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An integral transform related to quantization

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We study in some detail the correspondence between a function \( f \) on phase space and the matrix elements \( \langle Q \rangle (a, b) \) of its quantized \( Q \), between the coherent states \( |a \rangle \) and \( |b \rangle \). It is an integral transform: \( Q(a, b) = f[a, b, |f(v)|] dv \) which resembles in many ways the integral transform of Bargmann. We obtain the matrix elements of \( Q \) between harmonic oscillator states as the Fourier coefficients of \( f \) with respect to an explicit orthonormal system.

I. INTRODUCTION

Quantization is a word which should be used with caution, since it means many things to many people. We understand it here in the sense first sketched by Weyl\(^1\), where it describes a "harmonic analysis" procedure. It consists in Fourier analyzing a (fairly arbitrary) function on phase space, and then replacing the "elementary building blocks" (i.e., exponentials on phase space) by appropriate operators (which have since been known as Weyl operators, and are exponentials of linear combinations of the operators \( X \) and \( P \)).

A satisfactory and intrinsic description of the procedure became possible when von Neumann\(^2\) proved the uniqueness theorem (Steps towards the theorem can be found in Weyl's book)\(^,\) which states that for a given (finite) number of degrees of freedom there exists—up to unitary equivalence—essentially only one irreducible family of Weyl operators in Hilbert space. This theorem is a cornerstone of quantum mechanics for a finite number of degrees of freedom. It seems however to have appeared too late to be fully incorporated in the mainstream of textbooks on the subject.

The intrinsic and symplectic formulation of quantum mechanics, made possible by von Neumann's theorem, was developed by Segal\(^3\) and Kastler\(^4\), largely as a by-product of work aimed at systems with infinitely many degrees of freedom (The MIT thesis of R. Lavine\(^5\) is devoted to finite numbers of degrees of freedom). The ingredients are as follows:

1. A phase space \( E \) which is defined (without any "a priori" decomposition into position and momentum) as an even-dimensional vector space (dim \( E = 2v \)) with an antisymmetric nondegenerate bilinear form \( \sigma \).

2. A Weyl system \( W \) which is a family of unitary operators, labeled by points in phase space, acting irreducibly on a Hilbert space \( \mathcal{H} \) and satisfying

\[
W(v_1)W(v_2) = e^{i \sigma(v_1, v_2)} W(v_1 + v_2). \tag{1.1}
\]

Given \( E \) and \( \sigma \), von Neumann guarantees the existence and uniqueness (up to unitary equivalence) of \( W \), but does not commit us to any concrete realization of \( W \). The Weyl quantization procedure is then a two-step affair: (a) Fourier analysis: \( f(v) \) is written as

\[
f(v) = 2^{-v} \int e^{i(a\sigma, v)} f(v') dv'; \tag{1.2}
\]

(b) substitution of \( W(-v/2) \) for \( e^{i(a\sigma, v)} \), giving

\[
Q(f) = 2^{-v} \int W\left(-\frac{v}{2}\right) f(v) dv \tag{1.3}
\]

as the definition of the "quantized" of \( f \).

It was shown in Ref. 3 that the correspondence \( f \rightarrow Q(f) \) is inverted by

\[
\hat{f}(v) = 2^{-v} \mathrm{tr}\left( W\left(-\frac{v}{2}\right) Q(f) \right) = 2^{-v} \left( W\left(-\frac{v}{2}\right), Q(f) \right)_{\text{HS}}, \tag{1.4}
\]

where \( (\cdot, \cdot)_{\text{HS}} \) is the inner product in the Hilbert space \( \mathcal{L}^2_{\text{HS}} \) of Hilbert-Schmidt operators in \( \mathcal{H} \), and that the map \( f \rightarrow Q(f) \) is unitary from \( L^2(E) \) onto \( \mathcal{L}^2_{\text{HS}} \). Consequently,

\[
(f_1, f_2)_{L^2(E)} = ((Q(f_1), Q(f_2))_{\text{HS}}. \tag{1.5}
\]

If \( e^a \) denotes the function \( e^a)(v) = e^{i(a, v)} \) we have

\[
Q(e^a) = W(-a/2), \text{ and so, by extension of Eq. (1.5),}
\]

\[
((Q(e^a), Q(e^b)))_{\text{HS}} = \int e^{-i(a, v)} e^{ib(v)} dv = 2^{2v} \delta(a - b), \tag{1.5}
\]

in a sense to be made precise (see, e.g., Ref. 7). This map is discussed in more detail by Pool.\(^6\)

In Ref. 9, one of us made the remark that step (a) of the quantization procedure (Fourier analysis) can be avoided at the price of replacing the Weyl operators \( W(v) \) by Wigner operators \( W(v) \) which are simply Weyl operators multiplied by parity, i.e., if \( W(0) \) is the parity operator (which can be defined intrinsically up to a sign in any Hilbert space that carries a Weyl system), and if \( W(v) \) is defined by

\[
W(v) = W(2v) W(0) = W(v) W(-v), \tag{1.6}
\]

then \( Q(f) \) can be written directly as

\[
Q(f) = 2^{-\frac{1}{2}} \int f(v) W(v) dv. \tag{1.6}
\]
and we do not have to consider the Fourier transform $\tilde{f}$ of the function $f$. The reason for calling $\Pi(\omega)$ a Wigner operator is that the Wigner quasiprobability density $\rho_\omega(v)$ corresponding to a pure state $\psi$ is just the expectation value of $\Pi(\omega)$:

$$\rho_\omega(v) = 2\langle \psi | \Pi(\omega) | \psi \rangle$$

(see Ref. 7).

Equation (1.6) expresses $Q(f)$ as a superposition of Wigner operators, which are in some ways simpler than Weyl operators, namely, (i) Every Wigner operator $\Pi(\omega)$, in addition to being unitary $[\Pi(\omega)^* \Pi(\omega) = (\Pi(\omega)^* \Pi(\omega))^{-1}]$ is also self-adjoint $[\Pi(\omega)^* = \Pi(\omega)]$. Consequently, $\Pi(\omega)$ is involutive $[\Pi(\omega)^2 = I]$ and its spectrum consists of the numbers $+1$ and $-1$. (ii) The relationship (1.1) for Weyl operators is replaced by

$$\Pi(v_1)\Pi(v_2)\Pi(v_3) = e^{i\pi (v_1 v_2 v_3)} \Pi(v_1 - v_2 - v_3)$$

(1.7) (see Huqenin10), where $v(v_1 v_2 v_3) = 4\sigma(v_1 v_2) + \sigma(v_2 v_3) + \sigma(v_3 v_1)$ is the oriented area of the triangle spanned by $v_1$, $v_2$, $v_3$; thus, Eq. (1.7) is affine (i.e., independent of the choice of origin in phase space) while Eq. (1.1) is vectorial (i.e., dependent on the choice of origin).

