$$

Together with (3.18) this implies

$$\begin{aligned}&\operatorname*{d-lim}_{\nu\to\infty} 4\pi \exp[\nu s(t''-t')/2]\\ &\times\int \exp\left[is\int\cos\theta\,d\phi - i\int h(\theta,\phi)dt\right]d\mu^\nu_W(\theta,\phi)\\ &= (2s+1)\langle\Omega'';s|\exp[-iH(t''-t')]\,\Omega';s\rangle.\end{aligned}$$

Moreover, one can show, in a way completely analogous to the proof of Theorem 2.4, that the limit also holds pointwise. Note that no extra conditions on the function h are needed here, since h can always be chosen to be a continuous, bounded function.

Putting everything together, we can formulate our final result for spin path integrals.

Theorem 3.1: Let s be any integer or half-integer number (non-negative). Let H be any Hermitian operator on $\mathscr{H}_s$. Let H be generated by the smooth real function $h(\theta,\phi)$ by

$$H = (2s+1)\int \frac{d\Omega}{4\pi}|\Omega;s\rangle\langle\Omega;s|h(\Omega).$$

Then, for all Ω'',Ω' in S^2, and for all $t''>t'$,

$$\begin{aligned}&\lim_{\nu\to\infty}\frac{4\pi}{2s+1}\exp\left[\frac{\nu s(t''-t')}{2}\right]\int\exp\left[is\int\cos\theta\,d\phi - i\int h(\theta,\phi)dt\right]d\mu^\nu_W(\theta,\phi)\\ &= \langle\Omega'';s|\exp[-iH(t''-t')]\,\Omega';s\rangle.\end{aligned}$$

Here μ^ν_W is a Wiener measure on the sphere S^2, with diffusion constant ν, and pinned at $\Omega''=(\theta'',\phi'')$ for $t=t''$ and at $\Omega'=(\theta',\phi')$ for $t=t'$. The normalization of μ^ν_W is given by

$$\int d\mu_W^\nu(\theta,\phi) = \sum_{l=0}^{\infty} \exp\left[-\frac{\nu(t''-t')l(l+1)}{2}\right] \times \sum_{m=-l}^{l} Y_{lm}(\Omega'')Y_{lm}^*(\Omega').$$

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APPENDIX: CONNECTION WITH PREVIOUS PROOF

Theorem 2.4 had been announced by us earlier,[4] in a weaker version, namely (see Ref. 4)

$$\lim_{\nu\to\infty} \int \frac{dp''\,dq''}{2\pi} \int \frac{dp'\,dq'}{2\pi} \Big(\langle \psi | p'',q'' \rangle \Big\{ 2\pi e^{\nu T/2} \times \int \exp\left[\frac{i}{2}\int (p\,dq - q\,dp) - i \int h(p,q)dt\right] d\mu_W^\nu \Big\} \langle p',q' | \phi \rangle \Big) = \langle \psi, \exp(-iTH)\phi \rangle. \tag{A1}$$

To see that (A1) is weaker than Theorem 2.4 let us go back to the properties of the cs. It follows from (2.5) that for any (bounded) operator B on $\mathscr{H}$ and any $\psi,\phi \in \mathscr{H}$

$$\langle \psi, B\phi \rangle = \int \frac{dp''\,dq''}{2\pi} \int \frac{dp'\,dq'}{2\pi} \times \langle \psi | p'',q'' \rangle \langle p'',q'' | B\, p',q' \rangle \langle p',q' | \phi \rangle. \tag{A2}$$

It is therefore clear that (2.37) implies (A1). The reverse is not true, however; due to the overcompleteness of the cs, the matrix element $\langle p'',q'' | B\, p',q' \rangle$ in formula (A2) can be replaced by any element of a large equivalence class of functions.

