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Ingrid Daubechies and John R. Klauder

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Constructing measures for path integrals

Ingrid Daubechies[a),b)]
Department of Physics, Princeton University, Princeton, New Jersey 08544
John R. Klauder
Bell Laboratories, Murray Hill, New Jersey 07974

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The overcompleteness of the coherent states for the Heisenberg–Weyl group implies that many different integral kernels can be used to represent the same operator. Within such an equivalence class we construct an integral kernel to represent the quantum-mechanical evolution operator for certain dynamical systems in the form of a path integral that involves genuine (Wiener) measures on continuous phase-space paths. To achieve this goal it is necessary to employ an expression for the classical action different from the usual one.

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I. INTRODUCTION

As usually formulated, quantum mechanical path integrals are physically elegant but unfortunately are mathematically inelegant as well. The apparently closed form of solution path integrals provide to many problems is tempered by the ambiguities inherent in giving the path integral a meaningful definition, and this aspect has been carefully documented.[1] There have been several attempts to introduce genuine measures and thereby restore order in path-integral formulations. In the works of Albeverio and Høegh-Kröhn[2] and of Combe *et al.*,[3] for example, effort is concentrated on multiplicative potentials which have the property that their Fourier transform is a bounded measure. While this limitation leads to well-defined path integrals the measures involved are Poisson measures for which the paths are not continuous but rather entail discontinuities. In addition this limited class of potentials does not include the harmonic oscillator which, to be incorporated, must be dealt with in an alternative fashion.

In this paper we present a detailed analysis of a quantum mechanical path integral formulation that involves genuine (Wiener) measures concentrated on continuous paths, which deals in a natural way with harmonic-oscillator potentials; a summary of our principal results has already appeared in Ref. 4. We are able to handle directly an essentially arbitrary quadratic Hamiltonian of the harmonic-oscillator type involving quite general time-dependent coefficients, all with one and the same Wiener measure. Superpositions over the time-dependent coefficients significantly widen the class of systems we are able to consider.

Our approach and analysis is based on coherent states and their special properties, and differs considerably from the viewpoint adopted in Ref. 2 or Ref. 3. Before undertaking our detailed analysis we sketch the general mathematical setting of our approach.

[a)] On leave from Dienst voor Theoretische Natuurkunde, Vrije Universiteit Brussels, Belgium.
[b)] Scientific collaborator at the Interuniversitair Instituut voor Kernwetenschappen (Interuniversitary Institute for Nuclear Sciences), Belgium.

A. Consequences of coherent-state overcompleteness

Coherent states are conventionally defined in an abstract Hilbert space $\mathscr{H}$ by

$$|p,q\rangle \equiv e^{i(pQ-qP)}|0\rangle \tag{1}$$

for all real p and q, where Q and P are an irreducible Heisenberg pair, and $|0\rangle$ denotes the normalized solution of the equation $(Q+iP)|0\rangle = 0$.[5] These states admit the fundamental resolution of unity

$$\mathbb{1} = \int |p,q\rangle\langle p,q|(dp\,dq/2\pi) \tag{2}$$

when integrated over all phase space. As a consequence we may conveniently represent the vectors of the abstract Hilbert space by bounded, continuous functions

$$\psi(p,q) \equiv \langle p,q|\psi\rangle, \tag{3}$$

with an inner product given by

$$\langle\phi|\psi\rangle = \int \phi^*(p,q)\psi(p,q)(dp\,dq/2\pi). \tag{4}$$

If $|\phi\rangle = |p',q'\rangle$ it follows that each function $\psi(p,q)$ satisfies the identity

$$\psi(p',q') = \int \mathscr{K}(p',q';p,q)\psi(p,q)(dp\,dq/2\pi), \tag{5}$$

where

$$\begin{aligned}\mathscr{K}(p',q';p,q) &\equiv \langle p',q'|p,q\rangle \\ &= \exp\{\tfrac{1}{2}i(pq'-qp') - \tfrac{1}{4}[(p'-p)^2 \\ &\quad + (q'-q)^2]\}\end{aligned} \tag{6}$$

plays the role of a reproducing kernel. Thus the set of functions of the form (3) with the inner product (4) comprise a reproducing-kernel Hilbert space $\mathscr{C}_0$.[6] The reproducing kernel projects out a closed subspace of the space $L^2(\mathbb{R}^2)$ of all square-integrable functions, and there remain infinitely many linearly independent square-integrable functions orthogonal to all elements of $\mathscr{C}_0$. This feature has important consequences for the representation of operators on $\mathscr{C}_0$ by integral kernels.

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Consider the expression

$$\langle \phi | B | \psi \rangle = \int \phi^*(p'',q'')K_B(p'',q'';r',q') \times \psi(p',q')(dp''dq''/2\pi)(dp'dq'/2\pi) \quad (7)$$

for arbitrary vectors $|\phi\rangle$ and $|\psi\rangle$, and an arbitrary but fixed bounded operator B. One integral kernel that satisfies (7) is always given by

$$K_B(p'',q'';p',q') = \langle p'',q''|B|p',q'\rangle, \quad (8)$$

but in view of the foregoing remarks there are infinitely many other kernels that serve equally well to represent the operator B. As an example we note that all kernels of the form

$$F_\lambda(p'',q'';p',q') = \tfrac{1}{2}(1+4\lambda) \times \exp\{\tfrac{1}{2}i(p'q''-q'p'') - \lambda[(p''-p')^2 + (q''-q')^2]\}, \quad (9)$$

where $\lambda > -\frac{1}{8}$ serve to represent the unit operator, even including the limiting distribution as $\lambda \to \infty$,

$$F_\lambda(p'',q'';p',q') \to 2\pi\delta(p''-p')\delta(q''-q'), \quad (10)$$

which also serves the same purpose.

All kernels that satisfy (7) for a given B form an equivalence class labeled by the operator B, and which we shall denote by $\mathscr{C}(B)$. Thus the examples F_λ in (9) and (10) all belong to the equivalence class $\mathscr{C}(\mathbb{1})$. A generic element of $\mathscr{C}(B)$ is conveniently denoted by $\langle p'',q''|B|p',q'\rangle_{\text{E.C.}}$ (where E. C. represents equivalence class). Any such kernel can serve to represent the operator B in the context of (7), or stated otherwise, in the form

$$B = \int |p'',q''\rangle\langle p'',q''|B|p',q'\rangle_{\text{E.C.}}\langle p',q'| \times (dp''dq''/2\pi)(dp'dq'/2\pi). \quad (11)$$

It is by exploiting this freedom of representation that we shall achieve our goal of representing the quantum mechanical propagator by means of a path integral involving genuine (Wiener) measures.

In the next section, Sec. 2, we detail the construction of the path integral for a special class of dynamical systems, following closely but with significant differences, the usual method of construction. In Sec. 3 we evaluate the path integrals constructed in Sec. 2, while in Sec. 4 we prove that each of the evaluated path integrals is indeed an element of the equivalence class (in the sense described above) of the evolution operator for the particular Hamiltonian in question. A brief conclusion follows in Sec. 5, and the Appendix contains some details needed for Sec. 3.

2. CONSTRUCTION OF THE PATH INTEGRAL

We start by recalling some more properties of the coherent states and the Weyl operators.

A. Basic properties and notations

We take $\mathscr{H}$ to be a separable Hilbert space, on which we define the Weyl operators $W(p,q)$ as

$$W(p,q) = \exp[i(pQ - qP)], \quad (12)$$

where P,Q are an irreducible Heisenberg pair on $\mathscr{H}$, chosen in such a way that

$$W(p',q')W(p'',q'') = \exp[\tfrac{1}{2}i(p'q''-p''q')]W(p'+p'',q'+q''). \quad (13)$$

The operators $W(p,q)$ then act on $\mathscr{H}$ in an irreducible way. Additional properties of the $W(p,q)$ are

$$W(p,q)^\dagger = W(-p,-q) \quad (14)$$

and, for any operator formally written as $F(P,Q)$,

$$W(p,q)^\dagger F(P,Q)W(p,q) = F(P+p,Q+q). \quad (15)$$

We shall use the fact that any (bounded) operator is completely characterized by its diagonal matrix elements between coherent states

$$B \in \mathscr{B}(\mathscr{H}), \forall p,q: \langle p,q|B|p,q\rangle = 0 \Leftrightarrow B \equiv 0, \quad (16)$$

where the coherent states (c.s) are defined as (see Sec. 1)

$$|p,q\rangle = W(p,q)|0\rangle.$$

One can also use diagonal matrix elements between coherent states to evaluate traces.

$$A \text{ trace-class} \Rightarrow \operatorname{Tr} A = \int \frac{dp\,dq}{2\pi}\langle p,q|A|p,q\rangle. \quad (17)$$

Using the product rule (13) for the Weyl operators, one may show that (4) can be rewritten in the following form:

$$\forall \phi,\psi \in \mathscr{H}: \int \frac{dp\,dq}{2\pi} W(p,q)|\phi\rangle\langle\psi|W(p,q)^\dagger = \langle\psi|\phi\rangle \mathbb{1}_{\mathscr{H}} \quad (18)$$

[the easiest way to verify (18) is to check that the diagonal matrix elements between c.s. (coherent states) of the two sides are the same].

Defining $|n\rangle$, $n = 0,1,2,\ldots$, to be the $(n+1)$th normalized eigenstate of $\frac{1}{2}(P^2+Q^2-1)$ (which is consistent with the definition of $|0\rangle$ in the Introduction), we have in particular

$$\int \frac{dp\,dq}{2\pi} W(p,q)|n\rangle\langle m|W(p,q)^\dagger = \delta_{mn}\mathbb{1}_{\mathscr{H}}. \quad (19)$$

The usual technique in the construction of a c.s. path integral for an evolution operator U_t is to reexpress the evolution operator as a product $U_t = (U_{t/n})^n$, to insert the resolution of the identity (2) between each two factors, and to take the limit as $n \to \infty$ (see Ref. 1)

$$\langle p'',q''|U_t|p',q'\rangle = \lim_{n\to\infty}\int \frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi} \times \prod_{j=0}^{n-1}\langle p_{j+1},q_{j+1}|U_{t/n}|p_j,q_j\rangle, \quad (20)$$

$$p_n = p'',\ q_n = q'',$$
$$p_0 = p',\ q_0 = q'.$$

For a time-ordered product $T\exp[-i\int_{t'}^{t''}H(t)\,dt]$, the same technique is used [put $\epsilon = (t''-t')/n$]

$$T\exp\left[-i\int_{t'}^{t''}H(t)\,dt\right] = \lim_{n\to\infty}\{\exp[-iH(t''-\epsilon)\epsilon] \times \exp[-iH(t''-2\epsilon)\epsilon]\cdots\exp[-iH(t')\epsilon]\},$$

which then implies that

$$\langle p'',q''|T\exp\left[-i\int_{t'}^{t''}H(t)dt\right]|p',q'\rangle$$
$$=\lim_{n\to\infty}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}$$
$$\times\prod_{j=0}^{n-1}\langle p_{j+1},q_{j+1}|\exp[-iH(t'+j\epsilon)\epsilon]|p_j,q_j\rangle, \tag{21}$$
$$p_n=p'',q_n=q'';p_0=p',q_0=q'.$$

Basically we shall do the same here; however, instead of (2) we shall insert some more complicated object, and the results of our manipulations will no longer be the matrix elements $\langle p'',q''|U_{t'',t'}|p',q'\rangle$, but some other element of $\mathscr{C}(U_{t'',t'})$ {From now on, we shall use the symbol $U_{t'',t}$ to denote the evolution operator $T\exp[-i\int_{t'}^{t''}H(t)\,dt]$}.