We can again invert formula (1.6) to obtain an expression analogous to Eq. (1.4), but giving now a direct correspondence between $f$ and $Q_f$:

$$f(v) = 2\text{tr} \left( Q_f \Pi(\omega) \right)$$

(1.8)

Wigner operators (without the name) were already present in Ref. 11. They can also be found in Ref. 12, where a decomposition of operators with respect to Wigner operators is given, analogous to Eq. (1.6), and relation (1.8) is derived for the function used in this decomposition. The Wigner operators were however not discussed in Ref. 12 as a means to do Weyl quantization without having to pass through the Fourier analysis step. That the $\Pi(\omega)$ formulas (1.6) and (1.8) may be more convenient then the $W(\omega)$ formulas (1.3) and (1.4) was also implicitly recognized in Ref. 13, where indeed some nondiagonal matrix elements involving a parity operation were used rather than the diagonal ones to compute classical functions, which amounts exactly to preferring Eq. (1.8) as a direct formula to the indirect version (1.4) containing still a Fourier transform.

In this paper we will exploit Eq. (1.6) to study directly the relationship between the function $f$ and matrix elements of the corresponding operator $Q_f$. We are particularly interested in the matrix elements of $Q_f$ between coherent states. So the coherent state formalism will be the second main input in this paper.

The coherent state formalism has "a long and proud history in quantum theory." Coherent states can be considered as eigenstates of a displaced harmonic oscillator (it is in this form they historically made their first appearance; see Ref. 15), as wave packets satisfying the minimum-uncertainty conditions, or as the eigenfunctions of the annihilation operator associated to the harmonic oscillator. For more details concerning these different points of view, see Ref. 14 and the references quoted there. We will consider the coherent states as displacements of the harmonic oscillator vacuum $\Omega$:
Coherent states can be defined in any Hilbert space carrying an irreducible Weyl system, which means that the matrix elements (1.14) can be computed in any representation, and are representation independent. We will use this to choose one specific representation, namely, the coherent state representation (written in an intrinsic, i.e., coordinateless way), which is particularly well suited for calculations with coherent states; the matrix elements we compute will however be independent of this particular choice of representation. The kernel $|a, b |v\rangle$ (which was briefly discussed in Ref. 24) is studied in Ref. 25. We consider in particular a bilinear expansion for the matrix elements $h_{mn}(v)$ of the Wigner operators $\Pi(v)$ between harmonic oscillator states. These functions $h_{mn}$ are given explicitly by Eq. (3.28). The Fourier coefficients of an arbitrary function $f$ with respect to the basis $h_{nn}$ are the matrix elements of $Q_f$ between harmonic oscillator states.

The integral transform (1.4) which is discussed in Sec. 4 is analogous in many ways to the transform of Bargmann. This analogy and the differences are discussed in Sec. 6.

The discussion of the integral transform given here is not at all exhaustive: a deeper study will be carried out in a forthcoming paper; we will study in particular the correspondence between some classes of distributions and the corresponding operators. A first application of Eq. (1.14) can be found in the computation of the classical functions corresponding to linear canonical transformations in Ref. 25.

2. THE GEOMETRICAL SETTING

We find it convenient to work in phase space without coordinates whenever possible. We shall however also rewrite some of the main formulas in a notation with coordinates which may be more familiar to most readers.

A. Affine phase space (symplectic geometry)

We denote by $E$ a set which has the structure of an affine space (i.e., which can be identified to a real vector space after the choice of an origin). Assume that $E$ is even dimensional and that we have associated an "oriented area" $\varphi(a, b, c)$ to every triangle with vertices $a, b, c$ (taken in a given order). We assume the following:

(i) $\varphi$ does not change if all its arguments are shifted by the same vector.

(ii) If a point $o$ is chosen as the origin, then $\sigma(a, b)$ defined by

$$\sigma(a, b) = \frac{1}{2} \varphi(a, o, b)$$

is symplectic (i.e., bilinear, antisymmetric, and nondegenerate).

The function $\varphi$ can now be expressed in terms of $\sigma$:

$$\varphi(a, b, c) = 4(\sigma(b, a) + \sigma(a, c) + \sigma(c, b)).$$

We see that it is totally antisymmetric: it changes sign if any two arguments are interchanged.

B. Phase space with a symplectic and a Euclidean geometry

Consider in $E$ a reference frame, i.e., a family of vectors $a, b, \ldots, a_v, b_v$ that span $E$ and such that $\sigma(a, a_v) = \sigma(b, b_v) = 0$, and $\sigma(a, b_v) = \delta_{a b}$. For our purposes (the building of a representation space for the Heisenberg commutation relations) all the relevant information is contained in the map $J$ defined by

$$Ja_i = b_i, \quad Ja_v = b_v, \quad J b_i = -a_i \quad (i = 1, \ldots, v).$$

Notice that $J$ has the following properties:

$$J^2 = -1$$

(which is expressed by saying that $J$ is a complex structure),

$$\sigma(Ja, Jb) = \sigma(a, b),$$

and

$$\sigma(a, Ja) > 0, \quad \text{if} \quad a \not= 0.$$

It follows that the bilinear form

$$s(a, b) = \sigma(a, Jb)$$

defines a Euclidean geometry on $E$. A Triangle $a, b, c$ has now not only an oriented area $\varphi(a, b, c)$ but also side lengths $(s(a - b, a - b))^{1/2}, \ldots$, which however depend on the choice of $J$.

We shall also use the complex combination

$$h(a, b) = s(a, b) + i\sigma(a, b),$$

which makes $E$ into a $v$-dimensional Hilbert space.

Examples: (1) Take $E = \mathbb{C}^v$ with

$$\sigma(a, b) = \text{Im}(\bar{a}b),$$

$$Ja = ia.$$  

Then

$$s(a, b) = \text{Re}(\bar{a}b),$$

$$h(a, b) = \bar{a}b,$$

and all the conditions above are satisfied. (2) Take $E = \mathbb{R}^v \oplus \mathbb{R}^v$. Any $a$ in $E$ is written as $(x_a, p_a)$. Define

$$\sigma(a, b) = \frac{1}{2}(p_a x_b - p_b x_a),$$

$$J(x_a, p_a) = (p_a, -x_a).$$

Then

$$s(a, b) = \frac{1}{2}(x_a x_b + p_a p_b),$$

and all our conditions are fulfilled again. We can now use this structure to build a representation of the canonical commutation relations.
C. A representation space for canonical commutation relations

On $E$, we consider the space of holomorphic functions

$$\mathcal{L}(E) = \{ \varphi : E \to \mathbb{C} \mid \nabla^h \varphi = i \nabla \varphi \text{ for any } a \in E \},$$

