Two Theorems on Lattice Expansions

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Abstract—It is shown that there is a trade-off between the smoothness and decay properties of the dual functions, occurring in the lattice expansion problem. More precisely, it is shown that if \( g \) and \( \tilde{g} \) are dual, then 1) at least one of \( H^{1/2}g \) and \( H^{1/2}\tilde{g} \) is not in \( L^2(\mathbb{R}) \), 2) at least one of \( Hg \) and \( \tilde{g} \) is not in \( L^2(\mathbb{R}) \). Here, \( H \) is the operator \(-1/(4\pi^2) d^2/dx^2 + t^2 \). The first result is a generalization of a theorem first stated by Balian and independently by Battle. Battle suggests a theorem proved by Coifman and Semmes; a new, much shorter proof implicit assumptions made by Battle are removed. Result 2) is in the lattice expansion problem. More precisely, it is shown that if \( g \) is of type (i), but our result is stronger in the sense that certain basics, and time-frequency localization.

Index Terms—Gabor transformation, frame, orthonormal basics, and time-frequency localization.

I. INTRODUCTION

We consider in this note expansions of the type

\[
f(t) \sim \sum_{n,m} c_{nm} g_{nm}(t), \quad t \in \mathbb{R},
\]

where \( f \in L^2(\mathbb{R}) \),

\[
g_{nm}(t) = e^{-2\pi i nm \omega} g(t + n), \quad t \in \mathbb{R}, n, m \in \mathbb{Z},
\]

\( g \in L^2(\mathbb{R}) \) is a fixed function of time, usually well concentrated in time and frequency, and \( \omega \) is a fixed real number \( \neq 0 \).

For \( \omega \) < 1, many choices for \( g \) lead to convergent expansions for all \( f \in L^2(\mathbb{R}) \), [8]. In this paper, we restrict ourselves to the case \( \omega = 1 \), corresponding to lattices with the largest possible mesh size.

In (1.1), the coefficients \( c_{nm} \) depend on both \( f \) and \( g \). Problems related to the present one were considered (for the case of Gaussian \( g \)) by Von Neumann in a quantum mechanical context [16], by Gabor in the context of efficient data transmission [9], by Perelomov [17], Bargmann, Butera, Girardello, and Klauder [3], and by Bacry, Grossmann, and Zak [1], who all gave completeness properties of the set of \( g_{nm} \)'s. The problem of determining the coefficients \( c_{nm} \) in the expansion (1.1) became tractable notably through the work of Zak on solid-state physics related problems [1], [20], [21], of Bastiaans on optical signal description [4], [5], and of Janssen who gave rigorous proofs of existence and convergence of the expansions (1.1) in \( L^2(\mathbb{R}) \) and other spaces of (generalized) functions [11]–[13]. In this context, Daubechies and Grossmann [7] exploited the notion of frame that indicates a set of functions \( g_{nm} \) as in (1.2) such that

\[
A ||f||^2 \leq \sum_{n,m} |(f,g_{nm})|^2 \leq B ||f||^2,
\]

for all \( f \in L^2(\mathbb{R}) \), \( A = \| \cdot \| \) and \( (\cdot,\cdot) \) denote ordinary norm and inner product in \( L^2(\mathbb{R}) \). It is amply demonstrated in the comprehensive [8] that for numerically reliable expansion of \( f \) as in (1.1), one needs to consider \( g \)'s such that the set of \( g_{nm} \)'s constitute a frame. An even more desirable case occurs when the constants \( A \) and \( B \) are equal. Then the \( g_{nm} \)'s are said to constitute a tight frame, and the \( g_{nm} \)'s are orthogonal.

The procedure of finding the coefficients \( c_{nm} \) in (1.1) is as follows. Consider the mapping \( Z \) defined for \( f \in L^2(\mathbb{R}) \) by

\[
(Zf)(\tau,\Omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i (\tau t + \Omega \omega)} dt,
\]

\( \tau, \Omega \in \mathbb{R} \). (1.4)

This mapping has several names, such as Gel’fand mapping [10], [19], Weil–Brézin mapping [18], Zak transform [13], [14], while it seems that Gauss was already aware of some of its properties [18]. We shall call \( Z \) the Zak transform, since Zak seems to be the first one to exploit the transform systematically in the context of completeness and expansion problems. For a survey of the numerous properties of the Zak transform we refer to [14]. The relevant property of \( Z \) for the expansion problem is that

\[
(Zg_{nm})(\tau,\Omega) = e^{-2\pi i n + \pi i m \omega} (Zg)(\tau,\Omega),
\]

\( n, m \in \mathbb{Z} \). (1.5)

Hence, we have, at least formally,

\[
c_{nm} = \int \int e^{2\pi i \tau - 2\pi i n \Omega} (Zf)(\tau,\Omega) (Zg^*)(\tau,\Omega) d\tau d\Omega.
\]