B. The "big" space $\hat{\mathscr{H}}$ and the vectors $|p,q;\beta\rangle\rangle$

We define a "big" Hilbert space $\hat{\mathscr{H}}$ by

$$\hat{\mathscr{H}}=\bigoplus_{n=0}^{\infty}\mathscr{H}_n,$$

where each $\mathscr{H}_n$ is isomorphic with $\mathscr{H}$ (we shall not write out these isomorphisms explicitly, but shall always assume them tacitly understood). We define canonical projections P_m from $\hat{\mathscr{H}}$ to $\mathscr{H}$ as follows:

$$\forall\phi=\bigoplus_{n=0}^{\infty}\phi_n\in\hat{\mathscr{H}}:P_m\phi=\phi_m.$$

The conjugate operators to these P_m are the canonical injections I_m; these are the maps from $\mathscr{H}$ to $\hat{\mathscr{H}}$ defined as follows:

$$\forall\phi\in\mathscr{H}:I_m\phi=\bigoplus_{n=0}^{\infty}\psi_n,$$

where all but the mth ψ_n are zero:

$$\psi_n=\delta_{nm}\phi.$$

The following properties of the P_m, I_m are easy to check:

(a) $P_mI_n=\delta_{mn}\mathbf{1}_{\mathscr{H}}$,

(b) $I_mP_m=\bigoplus_{n=0}^{\infty}(\delta_{mn}\mathbf{1}_n)$

(this operator is zero on all the $\mathscr{H}_n$ with $n\neq m$, and $\mathbf{1}$ on $\mathscr{H}_m$; it is the orthogonal projection operator in $\hat{\mathscr{H}}$ with image $\{0\}\oplus\cdots\oplus\{0\}\oplus\mathscr{H}_m\oplus\{0\}\oplus\cdots$),

$$\text{(c) } \mathbf{1}_{\hat{\mathscr{H}}}=\bigoplus_{n=0}^{\infty}\mathbf{1}_n$$
$$=\sum_m I_mP_m \tag{22}$$

(as a sum of mutually orthogonal projection operators, this sum is well defined in the strong topology),

(d) $I_m=P_m^\dagger,\quad P_m^\dagger=I_m$.

For any $\beta\in[0,1)$ we define a set of normalized vectors $|p,q;\beta\rangle\rangle$ in $\hat{\mathscr{H}}$ by the rule

$$|p,q;\beta\rangle\rangle\equiv(1-\beta)^{1/2}\sum_{n=0}^{\infty}\beta^{n/2}I_n(W(p,q)|n\rangle)$$
$$=(1-\beta)^{1/2}\bigoplus_{n=0}^{\infty}[\beta^{n/2}W(p,q)|n\rangle]. \tag{23}$$

These vectors $|p,q;\beta\rangle\rangle$ have the following overlap function:

$$\langle\langle p'',q'';\beta|p',q';\beta\rangle\rangle$$
$$=(1-\beta)\sum_n\beta^n\langle n|W(p'',q'')^\dagger W(p',q')|n\rangle$$
$$=(1-\beta)\mathrm{Tr}[\beta^NW(p'',q'')^\dagger W(p',q')],$$

where $N=\frac{1}{2}(P^2+Q^2-1)$. For the evaluation of this expression we refer to the Appendix [see (A9)]; the result is

$$\langle\langle p'',q'';\beta|p',q';\beta\rangle\rangle=\exp\left\{\frac{i}{2}(p'q''-p''q')\right.$$
$$\left.-\frac{1+\beta}{4(1-\beta)}[(p''-p')^2+(q''-q')^2]\right\}. \tag{24}$$

One can easily calculate the overlap function $\langle\langle p'',q'';0|p',q';\beta\rangle\rangle$ as

$$\langle\langle p'',q'';0|p',q'';\beta\rangle\rangle=(1-\beta)^{1/2}\langle 0|W(p'',q'')^\dagger W(p',q')|0\rangle$$
$$=(1-\beta)^{1/2}\langle p'',q''|p',q'\rangle. \tag{25}$$

Another property of the $|p,q;\beta\rangle\rangle$ is the following [use (19)]:

$$\int\frac{dp\,dq}{2\pi}|p,q;\beta\rangle\rangle\langle\langle p,q;\beta|$$
$$=(1-\beta)\sum_{n,m}\beta^{(n+m)/2}I_n$$
$$\times\int\frac{dp\,dq}{2\pi}W(p,q)|n\rangle\langle m|W(p,q)^\dagger P_m$$
$$=(1-\beta)\sum_n\beta^nI_nP_n$$
$$=(1-\beta)\bigoplus_n(\beta^n\mathbf{1}_n). \tag{26}$$

This operator is a multiple of the identity on each of the $\mathscr{H}_n$-spaces with, however, difficult constants on different spaces. We shall use this "generalized effective resolution of the identity" to replace (2) in the construction of (20) or (21). Note that (26) holds for any $\beta\in[0,1)$ [for $\beta=0$, it essentially gives (2) again], which allows us to adjust β when needed; this feature will turn out to be important in our construction of the path integrals below.

C. Construction of elements of $\mathscr{C}(B)$ for $B\in\mathscr{B}(\mathscr{H})$

Let us now see how (26) can be useful for our purposes.

Every bounded operator $\mathbf{B}$ on $\hat{\mathscr{H}}$ is completely characterized by the sequence $(B_{kl}=P_k\mathbf{B}I_l)_{k,l=0}$.

Using (26) twice, we obtain

$$\int\frac{dp''dq''}{2\pi}\int\frac{dp'dq'}{2\pi}|p'',q'';\beta\rangle\rangle$$
$$\times\langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle\langle p',q'|p_1,q_1\rangle$$
$$=(1-\beta)^2\sum_{n,m}\beta^{n+m}I_nB_{nm}P_m,$$

Sandwiching this between $\langle\langle p_2,q_2;0|$ and $|p_1,q_1;0\rangle\rangle$ and using (25), we find

$$\int\frac{dp''dq''}{2\pi}\int\frac{dp'dq'}{2\pi}\langle p_2,q_2|p'',q''\rangle$$
$$\times\langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle\langle p',q'|p_1,q_1\rangle$$
$$=(1-\beta)\langle p_2,q_2|B_{00}|p_1,q_1\rangle.$$

Hence

$$B_{00}=\frac{1}{1-\beta}\int\frac{dp''dq''}{2\pi}\int\frac{dp'dq'}{2\pi}|p'',q''\rangle \langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle\langle p',q'| \qquad (27)$$

or, stated otherwise,

$$1/(1-\beta)\langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle\in\mathscr{C}(B_{00}).$$

What has happened in this construction is really a projection such as described in the Introduction: The matrix element $\langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle$ is a sum of different matrix elements:

$$\langle\langle p'',q'';\beta|\mathbf{B}|p',q';\beta\rangle\rangle = (1-\beta)\sum_{n,m}\beta^{(n+m)/2}\langle n|W(p'',q'')^\dagger B_{nm}W(p',q')|m\rangle.$$

By virtue of (19) all matrix elements with $n\neq 0$ and/or $m\neq 0$ give no contribution whatever when the projection is carried out:

$$n\neq 0 \quad\text{or}\quad m\neq 0$$
$$\Rightarrow\forall\phi,\psi\in\mathscr{H}:\int\frac{dp''dq''}{2\pi}\int\frac{dp'dq'}{2\pi} \times\langle\phi|W(p'',q'')|0\rangle\langle n|W(p'',q'')^\dagger B_{nm}W(p',q')|m\rangle \times\langle 0|W(p',q')^\dagger|\psi\rangle=0.$$

Therefore, all these terms drop out when the projection is performed, and the only relevant term is the B_{00} term.

To apply the same argument to a product of operators $\mathbf{B}_n\cdots\mathbf{B}_1$, we must restrict ourselves to diagonal operators. An operator $\mathbf{B}\in\mathscr{B}(\hat{\mathscr{H}})$ is called "diagonal" if $\forall k\neq l, B_{kl}=0$ (i.e., the operator $\mathbf{B}$ does not mix the different $\mathscr{H}_n$, $\mathbf{B}=\oplus_{n=0}^\infty B_{nn}$). Let $\mathbf{B}_1,\ldots,\mathbf{B}_n$ be diagonal operators on $\mathscr{H}$; then

$$P_0\Big[\frac{1}{(1-\beta)^{n-1}}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}\mathbf{B}_n|p_{n-1},q_{n-1};\beta\rangle\rangle \langle\langle p_{n-1},q_{n-1};\beta|\mathbf{B}_{n-1}|p_{n-2},q_{n-2};\beta\rangle\rangle\cdots|p_1,q_1;\beta\rangle\rangle\langle\langle p_1,q_1;\beta|\mathbf{B}_1\Big]I_0$$
$$=\sum_{l_1\cdots l_{n-1}}\beta^{(l_1+\cdots+l_{n-1})}P_0\mathbf{B}_nI_{l_{n-1}}P_{l_{n-1}}\mathbf{B}_{n-1}I_{l_{n-2}}\cdots P_{l_1}\mathbf{B}_1I_0=B_{n,00}B_{n-1,00}\cdots B_{1,00};$$

hence [apply (27)]

$$\frac{1}{(1-\beta)^n}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}\prod_{j=0}^{n-1}\langle\langle p_{j+1},q_{j+1};\beta|\mathbf{B}_{j+1}|p_j,q_j;\beta\rangle\rangle\in\mathscr{C}(B_{n,00}B_{n-1,00}\cdots B_{1,00}) \qquad (28)$$
$$(p_n=p'',q_n=q'';p_0=p',q_0=q').$$

Again, one can easily understand what has happened; since all the $\mathbf{B}_j$ are diagonal, the insertion of the generalized effective resolution of the identity (26) does not mix the $B_{j,kk}$ with different k and, as before in the linear combination of functions in the left-hand side of (27), only one term, the term corresponding to $B_{n,00}\cdots B_{1,00}$, is not orthogonal to $\{\langle\phi|p'',q''\rangle\langle p',q'|\psi\rangle;\ \psi,\phi\in\mathscr{H}\}$.

D. Application to the evolution operator

It is now easy to apply (28) to the propagator $U_{t'',t'}=T\exp[-i\int_{t'}^{t''}H(t)\,dt]$. We have

$$U_{t'',t'}=\lim_{n\to\infty}U_n(t'',t') \qquad (29)$$

with $U_n(t'',t')=\exp[-iH(t''-\epsilon)\epsilon]\exp[-iH(t''-2\epsilon)\epsilon]\times\exp[-iH(t')\epsilon]$ [where $\epsilon=(t''-t')/n$]. Let $\mathbf{H}(t)$ be a self-adjoint, diagonal operator on $\mathscr{H}$ satisfying

$$H_{00}(t)=H(t).$$

Then (28) implies

$$\frac{1}{(1-\beta)^n}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}\prod_{j=0}^{n-1} \times\langle\langle p_{j+1},q_{j+1};\beta|\exp[-i\mathbf{H}(t'+j\epsilon)\epsilon] \times|p_j,q_j;\beta\rangle\rangle\in\mathscr{C}(U_n(t'',t')) \qquad (30)$$
$$(p_n=p'',q_n=q'';p_0=p',q_0=q').$$

As yet we are still free to choose β and all the H_{kk} for $k\neq 0$; (30) holds for all possible choices. We shall see that, at least for certain quadratic Hamiltonians $H(t)$, it is possible to choose the β, H_{kk} in such a way that the functions (30) converge for $n\to\infty$, and give rise to an element of $\mathscr{C}(U_{t'',t'})$.