(2.13)

where

$$(\nabla^h \varphi)(v) = \lim_{\lambda \to 0} \frac{1}{\lambda} (\varphi(v + \lambda a) - \varphi(v)).$$

On the other hand, we define the Gaussian $\Omega$ by

$$\Omega(v) = \exp[-\frac{i}{2} \Im(v,v)].$$

(2.14)

We shall say that a function $\phi$ on $E$ is modified holomorphic if it is the product of a $\varphi \in \mathcal{L}$ and of the Gaussian:

$$\varphi(v) \phi(v).$$

This combination of both $\mathcal{L}$ and $\Omega$ gives us $Z$, the space of modified holomorphic functions:

$$Z(E) = \{ \varphi \Omega \mid \varphi \in \mathcal{L} \}.$$  

(2.15)

An alternative way of defining $Z$ is

$$Z(E) = \{ \phi : E \to \mathbb{C} \mid D^h \phi = i D \phi \},$$

(2.15')

with $(D \phi)(v) = (\nabla^h \phi)(v) + i \Im(a,v) \phi(v).$

The Hilbert space we will use in the sequel whenever we want to consider a concrete representation space is now given by

$$\mathcal{L}_0 = Z(E) \cap L^2(E, dv),$$

(2.16)

where $dv$ is the Lebesgue measure, translationally invariant, normalized by the requirement

$$\int \Omega^*(v) dv = 1.$$  

(2.17)

On this Hilbert space, we define a set of unitary operators $W(a)$ by

$$(W(a) \phi)(v) = \exp[i \phi(a,v)] \phi(v - a),$$

(2.18)

for any $a \in E$. These $W(a)$ are called Weyl operators. It is easy to check that

$$W(a)W(b) = \exp[i \phi(a,b)] W(a + b),$$

(2.19)

which implies we have a representation of the canonical commutation relations. One can easily see this in the example given above:

$$E = \mathbb{R}^n \oplus \mathbb{R}^n,$$

$$W((x_a,0))W(0,p_b)$$

$$\exp\left(-\frac{i}{2} x_a p_b\right) W((x_a,p_b))$$

$$\exp(-i x_a p_b) W((0,0)) W((x_a,0)).$$

This is exactly what one would have expected from

$$W((x_a, p_b)) = \exp(i(x_a \cdot X + p_b \cdot P)).$$

with

$$[X_a, P_b] = i \delta_{ab}.$$ 

The representation given by Eq. (2.18) in the space (2.16) is irreducible.

Owing to von Neumann's uniqueness theorem for representations of the canonical commutation relations for a finite number of degrees of freedom, any result we will obtain in our particular representation on $\mathcal{L}_0$ can be transcribed to any irreducible representation.

Some particular functions in $\mathcal{L}_0$ will play a special role in the sequel: They are called the coherent states and are defined as

$$\Omega^*(v) = \langle W(a) \Omega \rangle(v) = \exp(i \phi(a,v)) \Omega(v - a).$$

(2.20)

These coherent states have the following "reproducing" property\textsuperscript{16,17}:

$$(\Omega^*, \phi) = \int \Omega^*(v) \phi(v) dv = \phi(a)$$

for any $\phi \in \mathcal{L}_0$.  

(2.21)

Writing this otherwise, we have

$$(\phi, \psi) = \int \phi(v) \psi(v) dv = \int (\phi, \Omega^*) (\Omega^* \psi) dv;$$

hence

$$\int |\Omega^* \rangle (\Omega^*)^* \, dv = 1.$$  

(2.22)

It is now easy to see that the $\Omega^*$ are normalized elements of $\mathcal{L}_0^*$:

$$\langle \Omega^*, \Omega^* \rangle = \Omega^*(0) = \Omega(0) = 1.$$  

(2.23)

As we already mentioned in the Introduction it is often useful to introduce Wigner operators, i.e., products of Weyl operators with the parity operator. We define

$$\pi : \psi \to \tilde{\psi},$$

(2.24)

with $\tilde{\psi}(v) = \psi(-v)$. This operator conserves the modified holomorphy properties of and is thus an involutive unitary operator from $\mathcal{L}_0$ to itself. Moreover, one easily sees that

$$\pi W(v) = W(-v) \pi$$

or

$$\pi W(v) \pi = W(-v).$$

Hence, $\pi$ represents the parity $v \to -v$ on phase space. The Wigner operators $\pi v$ are now defined as

$$\pi v = W(2a) \pi$$

(2.25)

i.e.,

$$\pi v = \exp[i \phi(a, b, c)] \pi v = \exp[i \phi(a, b, c)] \pi v.$$  

(2.26)

It is easy to check that

$$\pi v \pi v = \exp[i \phi(a, b, c)] \pi v.$$  

(2.27)

3. THE FUNCTIONS $[a, b | v]$ 

Definition: Let $\mathcal{H}$ be a Hilbert space carrying an irreducible representation of the Weyl commutation relations for $v$ degrees of freedom. Denote by $\Omega^*$ the coherent state centered at $a \in E$, and by $\pi v$ the parity operator around $v$ (Wigner operator). Given $a, b, v \in E$, we define

$$[a, b | v] = \kappa^{-1} (\Omega^*, \pi v \Omega^*),$$  

(3.1)

with

$$\kappa = 2^{-v}.  $$  

(3.1')

The numbers $[a, b | v]$ can be easily calculated and have simple properties.
A. Explicit form, symmetries, and special values

One has
\[
\{a, b \mid v\} = 2\exp\left[ i\rho \left( \frac{a}{2}, \frac{v}{2}, \frac{b}{2} \right) \right] \Omega(a + b - 2v),
\]
(3.2)
i.e., the phase of \( \{a, b \mid v\} \) is the oriented area of the triangle with vertices \( a, v, b \). The number \( \{a, b \mid v\} \) is real if and only if the three points \( a, v, b \) are collinear.

The absolute value of \( \kappa(a, b \mid v) \) is the exponential of the negative of half the squared Euclidean distance from \( v \) to the midpoint \((a + b)/2\) of the segment \((a, b)\). It takes its maximum value (which is 1) when \( v \) is the midpoint of \((a, b)\).

If we denote by \( \xi \) the pair \( \{a, b\} \) we have
\[
| - \xi - v | = | - a, a - b, v | = \{a, b \mid v\} = |\xi|v| .
\]
(3.3)
Denote by \( \tilde{\xi} \) the pair \( \{b, a\} \). (This will be justified below.)