In (1.6), the integration is over any unit square \((Zf/Zg)\) is periodic with period 1 in both its variables). To introduce the notion of dual function we note the property that \( Z \) is a Hilbert space isomorphism between \( L^2(\mathbb{R}) \) and the set of all functions \( F(\tau,\Omega) \) such that

\[
F(\tau,\Omega + 1) = F(\tau,\Omega), \quad F(\tau + 1,\Omega) = e^{2\pi i \Omega} F(\tau,\Omega).
\]

(1.7)

In the latter set of functions the inner product of an \( F \) and \( G \) satisfying the (quasi-) periodicity relations in (1.7) is given by

\[
(F,G) = \int \int F(\tau,\Omega)G^*(\tau,\Omega) d\tau d\Omega,
\]

(1.8)
where the integral is over any unit square and the asterisk denotes complex conjugation. In particular, for any \( f_1, f_2 \in L^2(\mathbb{R}) \), we have

\[
(f_1, f_2) = (Zf_1, Zf_2).
\]

Now, the function \( 1/(Zg)^* \) satisfies the relations (1.11), and, if it is square integrable over a unit square, there is a unique \( \tilde{g} \in L^2(\mathbb{R}) \) such that

\[
Z\tilde{g} = \frac{1}{(Zg)^*}.
\]

This \( \tilde{g} \) is called the dual function, and \( \tilde{g}_{nm} \) constitute the dual frame. We observe that \( \tilde{g} = g \), and that

\[
(g, \tilde{g}_{nm}) = \delta_{n0}\delta_{m0} = (\tilde{g}, g_{nm}),
\]

with \( \delta \) the Kronecker function.

It follows from (1.5), (1.6), and (1.10) that the coefficients \( c_{nm} \) can be expressed as

\[
c_{nm} = (f, \tilde{g}_{nm}).
\]

Furthermore, the conditions of being a frame and a tight frame can be expressed in terms of the Zak transform as

\[
\text{ess sup}|Zg| < \infty, \quad \text{ess sup}|Z\tilde{g}| > 0
\]

and

\[
\text{ess sup}|Z\tilde{g}| = \text{ess inf}|Zg| < \infty,
\]

respectively, (see [8]).

The two main theorems of this paper read as follows. With the notation, \( H = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial t^2} + t^2 \) for the Hermite operator, we have the following.

**Theorem I:** If \( g \) and \( \tilde{g} \) are dual functions in the sense previously explained, then \( H^{1/2}g \) and \( H^{1/2}\tilde{g} \) cannot both be in \( L^2(\mathbb{R}) \).  

**Theorem II:** Under the same assumptions, \( Hg \) and \( \tilde{g} \) cannot both be in \( L^2(\mathbb{R}) \).  

Hence, in a sense, \( g \) and \( \tilde{g} \) cannot both be smooth and rapidly decaying. Such results can be expected is seen as follows. Assume that \( g \) is such that \( Zg \) is continuous; this holds when \( g \) is continuous and decays sufficiently rapidly, e.g., like \( 1/(1 + |t|)^{\alpha} \) with \( \alpha > 1 \). It is a curious property of the Zak transform that then \( Zg \) has at least one zero in the unit square [1], [13]. Hence, \( \text{ess sup}|Z\tilde{g}| = \infty \). And when \( Zg \) is continuously differentiable, we even have that \( 1/Zg \) is not square integrable over the unit square.

Note that nevertheless Theorems I and II are nontrivial since \( H^{1/2}g \in L^2(\mathbb{R}) \) does not imply that \( Zg \) is continuous, and \( Hg \in L^2(\mathbb{R}) \) does not imply that \( Zg \) is continuously differentiable.

Recently, some results like ours have been proved. Balian [2] and Low [15] both argued that at least one of \( tg(t) \) and \( g'(t) \) is not in \( L^2(\mathbb{R}) \) when \( g_{nm} \) constitutes a tight frame. Their argument was made rigorous and extended by Coifman and Semmes to include the case of nontight frames; this is presented by Daubechies in [8]. Finally, an independent, more elegant, proof of the Balian-Low result was given by Battle in [6].