To show how the Wiener integral emerges, we first study the case $H(t)=0$. In this simple case, we take $\mathbf{H}(t)=0$ [i.e., $\forall k$, $H_{kk}(t)=H_{00}(t)=H(t)=0$]; the function (30) becomes [use (24)]

$$\frac{1}{(1-\beta)^n}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}\times\prod_{j=0}^{n-1}\langle\langle p_{j+1},q_{j+1};\beta|p_j,q_j;\beta\rangle\rangle$$
$$=\frac{1}{(1-\beta)^n}\int\frac{dp_{n-1}dq_{n-1}}{2\pi}\cdots\int\frac{dp_1dq_1}{2\pi}\times\prod_{j=0}^{n-1}\exp\Big\{\frac{i}{2}(p_jq_{j+1}-p_{j+1}q_j) -\frac{(1+\beta)}{4(1-\beta)}[(p_{j+1}-p_j)^2+(q_{j+1}-q_j)^2]\Big\} \qquad (31)$$
$$(p_n=p'',q_n=q'';p_0=p',q_0=q').$$

Recall[7] that the joint probability density for a Wiener process $x(t)$ to be at the points x_j at times t_j ($j=1,\ldots,m$; $t_m\geqslant t_{m-1}\geqslant\cdots\geqslant t_1\geqslant t_0$), having started at x_0 at time t_0, is given by

$$p(x_n,t_n;x_{n-1},t_{n-1};\cdots;x_0,t_0)$$

$$=\prod_{j=1}^{n}\left\{\frac{1}{[2\pi(t_j-t_{j-1})]^{1/2}}\exp\left[-\frac{1}{2}\frac{(x_j-x_{j-1})^2}{t_j-t_{j-1}}\right]\right\}.$$

If all the time intervals are equal, $\forall j$: $t_j-t_{j-1}=\epsilon=(t_n-t_0)/n$ this becomes

$$p(x_n,t_0+n\epsilon;x_{n-1},t_0+(n-1)\epsilon;\cdots;x_0,t_0)$$

$$=\frac{1}{(2\pi\epsilon)^{n/2}}\prod_{j=1}^{n}\exp\left[-\frac{1}{2}\frac{(x_j-x_{j-1})^2}{\epsilon}\right]. \qquad (32)$$

In order to fit (31) with (32), we choose $t_0=t'$, $t_n=t''$ and $(1+\beta)/[2(1-\beta)]=1/\epsilon$ or $\beta=(1-\epsilon/2)/(1+\epsilon/2)$ [where $\epsilon=(t''-t')/n$]. With this choice for β, it is now clear that the Gaussian factors in (31) are correctly chosen to generate independent (non-normalized) pinned Wiener measures in p and q, pinned at the starting points so that $p(t')=p'$, $q(t')=q'$, and at the final points so that $p(t'')=p''$, $q(t'')=q''$. We shall denote these pinned Wiener measures by $d\mu_W{}^{t''\leftarrow t'}_{p''\leftarrow p'}(p)$ and $d\mu_W{}^{t''\leftarrow t'}_{q''\leftarrow q'}(q)$. It is also easy to see what the other factors in (31) become in the limit for $n\to\infty$, namely,

$$(2\pi\epsilon)^n\frac{1}{(2\pi)^{n-1}(1-\beta)^n}=2\pi(1+\tfrac{1}{2}\epsilon)^n=2\pi\left(1+\frac{t''-t'}{2n}\right)^n$$
$$\underset{n\to\infty}{\longrightarrow} 2\pi e^{(t''-t')/2},$$

$$\sum[p_jq_{j+1}-p_{j+1}q_j]=\sum[p_j(q_{j+1}-q_j)-(p_{j+1}-p_j)q_j]$$

$$\to\int[p(t)dq(t)-q(t)dp(t)].$$

Thus (31) becomes

$$2\pi e^{(t''-t')/2}\int d\mu_{W,}{}^{t''\leftarrow t'}_{p''\leftarrow p'}(p)d\mu_{W,}{}^{t''\leftarrow t'}_{q''\leftarrow q'}(q)$$
$$\times\exp\left\{\frac{i}{2}\int[p(t)dq(t)-q(t)dp(t)]\right\}. \qquad (33)$$

It is clear that the integrand in (33) may be given a well-defined meaning in terms of stochastic integrals. Moreover, since p and q are independent stochastic variables, all prescriptions for defining the stochastic integral in (33) are equivalent, which means that this integral is a perfectly well-defined path integral over a genuine measure. We shall evaluate this integral in the next section, and show that it is indeed an element of $\mathscr{C}(\mathbb{1})$.

In the simple case $H(t)=0$ above, we chose all the H_{kk} identical, i.e., $H_{kk}=H_{00}=0$. Although this is of course the simplest choice, there is no *a priori* reason to choose all the H_{kk} identical. Indeed, considering $H(t)\neq0$ below, we shall see that in general the choice of identical H_{kk} does not lead to a well-defined limit of the expressions (30) as $n\to\infty$. On the other hand, it may well be possible that two different sequences of H_{kk}, H'_{kk}, with the same zeroth component $H_{00}=H'_{00}=H$, both lead to well-defined but different limits of (30), both of which are elements of $\mathscr{C}(U_{t'',t'})$.

For H linear in P,Q,

$$H(t)=s(t)Q+r(t)P,$$

the choice $H_{kk}(t)=H_{00}(t)=H(t)$ is satisfactory, and the result is

$$2\pi e^{(t''-t')/2}\int d\mu_{W,p''p'}(p)d\mu_{W,q''q'}(q)\exp\left(\frac{i}{2}\int[p(t)dq(t)-q(t)dp(t)]\right.$$
$$\left.+\int[r(t)dq(t)-s(t)dp(t)]+\int dt\{-i[s(t)q(t)+r(t)p(t)]-\tfrac{1}{2}[s^2(t)+r^2(t)]\}\right), \qquad (34)$$

where we assume s and r to be square integrable. Again, we shall check below that (34) is indeed an element of $\mathscr{C}(T\exp\{-i\int_{t'}^{t''}\times dt[s(t)Q+r(t)P]\})$.

For quadratic Hamiltonians, the choice of identical H_{kk} leads to convergence problems. We illustrate this by means of the simple time-independent quadratic Hamiltonian ($\alpha=$ constant $\neq0$)

$$H=(\alpha/2)(P^2+Q^2-\mathbb{1}).$$

Let us first try the choice $H_{kk}=H_{00}=H$. We get $[\epsilon=(t''-t')/n]$

$$\langle\langle p_{j+1},q_{j+1};\beta|e^{-i\mathbf{H}\epsilon}|p_j,q_j;\beta\rangle\rangle$$
$$=(1-\beta)\mathrm{Tr}[\beta^NW(p_{j+1},q_{j+1})^\dagger e^{-i\alpha N\epsilon}W(p_j,q_j)]$$
$$=(1-\beta)(1-\beta e^{-i\alpha\epsilon})^{-1}\exp\left\{\frac{i}{2}[q_{j+1}(-q_j\sin\alpha\epsilon+p_j\cos\alpha\epsilon)-p_{j+1}(q_j\cos\alpha\epsilon+p_j\sin\alpha\epsilon)]\right.$$
$$\left.-\frac{1+\beta e^{-i\alpha\epsilon}}{4(1-\beta e^{-i\alpha\epsilon})}[(q_{j+1}-q_j\cos\alpha\epsilon-p_j\sin\alpha\epsilon)^2+(p_{j+1}+q_j\sin\alpha\epsilon-p_j\cos\alpha\epsilon)^2]\right\}; \qquad (35)$$

see (A8) in the Appendix.

To generate a measure in the limit $n\to\infty$, we have to choose

$$\beta=1-b\epsilon+o(\epsilon),$$

which leads to

$$\frac{1+\beta e^{-i\alpha\epsilon}}{1-\beta e^{-i\alpha\epsilon}}=\frac{1}{\epsilon}\left(\frac{2}{b+i\alpha}\,o(1)\right).$$

For $\beta\in[0,1)$, b is real, and the factor $2(b+i\alpha)^{-1}$ in front of ϵ^{-1} has a nonzero imaginary part as long as $\alpha\neq 0$, which means that (35) cannot generate a genuine path integral measure. At first sight, it seems that this problem can be circumvented by allowing β to be complex: $\beta=e^{-i\alpha\epsilon}(1-\epsilon/2)/(1+\epsilon/2)$; however, going back to (23), one sees that for β complex, the factor $(1+\beta^{-i\alpha\epsilon})/(1-\beta e^{-i\alpha\epsilon})$ would have to be replaced by $(1+|\beta|e^{-i\alpha\epsilon})/(1-|\beta|e^{-i\alpha\epsilon})$, which shows that a complex choice for β does not solve the convergence problems.

All convergence problems are avoided if the H_{kk} are chosen in the following way:

$$H_{kk}=(\alpha/2)(P^2+Q^2-1)-\alpha k\mathbf{1}=H-\langle H\rangle_k.$$

For this choice of (nonidentical!) H_{kk}, we obtain $[\epsilon=(t''-t')/n]$

$$\begin{aligned}&\langle\langle p_{j+1},q_{j+1};\beta|e^{-i\mathbf{H}\epsilon}|p_j,q_j;\beta\rangle\rangle\\&=(1-\beta)\mathrm{Tr}[\beta^N W(p_{j+1},q_{j+1})^\dagger e^{-i\alpha N\epsilon}W(p_j,q_j)e^{i\alpha N\epsilon}]\\&=\exp\left\{\frac{i}{2}[q_{j+1}(-q_j\sin\alpha\epsilon+p_j\cos\alpha\epsilon)-p_{j+1}(q_j\cos\alpha\epsilon+p_j\sin\alpha\epsilon)]\right.\\&\left.-\frac{1+\beta}{4(1-\beta)}[(q_{j+1}-q_j\cos\alpha\epsilon-p_j\sin\alpha\epsilon)^2+(p_{j+1}+q_j\sin\alpha\epsilon-p_j\cos\alpha\epsilon)^2]\right\};\end{aligned}\tag{36}$$

see (A8) in the Appendix.

We can now again choose $\beta=(1-\epsilon/2)/(1+\epsilon/2)\,[\epsilon=(t''-t')/n]$; the substitution of (36) into (30) again leads to an integral w.r.t. the pinned Wiener measure, and we obtain

$$\begin{aligned}&2\pi e^{(t''-t')/2}\int d\mu_{W,\,p''\leftarrow p'}^{\,t''\leftarrow t'}(p)\,d\mu_{W,\,q''\leftarrow q'}^{\,t''\leftarrow t'}(q)\\&\times\exp\left\{\left(\frac{i}{2}+\alpha\right)\int[p(t)\,dq(t)-q(t)\,dp(t)]-\frac{\alpha}{2}(i+\alpha)\int dt\,[p^2(t)+q^2(t)]\right\}.\end{aligned}\tag{37}$$

The same technique of choosing

$$H_{kk}(t)=H(t)-\langle H(t)\rangle_k$$

works also for the time-dependent quadratic Hamiltonian

$$H(t)=[\alpha(t)/2](P^2+Q^2-\mathbf{1}+s(t)Q+r(t)P.$$

For this Hamiltonian we obtain

$$\begin{aligned}&2\pi e^{(t''-t')/2}\int d\mu_{W,\,p''\leftarrow p'}^{\,t''\leftarrow t'}(p)\,d\mu_{W,\,q''\leftarrow q'}^{\,t''\leftarrow t'}(q)\\&\times\exp\left[\int\left\{\left[\frac{i}{2}+\alpha(t)\right][p(t)\,dq(t)-q(t)\,dp(t)]-\frac{\alpha(t)}{2}[i+\alpha(t)][p(t)^2+q(t)^2]\,dt\right.\right.\\&\left.\left.-[s(t)\,dp(t)-r(t)\,dq(t)]-[i+\alpha(t)][s(t)q(t)+r(t)p(t)]\,dt-\tfrac{1}{2}[s(t)^2+r(t)^2]dt\right\}\right];\end{aligned}\tag{38}$$

the trace to be calculated is slightly more complicated than for (38); see (A10) and (A11) in the Appendix. Note that to give a sense to (38) or (34) we have to take s and r square integrable in $[t',t'']$. For (38) and (37) additional conditions on α will be introduced in Sec. 3 where needed.

In the next section (Sec. 3) we shall evaluate the path integrals (33), (34), (37), and (38). In Sec. 4 we show that they are indeed elements of the corresponding $\mathscr{C}(U_{t'',t'})$ for the Hamiltonians in question.

Remark: It is not really essential in the construction of (28) that the states $|n\rangle$ are the eigenstates of the harmonic oscillator; the only properties used are

$1^\circ)\langle m|n\rangle=\delta_{mn}$,

$2^\circ)|p,q\rangle=W(p,q)|0\rangle$.

We could therefore replace the vectors $|n\rangle$ by any orthonormal set ϕ_n in $\mathscr{H}$, as long as $\phi_0=|0\rangle$; the functions β^n can also be replaced by a positive function $\rho(\lambda_n)$, where λ_n are the eigenvalues of an operator A with eigenvectors ϕ_n: $A\phi_n=\lambda_n\phi_n$ with, however, the restriction that $\Sigma_n\rho(\lambda_n)<\infty$. This would allow us to try the same technique

$$H_{kk}(t)=H(t)-\langle H(t)\rangle_k$$

or, even more generally,

$$H_{kk}(t)=H(t)-g(\lambda_k,t),\quad\text{with } g(\lambda_0,t)=0.$$

for Hamiltonians different from the harmonic oscillator; the problem is then to choose A,f, and g in such a way that the traces

$$\mathrm{Tr}[\rho(A)W(p_{j+1},q_{j+1})^\dagger e^{-iH(t)\epsilon}W(p_j,q_j)e^{ig(A,t)\epsilon}]$$

still have the right form to generate a genuine measure in the limit $n\to\infty$.