Then
\[
|\tilde{\xi}v| = \{b, a \mid v\} = \{\overline{a, b \mid v}\} = |\xi|v| .
\]
(3.4)
If the arguments of \( \{a, b \mid v\} \) are shifted, we have
\[
\{a + c, b + c \mid v + c\} = \exp[i\rho(c, a - b)] \{a, b \mid v\} .
\]
(3.5)
One has
\[
\{a, a \mid v\} = \{a \mid a \mid v\} = \kappa^{-1}\Omega(2v - 2a) \tag{3.6}
\]
and
\[
\frac{a + b}{2} = \kappa^{-1}. \tag{3.7}
\]

B. Expression of \( \{a, b \mid v\} \) in coordinates

1. Identify \( E \) with \( \mathbb{C}^n \). Then \( \sigma(a, b) = \text{Im}(\overline{a}b) \), \( J_a = ia \), and \( s(a, b) = \text{Re}(\overline{a}b) \).

So
\[
\{a, b \mid v\} = 2\exp[-i\{a|a|b|b|2|v|^2]
\quad - a + b + 2\rho v + 2\overline{a} v]. \tag{3.8}
\]

2. Identify \( E \) with \( \mathbb{R}^n \oplus \mathbb{R}^n \). So \( a \) is the pair
\[
\{a = (x_a, p_a)\} : \sigma(a, b) = \frac{1}{2}(p_a - p_b, x_a - x_b), J(x_a, p_a) = (p_a, -x_a), \text{ and } s(a, b) = \frac{1}{2}(p_a, p_b + x_a x_b).
\]
Then
\[
\{a, b \mid v\} = 2\exp[-i\{x_a + x_b - 2x_v\}^2]
\quad - i(p_a + p_b - 2p_v + i(x_a x_b - x_a x_b - p_a p_b)] . \tag{3.9}
\]

C. Analyticity and regularity properties

The expression (3.2) can be rewritten as
\[
\{a, b \mid v\} = 2\exp[2h(b, v) + 2h(v, a)\phi(h, a)] \times \Omega(a) \Omega(b) \Omega(2v), \tag{3.10}
\]
where \( h(a, v) \) is defined by Eq. (2.5). In coordinates, Eq. (3.10) is just Eq. (3.8).

Since Eq. (3.10) can be rewritten as
\[
\{a, b \mid v\} = 2\exp[2i\rho(a, v)] \Omega^{2\rho(b)}(a)
\quad = 2\exp[2i\rho(a, v)] \Omega^{2\rho(b)}(a)
\quad = 2\exp[i\rho(a, b)] \Omega^{-1}(a - b)
\quad \times \Omega^\sqrt{2\rho}(\sqrt{2v}) \Omega^\sqrt{2\rho}(\sqrt{2v}),
\]
(3.11)
we see that \( \{a, b \mid v\} \) is modified holomorphic in \( a \), modified antiholomorphic in \( b \), and a product of a modified holomorphic function with a modified antiholomorphic one in \( v \). In each of these variables it is infinitely differentiable and of Gaussian decrease at infinity.

D. Fourier transforms and integrals

The (symplectic) Fourier transform \( F \) can be defined by
\[
(Fg)(v) = \kappa \int \exp[i\rho(v, v')] g(v') dv'. \tag{3.12}
\]
Then
\[
F^* = 1, \quad \Omega \Omega = \Omega, \quad \Omega^* = \Omega^{-1}. \tag{3.13}
\]
If a function \( \phi \) is modified holomorphic [see Eq. (2.15)], it satisfies \((F\phi)(v) = \phi(-v)\). If it is modified antiholomorphic, it satisfies \((F\phi)(v) = \phi(v)\). So
\[
\kappa \int \exp[i\rho(a, a')] \{a', b \mid v\} da' = \{a, b \mid v\} \tag{3.14}
\]
In particular,
\[
\kappa \int \{a', b \mid v\} da' = \kappa^{-1} \exp[i\rho(b, 2v)] \Omega(2v - b). \tag{3.15}
\]
Similarly,
\[
\kappa \int \{a, b \mid v\} db' = \kappa^{-1} \exp[i\rho(2v, a)] \Omega(2v - a). \tag{3.16}
\]
The Fourier transform in the variable \( v \) can be computed directly. It is
\[
\kappa \int \exp[i\rho(v, v')] \{a, b \mid v'\} dv' = \kappa^{-1} \{a, -b \mid v\}. \tag{3.17}
\]
In particular,
\[
\kappa \int \{a, b \mid v\} dv' = \kappa^{-1} \Omega^{-b}(a). \tag{3.18}
\]
One has also
\[
\kappa \int \{a, a \mid v\} da = 1. \tag{3.19}
\]
We now consider integrals that are bilinear in the symbols \( \{a, b \mid v\} \). Particularly important is the relationship
\[
\kappa \int \{a, b \mid v\} \{b, a \mid v\} da db = \delta(v - v'), \tag{3.20}
\]
which can also be written as
\[
\kappa \int |\xi|v| |\xi|v'| d\xi = \kappa \int |\xi|v| |\xi|v'| d\xi = \delta(v - v'). \tag{3.21}
\]
We shall derive it here, in order to show how simple the calculations are:
\[
\kappa \int \{a, b \mid v\} \{b, a \mid v\} da db
\]
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Another useful relation is

\[ (\Omega^a \Pi(v) \Omega^b)(\Omega^a \Pi(v) \Omega^b) = \Omega^a \Pi(v) \Omega^b \Omega^a \Omega^b = \Omega^a \Pi(v) \Omega^b \Omega^a \Omega^b \]

We give finally two integrals of triple products:

\[ \int [la,b | v] [b,c | v'] [c,d | v''] \, db \, dc = \kappa^2 \exp[i \varphi(v, v', v'')] [la,b | v] [b,c | v'] [c,d | v''] \]

\[ \int \int [la,b | v] [b,c | v'] [c,a | v''] \, db \, dc \]

**E. Bilinear expansions of \( [\xi | v] \); The orthonormal family \( h_{mn}(v) \)**

Since \( [\xi | v] \) will be used as the kernel of an integral transform, it is natural to expand it into a sum of products of functions of \( v \) and of functions of \( \xi \). Such expansions can be found immediately since the \( [a, b | v] \)'s are matrix elements of irreducible families of operators.

**1. Generalities**

For any orthonormal base \( e_n \) in \( \mathcal{H} \), we define the functions \( e_{mn}(v) \) by

\[ e_{mn}(v) = \kappa^{-1}(e_m, \Pi(v)e_n) = \kappa^{-1}(e_m, \Pi(v)e_n) = \kappa^{-1}(e_m, \Pi(v)e_n) \]

These \( e_{mn} \) form an orthonormal base in \( L^2(E) \); orthogonality is a consequence of Eq. (3.22), and completeness follows from the fact that the family \( \{ \Pi(a) \} \) is an irreducible family of operators.