To see what the novelty of the present paper is, we give some further preliminary remarks. The conditions

a) \( H^{1/2}g \in L^2(\mathbb{R}) \),

b) \( tg(t) \in L^2(\mathbb{R}) \), \( g'(t) \in L^2(\mathbb{R}) \),

c) \( \frac{\partial^2}{\partial t^2} g \in L^2(\mathbb{R}) \), \( \frac{\partial^2}{\partial t^2} g' \in L^2(\mathbb{R}) \),

d) \( \frac{\partial^2}{\partial t^2} g' \in L^2(\mathbb{R}) \),

are equivalent. That a) and b) are equivalent is a standard fact; that b) and c) are equivalent follows from (2.1) and (2.9). Similarly, the conditions

e) \( Hg \in L^2(\mathbb{R}) \),

f) \( t^2g(t) \in L^2(\mathbb{R}) \), \( g''(t) \in L^2(\mathbb{R}) \),

g) \( \frac{\partial^2}{\partial t^2} g \in L^2(\mathbb{R}) \), \( \frac{\partial^2}{\partial t^2} g' \in L^2(\mathbb{R}) \), \( \frac{\partial^2}{\partial t^2} g'' \in L^2(\mathbb{R}) \)

are equivalent. Now, when \( g_{nm} \) constitutes a frame and \( Zg \in W^{2,1}(S) \), it follows from \( \text{ess inf}|Zg| > 0 \) and

\[
\frac{\partial}{\partial t} \left( \frac{1}{Zg} \right) = -\left( \frac{1}{Zg} \right)^2 \frac{\partial^2}{\partial t^2} Zg,
\]

\[
\frac{\partial}{\partial t} \left( \frac{1}{Zg} \right) = -\left( \frac{1}{Zg} \right)^2 \frac{\partial^2}{\partial t^2} Zg
\]

that

\[
\frac{1}{(Zg)^*} = Z\tilde{g} \in W^{2,1}(S).
\]

That is, Theorem I implies the Coifman-Semmes result, and, a fortiori, the Balian-Low-Battle result. The Theorem II is entirely new as far as we know. While our proof of Theorem I uses a little trick that can be found in Battle’s paper, the proof of Theorem II is based on the two facts that

a) when \( Zg \in W^{2,2}(S) \) then \( Zg \) is continuous and has a zero in \( S \),

b) when \( G \in W^{2,2}(S) \) has a zero then \( 1/G \not\in L^2(S) \).

We were unable to find the result (b) in the literature, and it may be of some independent interest.

Theorems I and II may be viewed as no-go theorems, excluding the possibility of numerically stable expansions of type (1.1) with respect to \( g_{nm} \) in (1.2) with \( \omega = 1 \), which are well-localized in both time and frequency. This can be avoided by using expansions with tighter lattices, corresponding to the choice \( \omega < 1 \). It is well known that the dual function \( g \) has many singular features [4], [5] if \( g \) is Gaussian. Our Theorem II generalizes the result in [13] that \( g \) is Gaussian integrable.

**II. PROOF OF THEOREM I**

As explained in Section I, we must take a \( g \in L^2(\mathbb{R}) \) with \( Zg, Z\tilde{g} \in W^{2,1}(S) \) and show that this leads to a contradiction. Denote

\[
(Qg)(t) = tg(t), \quad (Pg)(t) = \frac{1}{2\pi i} g'(t), \quad \text{etc.}
\]

As in Battle’s proof, we shall show that

\[
(Qg, P\tilde{g}) = (Qg, Q\tilde{g})
\]
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(by assumption, all four functions involved in (2.2) are in $L^2(\mathbb{R})$). This implies that $(g, g) = 0$, which is absurd by (1.11).

To demonstrate (2.2), we need the auxiliary results

\[ (Qg, P\tilde{g}) = \sum_{n,m} (Qg, \tilde{g}_{nm})(g_{nm}, P\tilde{g}), \]
\[ (Pg, Q\tilde{g}) = \sum_{n,m} (Pg, \tilde{g}_{nm})(g_{nm}, Q\tilde{g}). \]

In [6], the validity of the expansions (2.3), (2.4) was implicitly assumed (and not proved as is done here).

The relation (2.3) follows from the fact, to be proved below, that $ZQg/Zg, ZP\tilde{g}/Z\tilde{g} \in L^2(S)$. Indeed, it then follows from

\[ (Qg, \tilde{g}) = 0, \]

which is absurd by (1.11).

To demonstrate (2.2), we need the auxiliary results

\[ (Qg, P\tilde{g}) = (ZQg, ZP\tilde{g}) = (ZQg/Zg, ZP\tilde{g}/Z\tilde{g}). \]

The right-hand side of (2.5) equals the right-hand side of (2.3), since $(Qg, j_{nm})$ and $(Pn, g_{nm})$ are the Fourier coefficients of $ZQg/Zg$ and $ZP\tilde{g}/Z\tilde{g}$ by (1.6) and (1.12). Similarly, $ZP\tilde{g}/Zg, ZQg/Z\tilde{g} \in L^2(S)$ implies that (2.4) holds.