In the case of ν degrees of freedom ($\nu > 1$), a class of Hamiltonians for which the procedure above clearly works is given by

$$H(t) = \tfrac{1}{2}\alpha(t)\left\{\sum_{i,j=1}^{\nu}[A_{ij}(P_iP_j + Q_iQ_j) + 2B_{ij}P_iQ_j]\right\} + \sum_{j=1}^{\nu}[S_j(t)Q_j + R_j(t)P_j], \qquad (39)$$

where A, B are $\nu\times\nu$ matrices with $A^t = A$, $B^t = -B$. The path integral corresponding to the Hamiltonian (39) is given below [expression (44)] in a more intrinsic and shorter notation system than we have used up to now (see Sec. 3A). The proof, given in Sec. 4, that the path integral (38) really is an element of the equivalence class $\mathscr{C}(U_{t'',t'})$ for the Hamiltonian $H(t) = \frac{1}{2}\alpha(t)(P^2 + Q^2 - \mathbb{1}) + s(t)Q + r(t)P$, easily extends to this multidimensional case.

One can show that the path integral (44) also gives an element of $\mathscr{C}(U_{t'',t'})$ for the more general quadratic Hamiltonian

$$H(t) = \frac{1}{2}\sum_{i,j=1}^{\nu}[A_{ij}(t)(P_iP_j + Q_iQ_j) + 2B_{ij}(t)P_iQ_j] + \sum_{j=1}^{\nu}[S_j(t)Q_j + R_j(t)P_j],$$

with $A_{ji}(t) = A_{ij}(t)$, $B_{ij}(t) = -B_{ji}(t)$, and A_{ij}, B_{ij} almost everywhere differentiable and piecewise continuous. The only change needed in (44) is the replacing of the constant matrices A, B by time-dependent ones. Note, however, that this is a generalization on the level of the path integral only, while for the Hamiltonian (39) (i.e., constant matrices A, B) the complete construction in Sec. 2 can be generalized; this is not true for the case where A, B are time-dependent; it would then be necessary to choose also the basis vectors $|n\rangle$ time-dependent, and the evaluation of the resulting formulas as traces (see above) would no longer hold.

3. EVALUATION OF THE PATH INTEGRALS

Since the path integrals (37), (34), and (33) can all be obtained from (38) [by putting, respectively, $r = s = 0$ for (37), $\alpha = 0$ for (34), and $r = s = \alpha = 0$ for (33)], we shall only evaluate (38) here.

A. Notations

For reasons of convenience, and to shorten the calculations, we shall use the more condensed symplectic notation system, introduced in Ref. 8 and frequently used thereafter in, e.g., studies of Weyl quantization[9]:

$$\begin{aligned}
&(p,q) = v,\\
&\sigma(v',v'') = \tfrac{1}{2}(p',q'' - p'',q'),\\
&Jv = J(p,q) = (-q,p),\\
&s(v',v'') = \sigma(v',Jv'') = \tfrac{1}{2}(p'p'' + q'q'').
\end{aligned} \qquad (40)$$

Some simple and useful properties of σ, s, an J are

$$\begin{aligned}
&\sigma(v,v) = 0,\\
&J^2 = -\mathbb{1},\\
&\sigma(Jv',Jv'') = \sigma(v',v'') = s(Jv',v'') = -s(v',Jv''),\\
&e^{\gamma J} = \cos\gamma\mathbb{1} + \sin\gamma J\ (\gamma\in\mathbb{C}),\\
&\sigma(e^{J\gamma}v',e^{\gamma J}v'') = \sigma(v',v'')\quad(\gamma\in\mathbb{C}),\\
&s(e^{\gamma J}v',e^{\gamma J}v'') = s(v',v'')\quad(\gamma\in\mathbb{C}).
\end{aligned} \qquad (41)$$

We shall also use the following consequence of the properties of J:

$$(\mathbb{1} - e^{iJ\alpha})^{-1}(a - e^{iJ\alpha}b) = \frac{-i}{2}\coth\frac{\alpha}{2}J(a-b) + \frac{a+b}{2}. \qquad (42)$$

Furthermore, we introduce the notations

$$|v|^2 = s(v,v) = \tfrac{1}{2}(p^2 + q^2),$$

$$\omega(v) = \exp(-\tfrac{1}{2}|v|^2).$$

In these notations, (6) and (13), e.g., become

$$\langle v''|v'\rangle = e^{i\sigma(v',v'')}\omega(v' - v''),$$

$$W(v')W(v'') = e^{i\sigma(v',v'')}W(v' + v'').$$

In the symplectic notation system, we can rewrite (38) as

$$2\pi e^{(t''-t')/2}\int d\mu_{W,\,v''\leftarrow v'}^{\,t''\leftarrow t'}(v)\times\exp\left\{\int[(i+2\alpha)\sigma(v,dv) + 2\sigma(b,dv)] + \int dt\,[-\alpha(i+\alpha)s(v,v) - 2(i+\alpha)s(b,v) - s(b,b)]\right\}, \qquad (43)$$

where we have written $b(t)$ for $(r(t), s(t))$; in general, both α and b are time-dependent.

Almost the same integral can be written for the more complicated Hamiltonian (39); while this integral would be rather lengthy to write in the conventional notations, in the symplectic notation system it becomes

$$2\pi e^{(t''-t')/2}\int d\mu_{W,\,v''\leftarrow v'}^{\,t''\leftarrow t'}(v)\times\exp\left\{\int[\sigma(iv + 2\alpha Cv + 2b,dv)] + \int dt\,[-s(v,i\alpha Cv + \alpha^2C^2v) - 2s(ib + \alpha Cb,v) - s(b,b)]\right\}, \qquad (44)$$

where now v is a 2ν-dimensional pinned Wiener process, where b is the 2ν-dimensional vector $(R_1,\ldots,R_\nu; S_1,\ldots,S_\nu)$, and where C is the $2\nu\times2\nu$ matrix $\left(\begin{smallmatrix}A & B\\ -B & A\end{smallmatrix}\right)$.

In evaluating any of these Gaussian integrals, the result will be given by the contribution of the extremal path, multiplied by a suitable constant.

B. Contribution of the extremal path

To determine the extremal path, we proceed formally and extract the $\exp[-\frac{1}{2}(\dot{p}^2+\dot{q}^2)]$ from the Wiener measure, and (43) then becomes

$$2\pi e^{(t''-t')/2}\int Dv^{v''\leftarrow v'}_{t''\leftarrow t'}\exp\left\{\int_{t'}^{t''} dt\,[i\sigma(-\dot{v}+\alpha Jv+2Jb,v)-|-\dot{v}+\alpha Jv+b|^2]\right\}. \tag{45}$$

We can rewrite the integrand in the exponent as

$$\begin{aligned} F(v,\dot{v},t) &= -\alpha(i+\alpha)s(v,v)+s(v,-2(i+\alpha)b-(i+2\alpha)J\dot{v})+f(\dot{v},t)\\ &= -s(\dot{v},\dot{v})+s(\dot{v},2\alpha Jv+2Jb+iJv)+g(v,t). \end{aligned}$$

The variational equations are therefore (we assume α to be differentiable a.e.)

$$-2\alpha(i+\alpha)v-2(i+\alpha)b-(i+2\alpha)J\dot{v}-\frac{d}{dt}(-2\dot{v}+2\alpha Jv+2Jv+iJv)=0$$

or

$$\ddot{v}-(i+2\alpha)J\dot{v}-\alpha(i+\alpha)Jv-\dot{\alpha}Jv=(i+\alpha)b+J\dot{b},$$

which can be rewritten as

$$\left[\frac{d}{dt}-(i+\alpha)J\right]\left(\frac{d}{dt}-\alpha J\right)v=J\left[\frac{d}{dt}-(i+\alpha)J\right]b.$$

The extremal path is therefore given by

$$v(t)=e^{a(t)J}[c+e^{iJ(t-t')}d+JB(t)],$$

where

$$a(t)\equiv\int_{t'}^{t}ds\,\alpha(s),\quad B(t)\equiv\int_{t'}^{t}ds\,e^{-a(s)J}b(s) \tag{46}$$

(to define a, we assume α to be locally L^1), while the boundary conditions $v(t')=v'$, $v(t'')=v''$ impose

$$c+d=v',\quad c+e^{iJ(t''-t')}d=e^{-a(t'')J}v''-JB(t''). \tag{47}$$

We now evaluate the exponential in (45) for this extremal path. Since [from (46)]

$$-\dot{v}+\alpha Jv+Jb=-iJe^{[a(t)+i(t-t')]J}d,$$

we have [use (41)]

$$\begin{aligned} &i\sigma(\dot{v}+\alpha J+2Jb,v)-s(-\dot{v}+\alpha Jv+Jb,-\dot{v}+\alpha Jv+Jb)\\ &\quad=i\sigma(-iJe^{iJ(t-t')}d+J\dot{B}(t),c+e^{iJ(t-t')}d+JB(t))+s(d,d)\\ &\quad=i\sigma(\dot{B}(t),B(t))-i\frac{d}{dt}s(B(t),c+e^{iJ(t-t')}d)-s(e^{iJ(t-t')}d,c). \end{aligned}$$

Integrating this we obtain

$$\begin{aligned} &\int_{t'}^{t''}dt\,[i\sigma(-\dot{v}+\alpha Jv+2Jb,v)-|-\dot{v}+\alpha Jv+b|^2]\\ &\quad=i\int_{t'}^{t''}\sigma(\dot{B}(t),B(t))dt-is(B(t''),c+e^{iJ(t''-t')}d)-i\sigma(e^{iJ(t''-t')}d-d,c)\\ &\quad=i\int_{t'}^{t''}dt\,\sigma(\dot{B}(t),B(t))-is(B(t''),e^{-a(t'')J}v'')\\ &\quad\quad-i\sigma(e^{-a(t'')J}v''-JB(t'')-v',[1-e^{iJ(t''-t')}]^{-1}[e^{-a(t'')J}v''-JB(t'')-e^{iJ(t''-t')}v']). \end{aligned}$$

Using (42), this becomes

$$\begin{aligned} \int_{t'}^{t''}dt\cdots &= i\int_{t'}^{t''}dt\sigma(\dot{B}(t),B(t))-is(B(t''),v'+e^{-a(t'')J}v'')\\ &\quad-i\sigma(e^{-a(t'')J}v'',v')-\tfrac{1}{2}\coth\frac{t''-t'}{2}|e^{-a(t'')J}v''-JB(t'')-v'|^2. \end{aligned}$$

So finally (43) is equal to

$$\begin{aligned} (43) &= 2\pi e^{(t''-t')/2}A_{t'',t'}\cdot\exp\Big[i\int_{t'}^{t''}dt\,\sigma(\dot{B}(t),B(t))\\ &\quad-is(B(t''),v'+e^{-a(t'')J}v'')-i\sigma(e^{-a(t'')J}v'',v')\Big]\omega[e^{-a(t'')J}v''-v'-JB(t'')]^{\coth[(t''-t')/2]}, \end{aligned} \tag{48}$$

where the multiplicative constant $A_{t'',t'}$ still has to be determined, and where $a(t)$, $B(t)$ are given by (46).

C. Determination of the multiplicative constant

In this and in some of the following subsections we shall use the shorthand dv for the measure $dp\,dq/2\pi$.