It is now obvious that

\[ [a, b | v] = \sum_{m,n} (\Omega^a e_m, \Pi(v) \Omega^b e_n) \]

**2. In the representation on \( \mathcal{L}_0 \)**

Let \( \{ e_1, \ldots, e_n \} \) be a symplectic base on \( E, \sigma \), i.e., a set of \( \nu \) vectors satisfying

\[ \sigma(e_j, e_k) = \delta_{jk} \]

We define the normalized monomials \( h^{[n]} \) on \( E \) by

\[ h^{[n]}(v) = \frac{1}{\sqrt{[n]!}} \prod_{\alpha=1}^{[n]} (h(e_{\alpha}, v))^\alpha \]

Here \([n]\) is a multi-index, with \( [n] = \Pi_{\alpha=1}^{[n]} (n_{\alpha}) \).

The functions \( h^{[n]} \) form an orthonormal base in \( \mathcal{L}_0 \).

In fact, they are the eigenfunctions of the harmonic oscillat-
\[ \tilde{f}(v) = \kappa \int \exp[i\sigma(v,v')]f(v')\,dv'. \quad (4.1) \]

Let
\[ f(v) = \sum_{m,n} f_{mn}(v) \]
be the Fourier series expansion of \( f \) in the orthonormal system (3.28). By Eq. (3.30), the expansion of \( f \) is
\[ f(v) = \kappa^2 \sum_{m,n} \left( -1 \right)^n f_{mn} h_{mn} \left( \frac{v}{4} \right). \]

**Definition:** The \( Q \) transform of \( f \) is the function \( Q_f \) on \( E \times E \), defined by
\[ Q_f(a,b) = \int \{ a, b | v \} f(v)\,dv, \quad (4.3) \]
to be interpreted, if necessary, as the evaluation of the functional \( f \) on the testing function \( \{ a, b | \cdot \} \).

**Remark:** The above definition is more restrictive than necessary since the testing functions \( \{ a, b | \cdot \} \) can handle more general distributions "of type S." We shall not try here to study in detail the functional analysis associated with Eq. (4.3).

By Eq. (3.14), \( Q_f \) and \( Q_{\bar{f}} \) are related through
\[ Q_f(a,b) = \kappa^{-1} Q_{\bar{f}}(a,-b), \quad \text{with} \quad \bar{f}(v) = f(-4v). \quad (4.4) \]
In fact, this relationship between the matrix elements of \( Q_f \) and \( Q_{\bar{f}} \) is just a consequence of the equivalence of formulas (1.3) and (1.6). Indeed, we have from Eqs. (1.3) and (1.6)
\[ Q_f(a,b) = \kappa^{-1} \int dv f(v) (\Omega^{a}, \Omega^{b}) = \kappa \int dv f(v) (\Omega^{a}, W(-v/2)\Omega^{b}) = \kappa \int dv f(-v) (\Omega^{a}, \Omega^{b}) = \kappa^{-3} \int dv f(-4v) (\Omega^{a}, \Omega^{b}) = \kappa^{-3} Q_{\bar{f}}(a,-b). \quad (4.5) \]

Furthermore, by Eq. (3.29) and (4.3), the function \( Q_f(a,b) \) can be expressed in terms of the Fourier coefficients
\[ Q_f(a,b) = \int dv f(v) (\Omega^{a}, \Omega^{b}) \sum_{m,n} h^{m}(a) h^{-m}(b)f_{mn}, \]
where the functions \( h^{m}(a) \) and \( h^{-m}(b) \) are defined by Eq. (3.27).

An examination of either Eq. (4.3) or (4.5) shows that \( Q_f(a,b) \) is modified holomorphic in \( a \) and modified antiholomorphic in \( b \), i.e., it is holomorphic in \( \xi \) with respect to the complex structure \((\xi, -\xi)\).

**B. Inverting the map \( f \rightarrow Q_f \)**

Given \( Q_f(\xi) \), we can reconstruct \( f \) through
\[ f(v) = \sum_{a,b} \int d\xi |Q_f(\xi)|^2 \int d\xi |Q_f(\xi)|^2 = \int \{ a, b | v \} Q_f(b,a)\,db, \quad (4.6) \]
provided the integrals converge. This is an immediate consequence of (3.21).

In other words, the same kernel is used for (4.3) and its inverse just as in Fourier transforms and the integral transforms of Bargmann.

**C. Physical interpretation**

So far, we have only defined some integral transform \( f(v) \rightarrow Q_f(\xi) \) by means of the kernel \(|\xi|/|v|\). Of course, we always had in mind the physical interpretation of all this when we defined our map from one function space to another. This physical interpretation follows immediately from formula (1.6) and definition (3.1) of the kernel \(|a, b|v|\).

One has
\[ Q_f(a,b) = \kappa^{-1} \int dv f(v) (\Omega^{a}, \Omega^{b}) = \kappa^{-1} (\Omega^{a}, \Omega^{b}). \]
So for any \( a, b \) in \( E, Q_f(a,b) \) is the matrix element between the coherent states \( \Omega^{a} \) and \( \Omega^{b} \) of the quantum mechanical operator \( Q_f \) corresponding to the "classical observable" \( f \).

In an analogous way, the Fourier coefficient
\[ f_{mn} = \int h_{mn}(v)f(v)\,dv, \quad (4.7) \]
with
\[ h_{mn}(v) = \kappa^{-1} (h^{m}(\Omega^{a}), h^{-n}(\Omega^{b})), \]
the \( h^{m} \) being normalized monomials [see Eq. (3.27)], is equal to the matrix element of \( Q_f \) between harmonic oscillator states:
\[ \langle n|Q_f|m\rangle. \]

**D. Action of \( Q_f \) in \( \mathcal{L}(E) \)**

The action of the operator \( Q_f \) on \( \mathcal{L}(E) \) can be written explicitly with the help of the function \( Q_f(\xi) \)[see Eq. (2.21)]:
\[ \forall \psi \in \mathcal{L}(E) : (Q_f \psi)(a) = \int db Q_f(a,b)\psi(b). \quad (4.8) \]

**E. Unitarity of the correspondence \( f \rightarrow Q_f \)**

The function \( Q_f(\xi) \) is an element of \( \mathcal{L}(E \times E; (J, -J)) \), i.e., modified holomorphic in its first variable and modified antiholomorphic in its second variable.