The proof for (A1), sketched in Ref. 4, was different from the one given here. The main difference lies in the interpretation of the path integral expression for finite ν (corresponding to Sec. II B in the present paper). Let us restrict ourselves, in this discussion of the difference between Ref. 4 and the present paper, to the case where h is a bounded function, hence H_V a bounded operator. In Ref. 4, we introduced an abstract Hilbert space

$$\hat{\mathscr{H}} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n,$$

where each $\mathscr{H}_n$ was a copy of $\mathscr{H}_o$. It turns out that the Hilbert space $L^2(V)$ we have used here is nothing else than a concrete realization for this $\hat{\mathscr{H}}$; the subspace $\mathscr{H}_o \subset L^2(V)$ corresponds to the zeroth space $\mathscr{H}_o$ in the construction of $\hat{\mathscr{H}}$. The nth subspace $\mathscr{H}_n \subset \hat{\mathscr{H}}$ corresponds then to the closed linear span in $L^2(V)$ of the $(h_{nl}), l = 0,1,2,\ldots$. We also introduced vectors $|p,q;\beta\rangle\rangle$ in $\hat{\mathscr{H}}$ defined as

$$|p,q;\beta\rangle\rangle = \bigoplus_{n=0}^{\infty} \beta^{n/2} |p,q;n\rangle, \quad 0<\beta<1.$$

With the identification $\hat{\mathscr{H}} \leftrightarrow L^2(V)$, these can be written, in our present framework, as

$$|p,q;\beta\rangle\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \beta^{n/2}\, \overline{h_{nm}(p,q)}\, h_{nm}.$$

The operators, defined in Ref. 4,

$$\mathbf{A} = \bigoplus_{h=0}^{\infty} n\mathbb{1}_n, \quad \mathbf{I}(\beta) = \bigoplus_{n=0}^{\infty} \beta^n \mathbb{1}_n,$$

$$\mathbf{H} = \lim_{\beta\to 1} \int \frac{dp\,dq}{2\pi} |p,q;\beta\rangle\rangle\langle\langle p,q;\beta |\, h(p,q)$$

correspond to, respectively, A [as defined by (2.23)], $I(\beta) = \exp(A \ln\beta)$, and H_V. We also introduced in Ref. 4 the operators

$$\mathbf{E}_N = \int \frac{dp\,dq}{2\pi} e^{-i\epsilon_N h(p,q)} |p,q;\beta_N\rangle\rangle\langle\langle p,q;\beta_N|,$$

where

$$\epsilon_N = T/(N+1),$$

$$\beta_N = [1 - \nu T/2(N+1)]/[1 + \nu T/2(N+1)].$$

Finally, we rewrote $\mathscr{P}_\nu(h)$ as [using our present notations, and identifying $\hat{\mathscr{H}}$ and $L^2(V)$]

$$\int \frac{dp''\,dq''}{2\pi} \int \frac{dp'\,dq'}{2\pi} \langle \psi | p'',q'' \rangle \times \mathscr{P}_\nu(h;p'',q'',T;p',q',0) \langle p',q' | \phi \rangle = \lim_{N\to\infty} \langle \hat{U}\psi, I(\beta_N)(\mathbf{E}_N)^N I(\beta_N)\hat{U}\phi \rangle, \tag{A3}$$

with β_N as above, and $\hat{U}$ as defined in Sec. II A. In the limit $N\to\infty$, obviously, s-$\lim_{N\to\infty} I(\beta_N) = \mathbb{1}$. The limit of $(\mathbf{E}_N)^N$ was more tricky, because of the complicated N dependence of $\mathbf{E}_N$. Using a theorem of Chernoff,[17] one can show, however, that

$$\text{s-}\lim_{N\to\infty} (\mathbf{E}_N)^N = \exp[-T(\nu A + iH_V)]. \tag{A4}$$

This can intuitively be guessed already from the matrix elements of $\mathbf{E}_N$ between the h_{kl}

$$\langle h_{kl}, \mathbf{E}_N h_{rs} \rangle = \int \frac{dp\,dq}{2\pi}\, \overline{h_{kl}(p,q)} \exp[-i\epsilon_N h(p,q) - [(k+r)/2]\nu\epsilon_N + O(\epsilon_N^3)]\, h_{rs}(p,q).$$

Substituting (A4) into (A3) leads to the interpretation of $\mathscr{P}_\nu(h)$ as a generating function [in the sense of (A2)] for the operator $\hat{U}^* \exp[-T(\nu A + iH_V)]\hat{U}$. This conclusion is of course weaker than (2.33), and therefore led to the weaker result (A1) in Ref. 4.

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