With respect to this measure, we have

$$\int dv\, \omega^2(v) = 1.$$

$$\int dv\, e^{i\beta\sigma(v',v)}\omega(v)^\alpha = \frac{2}{\alpha}\,\omega(v')^{\beta^2/\alpha}. \tag{49}$$

We shall also use the following property:

Take any complex $2n\times 2n$ matrix A (matrix elements A_{ij}) satisfying

$$A^t = A,$$

$$\operatorname{Re} A = \frac{A + A^\dagger}{2} > 0;$$

let a_{ij} be the 2×2 matrices

$$a_{ij} = \begin{pmatrix} A_{2i-1,2j-1} & A_{2i-1,2j} \\ A_{2i,2j-1} & A_{2i,2j} \end{pmatrix},$$

then

$$\int dv_1\cdots\int dv_n\, \exp\Big[-\sum_{i,j=1}^{n} s(v_i,a_{ij}v_j)\Big] = \frac{1}{(\det A)^{1/2}}. \tag{50}$$

If A is real, the square root to be chosen is the positive one; if A is not a real matrix, the sign of $(\det A)^{1/2}$ is determined as follows:

$$(\det A)^{1/2} = \lim_{\lambda\to 1} f(\lambda),\ \text{with}\ \begin{cases} f:[0,1]\to\mathbb{C}\ \text{continuous}, \\ f(0)\in\mathbb{R}_+ \\ f(\lambda)^2 = \det(\operatorname{Re} A + i\lambda \operatorname{Im} A). \end{cases}$$

Let us now proceed to the determination of $A_{t'',t'}$ under the assumption that $\alpha(t)$ is a continuous function (this condition will be relaxed at the end of this section). As usual, the constant $A_{t'',t'}$ can be shown (by a variational argument) to be independent of the boundary conditions v',v'' and of the linear parts of the integrand in the exponential in (45). Hence

$$A_{t'',t'} = \int Dv_{0\leftarrow 0}^{t''\leftarrow t'} \times \exp\Big\{\int_{t'}^{t''} dt\,[i\sigma(-\dot v + \alpha J v, v) - |\dot v - \alpha J v|^2]\Big\}.$$

Writing this out as a limit, we obtain [as before $\epsilon = (t''-t')/n$]

$$\begin{aligned}
A_{t'',t'} &= \lim_{n\to\infty}\frac{1}{2\pi\epsilon^n}\int dv_1\cdots\int dv_{n-1}\prod_{j=0}^{n-1}\exp\Big[i\sigma(-v_{j+1}+\alpha(t'+\epsilon)Jv_j\epsilon, v_j) - \frac{1}{\epsilon}|v_{j+1}-v_j-\alpha(t'+j\epsilon)Jv_j\epsilon|^2\Big](v_n = v_0 = 0)\\
&= \lim_{n\to\infty}\frac{1}{2\pi\epsilon}\int du_1\cdots du_{n-1}\prod_{j=0}^{n-1}\exp[-is(Ju_{j+1}+\alpha(t'+j\epsilon)u_j\epsilon, u_j)\epsilon\\
&\quad - s(u_{j+1}-u_j-\alpha(t'+j\epsilon)Ju_j, u_{j+1}-u_j-\alpha(t'+j\epsilon)u_j)](u_n = u_0 = 0)\\
&= \lim_{n\to\infty}\frac{1}{2\pi\epsilon}\int du_1\cdots\int du_{n-1}\exp\Big[-\sum_{i,j=1}^{n-1} s(u_j, m_{i,j}u_j)\Big],
\end{aligned} \tag{51}$$

where $m_{i,j}$ are 2×2 matrices defined by

$$m_{k,k} = [2 + \epsilon^2\alpha(t'+k\epsilon)(i + \alpha(t'+k\epsilon))]\,\mathbb{1},$$

$$m_{k,k+1} = -\mathbb{1} + \Big[\frac{i}{2} + \alpha(t'+k\epsilon)\Big]\epsilon J,$$

$$m_{k+1,k} = -\mathbb{1} - \Big[\frac{i}{2} + \alpha(t'+k\epsilon)\Big]\epsilon J,$$

$$m_{k,l} = 0 \quad \text{if} \quad |k-l| > 1,$$

with J [as in (40)] given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Applying (50) to (51) we see that

$$A_{t'',t'} = \lim_{n\to\infty}\frac{1}{2\pi}\frac{1}{\epsilon(\det M_n)^{1/2}}, \tag{52}$$

where M_n is the $2(n-1)\times 2(n-1)$ matrix constituted by the 2×2 blocks $m_{i,j}$. We shall calculate the limit (52) in the standard way, i.e., by constructing a recursion formula for $\det M_n$. In the limit for $n\to\infty$, this recursion formula will become a differential equation, and the solution of this differential equation then gives an explicit expression for (52). Due to the particular structure of the matrix M_n, and the continuity of α, all $\alpha(t)$ dependence will cancel from the differential equation, leading to a constant $A_{t'',t'}$ independent of $\alpha(t)$!

For n fixed we define $M_{n,k}$ to be the $2k\times 2k$ matrix constructed from the 2×2 blocks $m_{i,j}$ with $i,j\leqslant k$ in the fashion

$$M_{n,k} = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,k} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,k} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m_{k,1} & m_{k,2} & \cdots & m_{k,k} \end{pmatrix}.$$

We furthermore define

$D_{n,k} = \det M_{n,k}$,

$D_{n,k,1}$ = minor in $M_{n,k}$ of the matrix element $(M_{n,k})_{2k,2k}$,

and

$D_{n,k,2}$ = minor in $M_{n,k}$ of the matrix element $(M_{n,k})_{2k-1,2k-1}$.

Developing the determinants $D_{n,k}$, $D_{n,k,1}$, and $D_{n,k,2}$ into products of matrix elements with the appropriate minors (we develop these determinants along the last row of columns), we obtain the following two recursion relations:

$$D_{nj} = [2+\alpha_j(i+\alpha_j)\epsilon^2](D_{n,j,1}+D_{n,j,2}) + [2+\alpha_j(i+\alpha_j)\epsilon^2]^2 D_{n,j-1} + \left[1+\left(\frac{i}{2}+\alpha_j\right)^2\epsilon^2\right]^2 D_{n,j-2}, \tag{53}$$

$$D_{n,j,1}+D_{n,j,2} = 2[2+\alpha_j i+\alpha_j)\epsilon^2]D_{n,j-1} - \left[1+\left(\frac{i}{2}+\alpha_j\right)^2\epsilon^2\right](D_{n,j-1,1}+D_{n,j-1,2}),$$

where we have written α_k for $\alpha(t'+k\epsilon)$. The quantity of interest to us is

$$f_n[t'+(j-1)\epsilon] = \epsilon^2 D_{n,j}. \tag{54}$$

In the limit for $n\to\infty$, this relation defines a (continuous) function f_∞ on $[t',t'']$, and we see from (52) that

$$A_{t'',t'} = \frac{1}{2\pi}\frac{1}{[f_\infty(t'')]^{1/2}}, \tag{55}$$

where the procedure discussed below (50) has to be applied to determine the sign of the square root. By analogy with (54), we define

$$g_n[t'+(j-1)\epsilon] = \epsilon^2(D_{n,j,1}+D_{n,j,2}).$$

In terms of the g_n, f_n the recursion equations (53) can be written as $[t_j = t'+(j-1)\epsilon]$

$$f_n(t_j) = \frac{g_n(t_{j+1}) + \{1+\epsilon^2[i/2+\alpha(t_{j+1})]^2\}g_n(t_j)}{2\{2+\alpha(t_{j+1})[i+\alpha(t_{j+1})]\epsilon^2\}},$$

$$g_n(t_j) = \frac{f_n(t_j) + \{2+\alpha(t_j)[i+\alpha(t_j)]\epsilon^2\}^2 f(t_{j-1}) - \{1+\epsilon^2[i/2+\alpha(t_j)]^2\}^2 f(t_{j-2})}{2+\alpha(t_j)[i+\alpha(t_j)]\epsilon^2}.$$

Substituting the expression for g_n into the equation for f_n, and grouping the terms of order $1, \epsilon^2, \epsilon^4, \cdots$ together, we obtain from these equations the relation

$$-2f_n(t_{j+1}) + 6f_n(t_j) - 6f_n(t_{j-1}) + 2f_n(t_{j-2}) = \epsilon^2[\xi_j f_n(t_{j+1}) + (2\zeta_{j+1} - 7\xi_{j+1} - 4\xi_j)f_n(t_j) + (4\zeta_{j+1} + 4\xi_{j+1} + 7\xi_j)f_n(t_{j-1}) + (-2\zeta_{j+1} - \xi_{j+1} - 4\xi_j)f_n(t_{j-2})] + O(\epsilon^4), \tag{56}$$

where

$$\xi_j = \alpha(t_j)[i+\alpha(t_j)], \quad \zeta_j = \left[\frac{i}{2}+\alpha(t_j)\right]^2.$$

Using the fact that $\zeta_j = \xi_j - \frac{1}{4}$, (56) can be rewritten as

$$-2[f_n(t_{j+1}) - 3f_n(t_j) + 3f_n(t_{j-1}) - f_n(t_{j-1})] = \epsilon^2\{-\tfrac{1}{2}[f_n(t_j) + 2f_n(t_{j-1}) - 3f_n(t_{j-2})] + \xi_{j+1}[-5f_n(t_j) + 8f_n(t_{j-1}) - 3f_n(t_{j-2})] + \xi_j[f_n(t_{j+1}) - 4f_n(t_j) + 7f_n(t_{j-1}) - 4f_n(t_{j-2})]\} + O(\epsilon^4). \tag{57}$$

Equation (57) holds for fixed n, and for j:$2\to n-1$, again with $\epsilon = (t''-t')/n$. In the limit for $n\to\infty$, (57) will lead to a differential equation for f_∞, and as an intermediate step we obtain

$$-2f_n'''(t_j)\epsilon^3 + O(\epsilon^4) = \epsilon^2[-\tfrac{1}{2}\cdot 4f_n'(t_j)\epsilon + O(\epsilon^2) + \xi_{j+1}(-2)f_n'(t_j)\epsilon + O(\epsilon^2) + \xi_j\cdot 2f_n'(t_j)\epsilon + O(\epsilon^2)] + O(\epsilon^4),$$

hence

$$f_n'''(t_j) = f_n'(t_j) + (\xi_{j+1} - \xi_j)f_n'(t_j) + O(\epsilon). \tag{58}$$

It is here that the "miracle" happens: If $\alpha(t)$ is a continuous function, then

$$\xi_{j+1} - \xi_j = \alpha(t_{j+1})[i+\alpha(t_{j+1})] - \alpha(t_j)[i+\alpha(t_j)] = o(1),$$

which means that the α-dependent terms ξ_j drop out of the equation. In the limit $n\to\infty$, Eq. (58) becomes

$$f'''_\infty(t) = f'_\infty(t),$$

where all the α dependence has vanished!

To determine the initial conditions for this differential equation, we go back to the definition (54) of the f_n, and we easily obtain

$$f_\infty(t') = \lim_{n\to\infty} f_n(t_0) = \lim_{n\to\infty} \epsilon^2 \cdot 1 = 0,$$

$$f'_\infty(t') = \lim_{n\to\infty} \frac{f_n(t_1) - f_n(t_0)}{\epsilon} = \lim_{n\to\infty} \epsilon[(2 + A_1\epsilon^2)^2 - 1] = 0,$$

$$f''_\infty(t') = \lim_{n\to\infty} \frac{f_n(t_2) - 2f_n(t_1) + f_n(t_0)}{\epsilon^2} = \lim_{n\to\infty} \{ [(2 + \xi_2\epsilon^2)(2 + \xi_1\epsilon^2) - \zeta_2\epsilon^2 - 1]^2 - 2(2 + \xi_1\epsilon^2)^2 + 1 \} = 2.$$

With these initial conditions, the solution of the differential equation is

$$f_\infty(t) = 2[\cosh(t - t') - 1] = 4\left(\sinh\frac{t - t'}{2}\right)^2.$$

One can easily check that the procedure sketched under (50) to determine the sign of the square root of det M_n gives

$$\lim_{n\to\infty} \sqrt{\det M_n} = \lim_{\lambda\to 1} \frac{2}{\lambda} \sinh\frac{(t'' - t')\lambda}{2} = 2 \sinh\frac{t'' - t'}{2};$$

so finally [from (55)] we find

$$A_{t'',t'} = \left(4\pi \sinh\frac{t'' - t'}{2}\right)^{-1}. \tag{59}$$

Substituting expression (59) for $A_{t'',t'}$ into (48), we have as a final result

$$\begin{aligned}
&2\pi e^{(t'' - t')/2} \int d\mu_{W,\, v'' \leftarrow v'}^{\, t'' \leftarrow t'} \exp\left[\int [(i + 2\alpha)\sigma(v,dv) + 2\sigma(b,dv)] + \int dt\, [-\alpha(i + \alpha)s(v,v) - 2(i + \alpha)s(b,v) - s(b,b)] \right] \\
&= \frac{1}{1 - e^{-(t'' - t')}} \exp\Big\{ i \int_{t'}^{t''} dt_1 \int_{t'}^{t_1} dt_2 [\sigma(b(t_1),b(t_2))\cos(\beta(t_1,t_2)) + s(b(t_1),b(t_2))\sin(\beta(t_1,t_2))] \\
&\quad - i \int_{t'}^{t''} dt\, [s(b(t),v')\cos(\beta(t,t')) - \sigma(b(t),v')\sin(\beta(t,t')) \\
&\quad + s(b(t),v'')\cos(\beta(t'',t')) + \sigma(b(t),v'')\sin(\beta(t'',t))] - i\sigma(v'',v')\cos(\beta(t'',t')) - is(v'',v')\sin\beta(t'',t') \Big\} \\
&\quad \times \omega\left\{ v''\cos[\beta(t'',t')] - Jv''\sin[\beta(t'',t')] - v' - \int_{t'}^{t} dt\, [Jb(t)\cos(\beta(t,t')) + b(t)\sin(\beta(t,t'))] \right\}^{\coth[(t'' - t')/2]},
\end{aligned} \tag{60}$$

where

$$\beta(t_1,t_2) = \int_{t_2}^{t_1} dt\, \alpha(t).$$

Putting $b(t) = [0,s(t)]$, $\alpha(t) = \alpha$ (time independent), (60) can easily be seen to lead to expression (15) in Ref. 4.