Define \( \mathcal{L}(E \times E) = L^2(E \times E; dv \otimes dv) \otimes \mathcal{L}(E \times E; (J, -J)) \). Equipped with the \( L^2 \) norm, this is a Hilbert space. Suppose \( f \) is square integrable. Then
\[ \int d\xi |Q_f(\xi)|^2 = \int \int da db Q_f(a,b)Q_f(b,a) = \int \int \int da db dv f^{*}(v)[b,a|f(v')|a,b|v'] dv dv' = \int \int dv f(v)^2 dv = \|f\|_{L^2}^2. \]

Hence the map \( f(.) \rightarrow Q_f(.) \) is a unitary map from \( L^2(E) \) to \( \mathcal{L}(E \times E) \); its inverse is defined by Eq. (4.6).

In fact, this unitarity is nothing else than the well-known unitarity of the correspondence between square integrable functions and Hilbert–Schmidt operators. Indeed, one can check that the operator \( Q_f \) is Hilbert–Schmidt if the function \( Q_f(.) \) is square integrable, and one has the equality
\[ \|Q_f\|_{HS} = Tr(Q_f Q_f^{*}) = \int d\xi |Q_f(\xi)|^2. \]

**F. Products**

Let \( A \) and \( B \) be operators on \( \mathcal{L}_0 \). Then
(A, B)(a, b) = (Ω^*ABΩ^*)

= \int (Ω^*AΩ^*(Ω^*BΩ^*)) \, dc

= \int (A(a, c)B(b, c)) \, dc.

Hence,

(Q_f, Q_g)(a, b) = \int Q_f(a, c)Q_g(c, b) \, dc.

We define the twisted product \( f \circ g \) by

\[ (f \circ g)(v) = \int f(v')g(v'') \, dv'dv'' \]

(4.9)

Hence,

\[ (f \circ g)(v) = \int \int J(a, b)Q_f(a, c)Q_g(c, b) \, db \, dc \]

(4.10)

which is a well-known expression. 10,28

G. Bounds on \( Q_f(a, b) \)

Define the following two regularized functions associated to \( f \):

\[ f_R(v) = \int \frac{f(v')\Omega(2(v' - v))}{\|(v')^2 \|} \, dv' \]

(4.11)

\[ f_\ast(v) = \int f(v')\Omega(\frac{v' - v}{2}) \, dv' \]

(4.12)

They can be obtained by choosing \( f \) as the initial value of a diffusion (heat) equation and the appropriate time.

Assume now that \( f \) is a (locally integrable) function so that \( |f| \) (the absolute value of \( f \)) is well defined. Denote by \( |f|_R \) the regularized Eq. (4.11) of \( |f| \). Then Eq. (4.13) gives

\[ |Q_f(a, b)| \leq |f|_R \left( \frac{a + b}{2} \right). \]

On the other hand, if the Fourier transform \( \widehat{f} \) of \( f \) is a function (here one should not think of \( Q_f \) as, say, a Hamiltonian but for example a resolvent), and if \( |\widehat{f}| \), is the regularized Eq. (4.12) of \( |\widehat{f}| \), we obtain from Eq. (4.14)

\[ |Q_f(a, b)| \leq |\widehat{f}|_R \left( \frac{a + b}{2} \right). \]

(4.13)

If both \( f \) and \( \widehat{f} \) are functions we obtain

\[ |Q_f(a, b)| \leq |\widehat{f}|_R \left( \frac{a + b}{2} \right). \]

(4.14)

For the diagonal matrix elements one obtains 24 an equality, which does not require any special assumption on \( f \),

\[ Q_f(a, a) = \langle f \rangle \left( \frac{a + b}{2} \right). \]

(4.15)

More generally, \( f \) can be assumed to be a measure.

H. Positivity of \( Q_f \)

Suppose that \( Q_f \) is positive, i.e.,

\[ \forall \psi \in L^2 : (\psi, Q_f \psi) = \int \int \overline{\psi}(a)\psi(b)Q_f(a, b) \, da \, db \geq 0. \]

(4.17)

Since

\[ |a, b \rangle \langle b, a| = \kappa^4 \exp[2i\sigma(v, a)] \Omega^{2v - \sigma(b)} \]

(see Sec. 3c), this is equivalent to

\[ \forall \psi \in L^2 : \int \int \overline{\psi}(a)\psi(2v - a)\exp[2i\sigma(v, a)] f(v) \, da \, dv \geq 0, \]

(4.18)

provided we are allowed to change the order of the integrations, which is certainly true, for example, for \( f \in L^2(\mathbb{R}) \).

We can rewrite condition (4.18) as

\[ \forall \psi \in L^2 : \int \int \overline{\psi}(a)\psi(2v - a)\exp[2i\sigma(v, a)] f(v) \, da \, dv \geq 0, \]

(4.19)

For \( f \in L^2(\mathbb{R}) \), Eq. (4.19) is a necessary and sufficient condition for \( Q_f \) to be positive.

If, moreover, we suppose \( f \) is essentially bounded \((\mathcal{L}^\infty)\) and integrable \((\mathcal{L}^1)\), then Eq. (4.19) is implied by

\[ \forall n \in \mathbb{N}, \forall a_1, ..., a_n : \text{positive } f_{a_1, ..., a_n} \]

(4.20)

So, for \( f \in \mathcal{L}^\infty \cap \mathcal{L}^1 \), Eq. (4.20) is a sufficient condition for \( Q_f \) to be positive.

A similar result, though in a different context, can be found in Ref. 29; the fact that the matrices

\[ \left[ \exp[i\sigma(a, a_k)] f \left( \frac{a_k}{2} \right) \right]_{j,k} \]

are considered in Ref. 29 and not

\[ \left[ \exp[i\sigma(a, a_k)] f \left( \frac{a_j + a_k}{2} \right) \right]_{j,k} \]

as here, is due to their studying the correspondence \( f \rightarrow f \mathcal{W}(v) \) and not \( f \rightarrow f \mathcal{W}(v)\mathcal{W}(v) \).

Examples: Using Eq. (4.20), one can easily check that the following functions yield positive operators:

\[ f(v) = \Omega^{-\alpha}(2v), \quad \alpha < 1. \]

In particular,

\[ f(v) = \Omega(2v) \quad \text{and} \quad f(v) = \Omega(v), \]

\[ f(v) = \exp[2\sigma(c, v)] \Omega(2v), \]

\[ f(v) = \exp[\sigma(c, v)] \Omega(2v). \]

5. EXAMPLES

A. Operators corresponding to elementary functions

For some functions we shall use Eq. (4) to compute both \( Q_f \) and \( Q_g \):

(1) \( f(v) = 1 \). Then \( Q_f(a, b) = \Omega^{-\alpha}(a) \) or \( Q_f = 1 \).

(2) \( f(v) = \exp[i\sigma(c, v)] \). This gives \( Q_f(a, b) = \exp[i\sigma(c, b)] \Omega^{2v - \sigma(b)}, \) hence \( Q_f = (c/4)\mathcal{H} \). Applying again Eq. (4), we see that
and
\[ Q_{\beta} = \kappa^{-1} H \circ (c). \]

(3) \( f(v) = \Omega (\alpha v) \) for \( \alpha \in \mathbb{R} \).