Remark: In what follows, we shall denote the right-hand side of (60) by $F_{t'',t'}(v'',v')$:

$$\begin{aligned}
F_{t'',t'}(v'',v') = {} & [1 - e^{-(t'',t')}]^{-1} \exp\Big[i \int_{t'}^{t''} dt\, \sigma(\dot{B}(t),B(t)) \\
& - i\sigma(v' + e^{-a(t'')J}v'', JB(t'')) - i\sigma(e^{-a(t'')J}v'',v') \Big] \omega(v' + JB(t'') - e^{-a(t'')J}v'')^{\coth[(t'' - t')/2]}.
\end{aligned}$$

One can easily show that these $F_{t'',t'}$ have an interesting property,

$$\forall \tilde{t} \in [t,t'']: \int d\tilde{v}\, F_{t',\tilde{t}}(v,\tilde{v}) F_{\tilde{t},t''}(\tilde{v},v'') = F_{t',t''}(v',v'').$$

This can be proven by direct computation; it can also be considered as a consequence of the fact that

$$\forall \tilde{t} \in [t',t'']: \int d\tilde{v} \int d\mu_{W,\, v'' \leftarrow \tilde{v}}^{\, t'' \leftarrow \tilde{t}}\, d\mu_{W,\, \tilde{v} \leftarrow v'}^{\, \tilde{t} \leftarrow t'} = d\mu_{W,\, v'' \leftarrow v'}^{\, t'' \leftarrow t'}.$$

This last property can be used to relax the continuity requirement on α (see above); for piecewise continuous α, we can cut the path integral into different pieces corresponding to time intervals on which α is continuous. For each of these pieces, our evaluations as carried out above hold without any problem. We can then use the "chain property" of the $F_{t'',t'}$ as stated above (the direct proof of which does not require α to be continuous) to show that even for piecewise continuous α the result (60) still holds.

Bringing all our conditions on α, r, and s together, we see now that
—α has to be piecewise continuous, a.e. differentiable and locally L^1
—r and s have to be in $L^2([t',t''])$.

Now that we have calculated the integral, we shall verify in the next section that the result is indeed an element of $\mathscr{C}(U_{t'',t'})$ for the corresponding Hamiltonian; we shall also discuss in what respect it differs from the matrix element $\langle v''|U_{t'',t'}|v'\rangle$.

4. THE PATH INTEGRALS YIELD ELEMENTS OF THE PROPAGATOR EQUIVALENCE CLASS

Let us denote the function defined by (60) by $F_{t'',t'}(v'',v')$. We claim that $F_{t'',t'}\in\mathscr{C}(U_{t'',t'})$, i.e., that

$$\int dv''\int dv'|v\rangle F_{t'',t'}(v'',v')\langle v'| = U_{t'',t'} = T\exp\left[-i\int_{t'}^{t''} H(t)\,dt\right], \tag{61}$$

with $H(t) = \frac{1}{2}\alpha(t)(P^2 + Q^2 - \mathbb{1}) + r(t)P + s(t)Q$. To prove (61) it is sufficient to show for all v that

$$\int dv''\int dv'\langle v|v''\rangle F_{t'',t'}(v'',v')\langle v'|v\rangle = \langle v|U_{t'',t'}|v\rangle. \tag{62}$$

The equivalence of (61) and (62) follows from a standard analyticity argument: Since $F_{t'',t'}$ is uniformly bounded,

$|F_{t'',t'}(v'',v')|\leqslant[1-e^{-(t''-t')}]^{-1}$ (we always assume $t''>t'$),

one can use [see (16)]

$$\langle a|b\rangle = e^{(p_a+ix_a)(p_b-ix_b)/2}e^{(x_a^2+p_a^2+x_b^2+p_b^2)/4}$$

to show that the function

$$\phi(v_1,v_2) = \int dv''\int dv'\langle v_2|v''\rangle F_{t'',t'}(v'',v')\langle v'|v_1\rangle$$

can be written as a product,

$$\phi(v_1,v_2) = f(v_1,v_2)e^{(x_1^2+p_1^2+x_2^2+p_2^2)/4},$$

where f is a complex analytic function in the variables p_1+ix_1, p_2-ix_2. The matrix element $\langle v_1|U_{t'',t'}|v_2\rangle$ is a function of the same type:

$$\langle v_1|U_{t'',t'}|v_2\rangle = g(v_1,v_2)e^{-(x_1^2+p_1^2+x_2^2+p_2^2)/4}$$

with g complex analytic in p_1+ix_1, p_2-ix_2. Equation (62) can be rewritten as

$$\forall v{:}f(v,v) = g(v,v).$$

Because of their analyticity, this condition forces f and g to be identical:

$$\forall v_1,v_2{:}f(v_1,v_2) = g(v_1,v_2)$$

or

$$\forall v_1,v_2{:}\int dv''\int dv'\langle v_2|v''\rangle F_{t'',t'}(v'',v')\langle v'|v_1\rangle = \langle v_2|U_{t'',t'}|v_1\rangle. \tag{63}$$

Now using the fact that $|F_{t'',t'}|$ is an L^1 function in $v''-v'$, and the density of the linear span of the c.s. in $\mathscr{H}$, one sees that (63) implies (61).

Note that the argument above only uses properties of the "small" space $\mathscr{H}$. The "big" space $\hat{\mathscr{H}}$ was only introduced as a device to define $F_{t'',t'}$ as a path integral with respect to a genuine measure. Once $F_{t'',t'}$ is found, we no longer concern ourselves with $\hat{\mathscr{H}}$ or the $|p,q;\beta\rangle\rangle$, but simply prove directly that $F_{t'',t'}\in\mathscr{C}(U_{t'',t'})$.

We now proceed to prove (62). Using $\langle v|v''\rangle\langle v'|v\rangle = e^{i\sigma(v,v'-v'')}\omega(v-v')\omega(v''-v)$ (see Sec. 3A and (60)), we have

$$\begin{aligned}\int dv''\int dv'\langle v|v''\rangle F_{t'',t'}(v'',v')\langle v'|v\rangle &= \int dv''\int dv' e^{i\sigma(v,v'-v'')}\omega(v-v')\omega(v-v'')[1-e^{-(t''-t')}]^{-1}\\ &\times\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot B(t),B(t)) - i\sigma(v'+e^{-a(t'')J}v'',JB(t''))\right]\\ &\times\exp[-i\sigma(e^{-a(t'')J}v'',v')]\,\omega[e^{-a(t'')J}v''-v'-JB(t'')]^{\coth[(t''-t')/2]},\end{aligned} \tag{64}$$

where

$$a(t) = \int_{t'}^{t} ds\,\alpha(s),\quad B(t) = \int_{t'}^{t} ds\,e^{-a(s)J}b(s).$$

Introducing the change of variable $u'' = e^{-a(t'')J}v''$, and using (41), Eq. (64) becomes

$$(64) = [1-e^{-(t''-t')}]^{-1}\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]\int du''\int dv'\exp[-i\sigma(u'',JB(t'')-e^{-a(t'')J}v)]$$
$$\times e^{-i\sigma(v',JB(t'')+v-u'')}\omega(v'-v)\omega[u''-v'-JB(t'')]^{\coth[(t''-t')/2]}\omega[u''-e^{-a(t'')J}v].$$

Taking the Gaussian in v' together and completing the squares, we have

$$\omega(v'-v)\omega[v'-u''+JB(t'')]^{\coth[(t''-t')/2]}$$
$$=\omega\left\{v'-\left[v+\coth\frac{t''-t'}{2}\cdot(u''-JB(t''))\right]\Big/\left[1+\coth\frac{t''-t'}{2}\right]\right\}^{1+\coth[(t''-t')/2]}$$
$$\cdot\,\omega[v-(u''-JB(t''))]^{\coth[(t''-t')/2]/\{1+\coth[(t''-t')/2]\}}.$$

Substituting this in the integral above, and making the change of variable

$$u'=v'-\frac{1}{1+\coth[(t''-t')/2]}\left[v+\coth\frac{t''-t'}{2}(u''-JB(t''))\right],$$

(64) becomes

$$(64) = [1-e^{-(t''-t')}]^{-1}\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]\int du''\int du'\exp[-i\sigma(u'',JB(t'')-e^{-a(t'')J}v)]e^{-i\sigma(v,JB(t'')-u'')}$$
$$\times e^{-i\sigma(u',JB(t'')-u''+v)}\omega[u''-e^{-a(t'')J}v]\omega(u')^{1+\coth[(t''-t')/2]}\cdot\omega[v-u''+JB(t'')]^{\coth[(t''-t')/2]/\{1+\coth[(t''-t')/2]\}}.$$

Applying (49) to the u' integral yields

$$(64) = \frac{2}{[1-e^{-(t''-t')}](1+\coth[(t''-t')/2])}\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]$$
$$\times\int du''\exp[-i\sigma(u'',JB(t'')-e^{-a(t'')J}v+v)]e^{-i\sigma(v,JB(t''))}\omega[u''-e^{-a(t'')J}v]\omega[v-u''+JB(t'')].$$

Again we group the Gaussians in u'', and complete the squares

$$\omega[u''-e^{-a(t'')J}v]\omega[u''-v-JB(t'')]=\omega\{u''-\tfrac{1}{2}[e^{-a(t'')J}v+v+JB(t'')]\}^2\omega[v+JB(t'')-e^{-a(t'')J}v]^{1/2}.$$

Substituting this into the integral, and making the change of variable

$$u=u''-\tfrac{1}{2}[e^{-a(t'')J}v+v+JB(t'')],$$

we obtain

$$(64) = \frac{2[e^{(t''-t')/2}-e^{-(t''-t')/2}]}{[1-e^{-(t''-t')}]\cdot 2e^{(t''-t')/2}}\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]\int du\,\exp[-i\sigma(e^{-a(t'')J}v,v+JB(t''))]e^{-i\sigma(v,JB(t''))}$$
$$\times\omega[v+JB(t'')-e^{-a(t'')J}v]^{1/2}\exp[-i\sigma(u,v+JB(t'')-e^{-a(t'')J}v)]\omega^2(u).$$

Applying (49) again, this becomes

$$(64) = \exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]e^{-i\sigma(v,JB(t''))}\exp[-i\sigma(e^{-a(t'')J}v,v+JB(t''))]\omega[v+JB(t'')-e^{-a(t'')J}v]$$
$$= \exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]e^{-i\sigma(v,JB(t''))}\langle e^{-a(t'')J}v|v+JB(t'')\rangle \tag{65}$$

(see Sec. 3A).

Since (see Appendix) $e^{-i\beta N}|v\rangle = |e^{\beta J}v\rangle$, we thus have

$$\int dv''\int dv'\langle v|v''\rangle F_{t'',t'}(v'',v')\langle v'|v\rangle = \exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]e^{-\sigma(v,JB(t''))}\langle v|e^{-ia(t'')N}|v+JB(t'')\rangle$$
$$= \exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]\langle v|e^{-ia(t'')N}W(JB(t''))|v\rangle.$$

Our claim (61) thus reduces to

$$\exp\left[i\int_{t'}^{t''} dt\,\sigma(\dot{B}(t),B(t))\right]e^{-ia(t'')N}W(JB(t''))=U_{t'',t'}=T\exp\left[-i\int_{t'}^{t''} dt\,H(t)\right], \tag{66}$$

where [see (46)]

$$a(t)=\int_{t'}^{t} ds\,\alpha(s),\quad B(t)=\int_{t'}^{t} ds\,e^{-a(s)J}b(s).$$

We shall prove (66) by differentiation with respect to t. We have, first of all,

$$\frac{d}{dt}W(JB(t)) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[W(JB(t+\epsilon))W(-JB(t)) - \mathbf{1}]W(JB(t))$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}[e^{-i\sigma(B(t+\epsilon),B(t))}W(JB(t+\epsilon) - JB(t)) - \mathbf{1}]W(JB(t))$$
$$= -i\sigma(\dot B(t),B(t))W(JB(t)) - 2is(\dot B(t),V)W(JB(t))$$

[the second term can be obtained by putting $B = (R,S)$]; then

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}W(JB(t+\epsilon) - JB(t))$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\exp(-i\{[R(t+\epsilon) - R(t)]P + [S(t+\epsilon) - S(t)]Q\}) = -i[\dot R(t)P + \dot S(t)Q] \equiv -2is(\dot B(t),V)\}.$$

Moreover, $e^{-i\beta N}s(B,V) = s(e^{\beta J}B,V)e^{-i\beta N}$ [this can be obtained from (A5) by differentiation].

After these preliminaries we are now ready to evaluate the time derivative of the left-hand side of (66)

$$i\frac{d}{dt}\left\{\exp\left[i\int_{t'}^{t}ds\,\sigma(\dot B(s),B(s))\right]e^{-i\alpha(t)N}W(JB(t))\right\}$$
$$= [-\sigma(\dot B(t),B(t)) + \dot a(t)N + \sigma(\dot B(t),B(t)) + 2s(e^{a(t)J}\dot B(t),V)]\exp\left[i\int_{t'}^{t}ds\,\sigma(\dot B(s),B(s))\right]e^{-a(t)N}W(JB(t))$$
$$= [\alpha(t)N + 2s(b(t),V)]\exp\left[i\int_{t'}^{t}ds\,\sigma(\dot B(s),B(s))\right]e^{-ia(t)N}W(JB(t)). \tag{67}$$

Since $\alpha(t)N + 2s(b(t),V) = \frac{1}{2}\alpha(t)(P^2 + Q^2 - \mathbf{1}) + r(t)P + s(t)Q = H(t)$, we see that $\exp[i\int_{t'}^{t}ds\,\sigma(\dot B(s),B(s))]e^{-ia(t)N}W(JB(t))$ and $T\exp[-i\int_{t'}^{t}H(t)dt]$ satisfy the same first-order differential equation in t. Since both operators have the same initial value (at $t = t'$, they are both equal to $\mathbf{1}$), they are therefore equal for all times

$$\exp\left[i\int_{t'}^{t''}ds\,\sigma(\dot B(s),B(s))\right]e^{-ia(t'')N}W(JB(t'')) = T\exp\left[-i\int_{t'}^{t''}H(t)dt\right].$$

This completes our proof that $F_{t'',t'}\in\mathscr{C}(U_{t'',t'})$.

Remarks: Comparing $F_{t'',t'}(v'',v')$,

$$F_{t'',t'}(v'',v')$$
$$= \frac{1}{1 - e^{-(t''-t')}}\exp\left[i\int_{t'}^{t''}dt\,\sigma(\dot B(t),B(t))\right]e^{-i\sigma(v',JB(t''))}\exp[-i\sigma(e^{-a(t'')J}v'',v' + JB(t''))]$$
$$\times\omega(v' + JB(t'') - e^{-a(t'')J}v'')^{\coth[(t''-t')/2]},$$

with the true matrix element $\langle v''|T\exp[-i\int_{t'}^{t''}H(t)dt]|v'\rangle$ [see (65)], we immediately see that the two expressions are very similar; there are only two differences: an overall extra factor $[1 - e^{-(t''-t')}]^{-1}$ in $F_{t'',t'}$, and an exponent $\coth[(t''-t')/2]$ for the ω-factor in $F_{t'',t'}$, where this exponent is 1 in the true matrix element. [These similarities were already noticed in Ref. 4 for the slightly simpler Hamiltonian $H(t) = \alpha(P^2 + Q^2 - 1)/2 + s(t)Q$]. In the limit where the time integral diverges, $t'' - t'\to\infty$, both these differences disappear

$$1 - e^{-(t''-t')}\to 1,\quad \coth\frac{t''-t'}{2}\to 1,$$

which means that as $t'' - t'\to\infty$, the function $F_{t'',t'}$ approaches the true matrix element; its component orthogonal to $\{\langle p'',q''|\phi\rangle\langle\psi|p',q'\rangle;\ \phi,\psi\in\mathscr{H}\}$ vanishes!

This is easily understood if one tries to analyze what happens for $t'' - t'\to\infty$ to the construction we made in Sec. 2. As an example we take the time-independent Hamiltonian $H = (\alpha/2)(P^2 + Q^2 - \mathbf{1})$. Then

$$F_{t'',t'}(p'',q'';p',q') = \lim_{k\to\infty}\frac{1}{(1-\beta)^k}\sum_{l=0}^{\infty}\left[\beta^{kl}(1-\beta)^k\int dv_{k-1}\cdots dv_1\prod_{j=0}^{k-1}\langle l|W(v_{j+1})^\dagger e^{-i\alpha(N-l)\epsilon}W(v_j)|l\rangle\right.$$

and

$$= \lim_{k\to\infty}\sum_{l=0}^{\infty}\beta^{kl}\langle l|W(v'')^\dagger e^{-i\alpha(N-l)(t''-t')}W(v')|l\rangle.$$

The term corresponding to $l = 0$ is simply $\langle v''|U_{t'',t'}|v'\rangle$. For $l\neq 0$, however, we also have to take into account a factor

$$\lim_{k\to\infty}\beta^{kl} = \lim_{k\to\infty}\left(1 - \frac{t''-t'}{2k}\right)^{kl}\left(1 + \frac{t''-t'}{2n}\right)^{-kl}$$
$$= e^{-(t''-t')l}.$$

In the limit $t'' - t'\to\infty$, these factors $e^{-(t''-t')l}\to 0$ for $l\neq 0$, which means that all the contributions to $F_{t'',t'}$ from terms with $l\neq 0$ disappear. Only the $l = 0$ term is left over; since this $l = 0$ term is exactly $\langle v''|U_{t'',t'}|v'\rangle$, we see that

$$F_{t'',t'}(v'',v') \underset{t''-t'\to\infty}{\longrightarrow} \langle v''|U_{t'',t'}|v'\rangle.$$

In the limit where $t'' - t' \to 0$, $F_{t''-t'}(v'',v')$ approaches $\delta(v''-v')$, i.e., a specific member of the equivalence class of the unit operator (see Sec. 1). Again, this can easily be understood from our construction. For the example $H = (\alpha/2) \times (P^2 + Q^2 - \mathbb{1})$, we now have

$$F_{t'',t'}(v'',v') = \sum_l e^{-(t''-t')l} \times \langle l | W(v'')^\dagger e^{-i\alpha(N-l)(t''-t')} W(v') | l \rangle .$$

As $t''-t' \to 0$, we see that $e^{-(t''-t')l} \to 1$, $e^{-i\alpha(N-l)(t''-t')} \to \mathbb{1}$, and thus as a distributional limit

$$\begin{aligned} F_{t'',t'}(v'',v') &\to \sum_l \langle l | W(v'')^\dagger W(v') | l \rangle \\ &= \text{"Tr"}(W(v'')^\dagger W(v')) \\ &= \delta(v''-v'). \end{aligned}$$

5. CONCLUSION

In this paper we have stressed the basic feature of overcompleteness of coherent states, and have used this fact to construct integral kernels to represent the evolution operator for a limited class of dynamical systems in the form of path integrals expressed in terms of Wiener measure.

Equation (38) presents the path integral representation for the most general one-dimension dynamical system that we are able to treat. This equation provides a novel formulation of the (equivalence-class) propagator and suggests a variety of further directions for study in addition to providing an alternative computational scheme for such propagators. However, our results are less than optimal in one sense. The necessary restriction that s and r be square integrable prohibits our results from describing local-in-time potentials when integrated over the external fields but only leads to nonlocal potentials. It is important to learn if and how this limitation can be overcome, and this problem may be clarified by using basic states other than the harmonic oscillator eigenstates. As observed in Ref. 4, the complex expression that plays the role of the classical action in the Wiener measure formulation of quantum mechanical path integrals may be formally interpreted in a natural way: The phase of the integrand is such as to form a martingale in which the phase-space motion is driven by the classical equations of motion. It is interesting to add that an entirely analogous type of construction can be given for kinematical groups other than the Heisenberg–Weyl group, and in particular for the kinematics of the SU(2) spin group.[10]

APPENDIX

We calculate the traces needed in Sec. 2:

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger W(p',q')], \tag{A1}$$

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger e^{-i\alpha N\epsilon} W(p',q')], \tag{A2}$$

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger e^{-i\alpha N\epsilon} W(p',q') e^{i\alpha N\epsilon}], \tag{A3}$$

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger e^{-i(\alpha N + sQ + rP)\epsilon} W(p',q') e^{i\alpha N\epsilon}], \tag{A4}$$

where $N = \frac{1}{2}(P^2 + Q^2 - \mathbb{1})$. The last trace (A4) corresponds to the choice $H_{kk}(t) = H(t) - \langle H(t) \rangle_k$ with $H(t) = \frac{1}{2}\alpha(t)(P^2 + Q^2 - \mathbb{1}) + s(t)Q + r(t)P$ (see the end of Sec. 2); since $\langle Q \rangle_k = \langle P \rangle_k = 0$, the term $-\langle H(t) \rangle_k$ leads to the factor $e^{i\alpha N\epsilon}$ in (A4). We start by proving two lemmas.

Lemma 1:

$$\text{(i) } e^{-itN} W(p,q) = W(p_t,q_t) e^{-itN} \tag{A5}$$

with

$$p_t = -q \sin t + p \cos t,$$

$$q_t = q \cos t + p \sin t.$$

$$\text{(ii) } e^{-it(\alpha N + sQ + rS)} = e^{(i/2\alpha)(s^2+r^2)t} W\left(-\frac{r}{\alpha}, \frac{-s}{\alpha}\right) e^{-i\alpha t N} W\left(\frac{r}{\alpha}, \frac{s}{\alpha}\right). \tag{A6}$$

Proof:

(i) This property can be proved by direct differentiation, but it can also be considered to be a consequence of the properties of homogeneous quadratic Hamiltonians in general. Indeed, it is well known that for a homogeneous quadratic Hamiltonian H, the quantum evolution of a coherent state $|p,q\rangle$ is given by the classical evolution, under the same Hamiltonian, of the labels p,q

$$e^{-itH(P,Q)} | p,q \rangle = | p_t,q_t \rangle ,$$

where p_t,q_t are the solutions for the Hamiltonian equations for $H(p,q)$, with initial conditions $p_0 = p$, $q_0 = q$. Hence $e^{-itN} | p,q \rangle = | p_t,q_t \rangle$, with p_t,q_t as in (A5). Consequently

$$\begin{aligned} e^{-itN} W(p,q) | p',q' \rangle &= e^{-itN} e^{(1/2)i(pq'-p'q)} | p+p', q+q' \rangle \\ &= e^{(1/2)i(pq'-p'q)} | p_t + p'_t, q_t + q'_t \rangle \\ &= e^{(1/2)i(pq'-p'q)} e^{-(1/2)i(p_t q'_t - p'_t q_t)} W(p_t,q_t) | p'_t, q'_t \rangle \\ &= W(p_t,q_t) e^{-itN} | p',q' \rangle . \end{aligned}$$

Since the linear span of the c.s. is dense, (A5) follows.

(ii) We prove (A6) by differentiation. Take any ψ in the linear span of the c.s. Then

$$\begin{aligned} & i\frac{d}{dt}\left[e^{(i/2\alpha)(s^2+r^2)t} W\left(-\frac{r}{\alpha}, -\frac{s}{\alpha}\right) e^{-i\alpha t N} W\left(\frac{r}{\alpha}, \frac{s}{\alpha}\right)\right]\psi \\ &= -\frac{1}{2\alpha}(s^2+r^2)[\cdots]\psi + e^{(i/2\alpha)(s^2+r^2)t}\, W\left(-\frac{r}{\alpha}, -\frac{s}{\alpha}\right)\frac{\alpha}{2}(P^2+Q^2-\mathbb{1}) e^{-i\alpha t N} W\left(\frac{r}{\alpha}, \frac{s}{\alpha}\right)\psi \\ &= -\left\{\frac{1}{2\alpha}(s^2+r^2) + \frac{\alpha}{2}\left[\left(P+\frac{r}{\alpha}\right)^2 + \left(Q+\frac{s}{\alpha}\right)^2 - \mathbb{1}\right]\right\}[\cdots]\psi \\ &= \left[\frac{\alpha}{2}(P^2+Q^2-\mathbb{1}) + rP + sQ\right]\left[e^{(i/2\alpha)(s^2+r^2)t} W\left(-\frac{r}{\alpha} - \frac{s}{\alpha}\right) e^{-i\alpha t N} W\left(\frac{r}{\alpha}, \frac{s}{\alpha}\right)\right]\psi. \end{aligned}$$

Since the linear span of the c.s. is a core for N, (A6) follows.