Then
\[
Q_f(a,b) = \left[ \frac{4}{4 + \alpha^2} \right] \exp \left[ i0(a,b) \right] \left[ \frac{1}{4 \alpha^2 - \alpha^2} \right] \times [\Omega (a + b)]^m \left[ \frac{1}{4 \alpha^2 + \alpha^2} \right] [\Omega (a - b)]^r \left[ \frac{1}{4 \alpha^2 + \alpha^2} \right].
\]

As a consequence of the fact
\[ [F (\Omega (\alpha)) (v)] = \alpha - 2 \Omega (\alpha^{-1} v), \]
we see that
\[ Q_{\Omega (\alpha^2)} (a, b) = (2 \alpha)^n Q_{\Omega (\alpha^{-1})} (a, -b). \]

(4) As a special case of (3), we have
\[ Q_{\Omega (2)} (a, b) = \kappa \Omega (a) \Omega (b); \]

hence,
\[ Q_{\Omega (\alpha^2)} (a, b) = \kappa \left[ \Omega (a) \Omega (b) \right]. \]

(5) \( f(v) = \delta(v, v) \). This is the Hamiltonian of the harmonic oscillator. We have
\[ Q_f(a,b) = \left[ \frac{\sqrt{2}}{2 + h (a, b)} \right] \Omega^b (a), \]

or
\[ Q_f = \frac{\sqrt{2}}{2 + N}, \]

with
\[ N: \Omega^b \mapsto h (b, \cdot) \Omega^b. \]

We see here the expected vacuum energy term \( v/2 \); moreover, one can easily check that for \( u_n = h^* \Omega \), one has
\[ N \cdot u_n = n \cdot u_n, \]

which is in accordance with the well-known fact that the \( u_n \) are the harmonic oscillator eigenfunctions.

(6) \( f(v) = \sigma(c, v) \). This gives \( Q_f(a, b) = (i/2) \Omega^b (a) \times [h (b, c) - h (c, a)] \). Define \( H_c: \psi \mapsto h (c, \psi) \). Then \( H_c^*: \psi \mapsto \Omega (\psi) \).\( \Omega (\psi) \mapsto (\nabla, \varphi) \Omega^j \) or \( \Omega^j \mapsto (\nabla, \varphi) \Omega^j \) or \( \Omega^j \mapsto h (b, c) \Omega^j \), and \( Q_{\Omega (\alpha^2)} = \left( i/2 \right) \left( \Omega^j - H_c \right) \). Analogously, \( Q_{\Omega (\alpha^2)} = \Omega (h^* + H_c) \) and \( Q_{\Omega (\alpha^2)} = H_c \).

(7) \( f(v) = \sigma(c, v) \sigma(d, v) \). Then
\[ Q_{\sigma(c, v) \sigma(d, v)} = \frac{1}{2} \left( H_c^* + H_d \right) \Phi \left( c, h^* + H_c \right) - H_c \Phi \left( c, h^* + H_c \right). \]

Analogously,
\[ Q_{\sigma(c, v) \sigma(d, v)} = H_c \Phi \left( c, h^* + H_c \right). \]

B. Functions corresponding to elementary operators

1. Dyadics

Take \( A = \delta(c, d) \). Then
\[ f_A (v) = \int \int [a, b \mid v] \Phi (b, a) \overline{\Phi} (a, b) \, da \, db \]
\[ = (\psi, \Omega (\psi) \Phi). \]

In particular,
\[ f_{j + \alpha H (j + 1)} (v) = |c, d \mid v|, \]
\[ f_{j + \alpha H (j + 1)} (v) = \kappa^{-1} \Omega (2v - 2c), \]
\[ f_{2j + \alpha H (2j + 1)} (v) = \kappa^{-1} \Omega (2v), \]
\[ f_{2j + \alpha H (2j + 1)} (v) = \kappa^{-1} \Omega (2v + 2c). \]

From this last example we see that \( h_{\alpha v} \) [given for instance by Eq. (3.31)] is the classical function corresponding to the projection onto the subspace \( C_{\alpha v} \). For the special case \( v = 1 \), this is the projection onto the \( n \)th eigenstate of the harmonic oscillator (a similar expression, obtained in a different way, can be found in Ref. 19).

2. Multiplication operators by holomorphic functions

Consider \( A: \psi \mapsto F \Phi \), where \( F \) is some (holomorphic) function such that \( F \Omega \in \mathcal{L} \) for any \( \alpha \).

Then
\[ A (a, b) = F (a) \Omega^b (a), \]

and
\[ f_A (v) = \kappa^{-1} \int \Omega (2v - 2b) F (b) \, db; \]

hence,
\[ f_A = \kappa^{-1} \Omega (2v) \Omega (2v - 2b) \Omega (2v + 2c), \]

3. Permutation operators

Suppose \( E = E_1 \oplus \cdots \oplus E_n \), with \( J E_i \subset C E_i \forall j, \sigma (E_j, E_k) = 0 \) for \( j \neq k \); this is the phase space for the simultaneous description of \( n \) particles (dim \( E_i = 2v \) for any \( j, v \neq v \)).

Consider \( A: \psi \mapsto \Phi \), where \( F \) is some (holomorphic) function such that \( F \Omega \in \mathcal{L} \) for any \( \alpha \).

Then
\[ A (a, b) = F (a) \Omega^b (a), \]

and
\[ f_A (v) = \kappa^{-1} \int \Omega (2v - 2b) F (b) \, db; \]

hence,
\[ f_A = \kappa^{-1} \Omega (2v) \Omega (2v - 2b) \Omega (2v + 2c). \]

Clearly,
\[ Q_{(a, b)} = \Omega^b (P_r (a)). \]

To compute the classical function corresponding to \( Q_{(a, b)} \), we split up \( \pi \) into a product of independent cyclic permutation operators. The classical function splits up in a product of independent functions, corresponding to these cyclic permutations. For the cyclic permutation \( \pi = (1, ..., m) \) (this permutation maps 1 to 2, 2 to 3, ..., \( m - 1 \) to \( m \) to 1), we get
\[ f_{(a, b)} (v_1, ..., v_m) = 2^{m \cdot (m - 1)^{1/2} (m - 1)} \prod_{j=1}^{m-1} \exp \left[ i \varphi (V_{j-1}^{-1} (P_{\pi^{-1}} (v_j)) P_{\pi^{-1}} (v_{j+1})) \right] \]

for \( m \) odd,

and
\[ f_{(a, b)} (v_1, ..., v_m) = 2^{m \cdot (m - 1)^{1/2} (m - 1)} \prod_{j=1}^{m-1} \exp \left[ i \varphi (V_{j-1}^{-1} (P_{\pi^{-1}} (v_j)) P_{\pi^{-1}} (v_{j+1})) \right] \delta (V_m), \]

for \( m \) even, with
\[ V_k = \sum_{j=1}^{k} (-1)^{j-k} (P_{\sigma})^{j-k}(v_j). \]