Note: (A6) is still true for $\alpha\to 0$; for $\alpha=0$ the left-hand side is equal to $W(-st,-rt)$, while the right-hand side gives

$$w-\lim_{\alpha\to 0} e^{(i/2\alpha)(s^2+r^2)t}W\left(-\frac{r}{\alpha},\frac{-s}{\alpha}\right)W\left(\frac{r_t}{\alpha},\frac{s_t}{\alpha}\right)e^{-i\alpha tN}$$
$$=w-\lim_{\alpha\to 0} e^{(i/2\alpha)(s^2+r^2)(t-(1/\alpha)\sin\alpha t)}W\left(\frac{r_t-r}{\alpha},\frac{s_t-s}{\alpha}\right)$$
$$=W(-st,-rt).$$

Lemma 2: For all $\gamma\in\mathbb{C}$ such that $|\gamma|<1$: γ^N is trace-class, and

$$\mathrm{Tr}(\gamma^N W(p,q))=\frac{1}{1-\gamma}\exp\left[-\frac{1+\gamma}{4(1-\gamma)}(p^2+q^2)\right]. \tag{A7}$$

Proof: Since γ^N has only a discrete spectrum, with eigenvalues γ^n, it is obvious that γ^N is trace-class for $|\gamma|<1$. On the other hand, direct calculation from (A5) yields

$$\langle p'',q''|e^{-itN}|p',q'\rangle=\exp[\tfrac{1}{2}e^{-it}(p''+iq)(p'-iq')-\tfrac{1}{4}(p''^2+q''^2+p'^2+q'^2)],$$

hence, by analytic continuation (the c.s. are analytic vectors for N),

$$\langle p'',q''|\gamma^N|p',q'\rangle=\exp[(\gamma/2)(p''+iq'')(p'-iq')-\tfrac{1}{4}(p''^2+q''^2+p'^2+q'^2)].$$

Using (17) we now evaluate $\mathrm{Tr}(\gamma^N W(p,q))$:

$$\mathrm{Tr}(\gamma^N W(p,q))=\int\frac{dp'dq'}{2\pi}\langle p',q'|\gamma^N W(p,q)|p',q'\rangle=\int\frac{dp'dq'}{2\pi}e^{(i/2)(pq'-p'q)}\langle p',q'|\gamma^N|p+p',q+q'\rangle$$
$$=\int\frac{dp'dq'}{2\pi}\exp\left\{\frac{i(\gamma+1)}{2}(pq'-qp')-\frac{1-\gamma}{2}[(p'+p/2)^2+(q'+q/2)^2]-\frac{1+\gamma}{8}(p^2+q^2)\right\}$$
$$=\frac{1}{1-\gamma}\exp\left[-\frac{1+\gamma}{8}(p^2+q^2)-\frac{(1+\gamma)^2}{8(1-\gamma)}(p^2+q^2)\right]=\frac{1}{1-\gamma}\exp\left[-\frac{1+\gamma}{4(1-\gamma)}(p^2+q^2)\right].$$

Using this result it is now very easy to calculate the traces (A1)$\to$(A4). Since (A1) can be obtained from (A2) by taking the limit $\alpha\to 0$, and (A3) from (A4) by putting $r=s=0$, we shall only evaluate (A2) and (A4) explicitly. For (A2) we get

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger e^{-i\alpha N\epsilon}W(p',q')]=\mathrm{Tr}[\beta^N W(p'',q'')^\dagger W(p'_\alpha,q'_\alpha)e^{-i\alpha N\epsilon}]$$

[with $p'_\alpha=p'\cos\alpha\epsilon-q'\sin\alpha\epsilon,\quad q'_\alpha=p'\sin\alpha\epsilon+q'\cos\alpha\epsilon$]

$$=e^{(i/2)(p'_\alpha q''-q'_\alpha p'')}\mathrm{Tr}[(\beta e^{-i\alpha\epsilon})^N W(p'_\alpha-p'',q'_\alpha-q'')]$$
$$=\frac{1}{1-\beta e^{-i\alpha\epsilon}}e^{(i/2)(p'_\alpha q''-q'_\alpha p'')}\exp\left\{-\frac{1+\beta e^{-i\alpha\epsilon}}{4(1-\beta e^{-i\alpha\epsilon})}[(p''-p'_\alpha)^2+(q''-q'_\alpha)^2]\right\}. \tag{A8}$$

This is exactly what was used in (35); in the limit for $\alpha\to 0$, we have

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger W(p',q')]=\frac{1}{1-\beta}e^{(i/2)(p'q''-p''q')}\exp\left\{-\frac{1+\beta}{4(1-\beta)}[(p''-p')^2+(q''-q')^2]\right\}, \tag{A9}$$

which yields (24). The evaluation of (A4) gives

$$\mathrm{Tr}[\beta^N W(p'',q'')^\dagger e^{-i(\alpha N+sQ+rP)\epsilon}W(p',q')e^{i\alpha N\epsilon}]$$
$$=\mathrm{Tr}\left[\beta^N W(-p'',-q'')e^{(i/2\alpha)(s^2+r^2)\epsilon}W\left(-\frac{r}{\alpha},-\frac{s}{\alpha}\right)e^{-i\alpha N\epsilon}W\left(\frac{r}{\alpha},\frac{s}{\alpha}\right)W(p',q')e^{i\alpha N\epsilon}\right]$$
$$=\exp\left[\frac{i}{2\alpha}(s^2+r^2)\epsilon+\frac{i}{2\alpha}(p''s-q''r)+\frac{i}{2\alpha}(rq'-sp')\right]\mathrm{Tr}\left[\beta^N W\left(-p''-\frac{r}{\alpha},-q''-\frac{s}{\alpha}\right)\right.$$
$$\left.\times e^{-i\alpha N\epsilon}W\left(p'+\frac{r}{\alpha},q'+\frac{s}{\alpha}\right)e^{i\alpha N\epsilon}\right]$$
$$=\exp\left[\frac{i}{2\alpha}(s^2+r^2)\epsilon+\frac{i}{2\alpha}(p''s-q''r)+\frac{i}{2\alpha}(rq'-sp')\right]$$
$$\times\mathrm{Tr}\left\{\beta^N W\left(-p''-\frac{r}{\alpha},-q''-\frac{s}{\alpha}\right)W([p'+r/\alpha]\cos\alpha\epsilon-[q'+s/\alpha]\sin\alpha\epsilon,\right.$$

$$[p' + r/\alpha]\sin\alpha\epsilon + [q' + s/\alpha]\cos\alpha\epsilon)\Big\}$$

$$= \exp\left[\frac{i}{2\alpha}(s^2 + r^2)\epsilon + \frac{i}{2\alpha}(p''s - q''r) + \frac{i}{2\alpha}(rq' - sp')\right]\cdot\frac{1}{1-\beta}$$

$$\times\exp((i/2)\{q'' + s/\alpha)[(p' + r/\alpha)\cos\alpha\epsilon - (q' + s/\alpha)\sin\alpha\epsilon]\} - (p'' + r/\alpha)[(p' + r/\alpha)\sin\alpha\epsilon + (q' + s/\alpha)\cos\alpha\epsilon]\}$$

$$-\frac{1+\beta}{4(1-\beta)}\{[(p'' + r/\alpha) - (p' + r/\alpha)\cos\alpha\epsilon + (q' + s/\alpha)\sin\alpha\epsilon]^2$$

$$+ [(q'' + s/\alpha) - (p' + r/\alpha)\sin\alpha\epsilon - (q' + s/\alpha)\cos\alpha\epsilon]^2\}) \tag{A10}$$

[use (A5) and (A9)]. Putting $r = s = 0$, this leads to (35).

Finally, we show how, under the assumption $\beta = (1 - \epsilon/2)/(1 + \epsilon/2)$, and in the approximation that ϵ is small, (A10) leads to (38). (A10) becomes now

$$\exp\left[\frac{i}{2\alpha}(s^2 + r^2) + \frac{i}{2\alpha}(p''s - q''r) + \frac{i}{2\alpha}(rq' - sp')\right]\frac{1}{1-\beta}$$

$$\times\exp\{(i/2)[(q'' + s/\alpha)(p' + r/\alpha) - (q'' + s/\alpha)(q' + s/\alpha)\alpha\epsilon - (p'' + r/\alpha)(p' + r/\alpha)\alpha\epsilon - (p'' + r/\alpha)(q' + s/\alpha)]\}$$

$$\times\exp(-(1/2\epsilon)\{[p'' - p' + (q' + s/\alpha)\alpha\epsilon]^2 + [(q'' - q') - (p' + r/\alpha)\alpha\epsilon]^2\})$$

$$= \frac{1}{1-\beta}\exp\{-(1/2\epsilon)[(p'' - p')^2 + (q'' - q')^2] + (i/2 + \alpha)[p'(q'' - q') - q'(p'' - p')]$$

$$-(\alpha^2/2)(q'^2 + p'^2)\epsilon - (i\alpha/2)(p''p' + q''q')\epsilon - (i/2)[s(q'' + q') + r(p'' + p')]\epsilon$$

$$-\alpha(sq' + rp')\epsilon - s(p'' - p') + r(q'' - q') - \tfrac{1}{2}(s^2 + r^2)\epsilon\}.$$

Putting $p_{j+1} = p''$, $q_{j+1} = q''$ and $p_j = p'$, $q_j = q'$, and assuming $p_{j+1} - p_j = O(\sqrt{\epsilon})$, $q_{j+1} - q_j = O(\sqrt{\epsilon})$, we finally obtain

$$\mathrm{Tr}\{\beta^N W(p_{j+1},q_{j+1})^\dagger e^{-i[\alpha(t_j)N + r(t_j)P + s(t_j)Q]\epsilon} W(p_j,q_j)e^{i\alpha(t_j)N\epsilon}\}$$

$$= (1 + \epsilon/2)\frac{1}{\epsilon}\exp\{-\frac{1}{2\epsilon}[(p_{j+1} - p_j)^2 + (q_{j+1} - q_j)^2]\}$$

$$+ [i/2 + \alpha(t_j)][p_j(q_{j+1} - q_j) - q_j(p_{j+1} - p_j)] - \tfrac{1}{2}\alpha(t_j)[i + \alpha(r_j)][p_j^2 + q_j^2]\epsilon$$

$$-[s(t_j)(p_{j+1} - p_j) - r(t_j)(q_{j+1} - q_j)] - [i + \alpha(t_j)][s(t_j)p_j + r(t_j)q_j]\epsilon - \tfrac{1}{2}[s(t_j)^2 + r(t_j)^2]\epsilon + O(\epsilon^{3/2}), \tag{A11}$$

which can easily be seen to lead to (38).

[1] See, e.g., J. R. Klauder, Acta Phys. Austriaca, Suppl. XXII, 3 (1980).

[2] S. Albeverio and R. Høegh-Kröhn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics, No. 523 (Springer, Berlin, 1976).

[3] Ph. Combe, R. Høegh-Kröhn, R. Rodrigues, M. Sirugue, and M. Sirugue-Collin, Commun. Math. Phys. **77**, 269 (1980); J. Math. Phys. **23**, 405 (1982); see also *Proceedings of the International Workshop "Functional Integration: Theory and Applications,"* edited by J. P. Antoine and E. Tirapegui (Plenum, New York, 1980).

[4] J. R. Klauder and I. Daubechies, Phys. Rev. Lett. **48**, 117 (1982); see also "Wiener Measures for Quantum Mechanical Path Integrals," in *Proceedings of the International Workshop Stochastic Processes in Quantum Theory and Statistical Physics: Recent Progress and Applications*, Lecture Notes in Physics (Springer, New York) (to be published).

[5] See, e.g., J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), Chap. 7.

[6] J. McKenna and J. R. Klauder, J. Math. Phys. **5**, 878 (1964).

[7] See, e.g., B. Simon, *Functional Integration and Quantum Mechanics*, (Academic, New York, 1979).

[8] D. Kastler, Commun. Math. Phys. **1**, 14 (1965).

[9] A. Grossmann, Commun. Math. Phys. **48**, 191 (1976).

[10] J. R. Klauder, "Constructing measures for spin-variable path integrals," J. Math. Phys. **23**, 1797 (1982).