In particular, if we describe two particles, and we want to consider the operator \( Q \) for \( \pi = 1 = 2 \), we have
\[ f(12)(v_1,v_2) = \delta(v_1 - P_{\sigma}(v_2)). \]

For three particles, we see that
\[ f(123) = 2^{3} \exp \left[ i \phi(v_1, P_{\sigma}^{-1}(v_2), P_{\sigma}^{-2}(v_3)) \right] \]
and
\[ f(123) = \delta(v_1 - P_{\sigma}(v_2)). \]

These different expressions can be considered as special cases of the classical functions corresponding to general symplectic transformations computed in Refs. 28 and 25.

6. A COMPARISON WITH BARGMANN'S INTEGRAL TRANSFORM REF. 17

In Ref. 17 some explicit expressions are given for the unitary operator intertwining the Schrödinger representation with the coherent state representation of the Weyl commutation relations. We rewrite this result in our notations.

Identify \( E \) with \( \mathbb{R}^{2n} = \mathbb{R}^{n} \oplus \mathbb{R}^{n} \) as space \( p \) space. Let us denote the \( n \) space by \( E_1 \). In what usually called the Schrödinger representation of the Hilbert space used is \( L^2(E_1) \), i.e., the space of square integrable functions on \( E_1 \), with respect to a Lebesgue measure on \( E_1 \).

Bargmann’s integral transform is a unitary map \( A \) from \( L^2(E_1) \) to \( \mathcal{L}_0(E, J) \) which can be represented by a kernel:
\[ A: L^2(E_1) \to \mathcal{L}_0(E, J), \]
\[ \forall \psi \in L^2(E_1) : (A\psi)(v) = \int dx \, A(v,x) \, \psi(x). \]

The kernel \( A(v,x) \) has many interesting properties. For fixed \( x \), it is an element of \( Z \), and for fixed \( x \) it is square integrable on \( E_1 \). This is analogous with our kernel \( [\xi | v] \) which for fixed \( \xi \) is square integrable on \( E \), and for fixed \( v \) an element of \( Z \) \( \{E \oplus (J, - J)\} \). Moreover, we know (see Sec. 4.E) that our integral transform \( Q \) is unitary from \( L^2(E) \) on \( \mathcal{L}_0(E \oplus E, (J, - J)) \). So we would seem that our integral transform is just a double Bargmann transform:
\[ A: L^2(E) \to \mathcal{L}_0(E, J), \]
\[ Q: L^2(E) \cong L^2(E_1) \otimes L^2(E_1) \to \mathcal{L}_0(E, J) \otimes \mathcal{L}_0(E, J)^* \]
\[ \cong \mathcal{L}_0(E \oplus E, (J, - J)). \]

We denote here by \( X^* \) the dual of \( X \); the isomorphism \( \mathcal{L}_0(E, J) \otimes \mathcal{L}_0(E, J)^* \cong \mathcal{L}_0(E \oplus E, (J, - J)) \) follows from the fact that \( \mathcal{L}_0(E \oplus E, (J, - J)) \) is isomorphic to the Hilbert space of Hilbert–Schmidt operators on \( \mathcal{L}_0(E, J) \) (see Sec. 4.E). It is however not altogether true that \( Q \) is just twice \( A \). Indeed, on has
\[ A(v,x) = \sum_{n \in \mathbb{Z}} u_{n}(x) \phi_{n}(v), \]
(6.2)
where \( u_{n}(x) = h^{n} \Omega \) and \( \phi_{n}(v) \) are the eigenfunctions of the harmonic oscillator, respectively, in \( \mathcal{L}_0(E, J) \) and \( L^2(E_1) \), on the other hand [see Eq. (3.29)],

\[ [a,b | (x,v \rho_{x})] = \sum_{n \in \mathbb{Z}} u_{n}(a) u_{n}(b) h_{n}(x \rho_{x}). \]

where \( h_{n}(x \rho_{x}) \) denotes the Fourier transform of \( h_{n}(x \rho_{x}) \) and is given by Eq. (3.29) and is definitely different from \( \phi_{n}(2^{1/2} \rho_{x}) \phi_{n}(2^{1/2} \rho_{x}) \) (the factor \( 2 \) has to be introduced because of a difference in normalization in the measures on \( E_1 \) and \( E \)). This can readily be checked in an example. Take \( v = 1, m = n = 1 \). Then
\[ \phi(2^{1/2} \rho_{x}) \phi(\sqrt{2} \rho_{x}) = x \rho_{x} e^{-\left( x^2 + \rho^2 \right)} \]
(6.3)
and
\[ h_{x}(x \rho_{x}) = 2 e^{-\left( x^2 + \rho^2 \right)} \]
(6.4)
Another way of seeing that the integral transform \( Q \) is not merely a double Bargmann transform is to look at the explicit expressions for the kernels \( A(v,x) \) and \( [\xi | v] \). We have
\[ A(v,x) = \Omega^{-1/2} e^{-(1/2) v x} e^{-(1/2) x \rho_{x}} \]
(6.5)
which is again very different from the expected
\[ A(a,\sqrt{2} x \rho_{x}) A(b,\sqrt{2} \rho_{x}) \]
(6.6)
In a certain sense these differences are due to the fact that the integral transform \( Q \) has to do with quantization, while \( A \) is just a unitary map from one quantum mechanical realization to another. Indeed, if we look at Eqs. (6.2) and (6.3), we see that on the \( \mathcal{L}_0 \) side everything is all right: Eq. (6.3) contains a straight copy \( u_{n}(a) \) and one complex conjugate copy \( u_{n}(b) \) of the \( \mathcal{L}_0 \) function \( u_{n}(v) \) in Eq. (6.2); but things go wrong with the \( L^2 \) function. This is precisely because \( L^2(E) \), the initial space of \( Q \), has to be considered as a space of classical functions, while the initial space of \( A \) is a quantum Hilbert space.

Another way of seeing this is the following: By taking a double Bargmann transform one treats \( x \) and \( \rho \) as two equivalent but independent (“commuting”) variables: in Eq. (6.6) \( x \) is only linked with \( x \) and \( \rho \) only with \( x \) and \( \rho \). However, this is not what happens in a quantization procedure; there \( x \) and \( \rho \) are linked with \( x \) and \( \rho \) as well as with \( x \) and \( \rho \); some mixing has taken place.

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