We shall show now that $ZP\tilde{g}/Zg, ZQg/Z\tilde{g} \in L^2(S)$. We have

\[ \left( \frac{1}{Zg} \right)^2 \frac{\partial Zg}{\partial \Omega} = -\frac{\partial}{\partial \Omega} \left( \frac{1}{Zg} \right) = -\frac{\partial}{\partial \Omega} (Zg)^* \in L^2(S). \]

It follows from the Cauchy–Schwarz inequality that

\[ \left| \frac{1}{Zg} \frac{\partial Zg}{\partial \Omega} \right| = \left| \left( \frac{1}{Zg} \right)^2 \frac{\partial Zg}{\partial \Omega} \right|^{1/2} \left| \frac{\partial Zg}{\partial \Omega} \right|^{1/2} \in L^2(S), \]

and, similarly,

\[ \left| \frac{1}{Zg} \frac{\partial Zg}{\partial r} \right| \in L^2(S). \]

Since

\[ ZQg = \frac{1}{2\pi i} \frac{\partial Zg}{\partial \Omega} + \tau(Zg)(r, \Omega), \quad ZP\tilde{g} = \frac{1}{2\pi i} \frac{\partial Z\tilde{g}}{\partial r}, \]

it follows that $ZQg/Zg, ZP\tilde{g}/Z\tilde{g} \in L^2(S)$, as claimed.

To show (2.2), it suffices to prove that

\[ (Qg, \tilde{g}_{nm}) = (g_{nm}, P\tilde{g}), \]
\[ (Pn, \tilde{g}_{nm}) = (g_{nm}, P\tilde{g}). \]

We have

\[ (Qg, \tilde{g}_{nm}) = \int t\frac{d}{dt}(g(t)g^*(t)) \]
\[ = \int (t-n)g(t-n)e^{2\pi nm}g^*(t) dt \]
\[ = (g_{nm}, Q\tilde{g}) - n(g_{nm}, \tilde{g}). \]

Together with (1.11), this implies the first part of (2.10). The second part of (2.10) follows from the first part by noting that, with $\mathcal{F}$ the Fourier transform,

\[ \mathcal{F}P = Q\mathcal{F}, \quad \mathcal{F}g_{nm} = (\mathcal{F}g)_{-m,-n}. \]

and the fact that $\mathcal{F}g$ and $\mathcal{F}\tilde{g}$ are dual. This establishes (2.2).

We conclude the proof of Theorem I by showing that (2.2) implies that $(g, \tilde{g}) = 0$. We have by assumption

\[ t^2 \frac{d}{dt}(g(t)\tilde{g}^*(t)) = t\tilde{g}(t)\tilde{g}(t)^* + t\tilde{g}(t)\tilde{g}(t) \in L^1(\mathbb{R}). \]

The right-hand side function in (2.13) equals

\[ -2\pi i(Qg \cdot (P\tilde{g})^* - P\cdot (Q\tilde{g})) \]

We now have, for all $a < b$

\[ \int_a^b t^2 \frac{d}{dt}(g(t)\tilde{g}^*(t)) dt = t\tilde{g}(t)\tilde{g}(t)^* \bigg|_a^b \int_a^b t\tilde{g}(t)\tilde{g}(t) dt. \]

When $a \to -\infty, b \to \infty$, the left-hand side of (2.15) tends to 0 by (2.2), (2.13) and (2.14), and the integral on the right-hand side tends to $(g, \tilde{g})$. Hence,

\[ \lim_{b \to \infty, a \to -\infty} t\tilde{g}(t)\tilde{g}(t)^* \]

exists as well and equals 0 since $t\tilde{g}(t)\tilde{g}(t) \in L^1(\mathbb{R})$. Therefore, $(g, \tilde{g}) = 0$, and the proof of Theorem I is complete.

III. PROOF OF THEOREM II

As already explained at the end of Section I, it is sufficient to show the following result.

Proposition: Assume $G \in W^{2,2}(S)$, where $S = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$, and $G(0,0) = 0$. Then, $1/G \not\in L^2(S)$.

Proof: For notational convenience we write $x = (\tau, \Omega) \in \mathbb{R}^2$, and we denote by $|\cdot|$ and · Euclidean norm and inner product in $\mathbb{R}^2$, respectively. Let $0 < r < \frac{1}{2}$. We (re)define $G(x)$ for $|x| \geq r$ such that the resulting function, again denoted by $G$, is in $W^{2,2}(\mathbb{R}^2)$. When $G(\xi), \xi \in \mathbb{R}^2$ is the Fourier transform if $G$, we have

\[ \iint (1 + |\xi|^2)^2 |\hat{G}(\xi)|^2 d\xi < \infty. \]

Now, by Fourier inversion,

\[ G(x) = G(x) - G(0) = \int (e^{2\pi \xi} - 1)\hat{G}(\xi) d\xi. \]

Hence, by the Cauchy–Schwarz inequality,

\[ |G(x)|^2 \leq \int \frac{|e^{2\pi \xi} - 1|^2}{1 + |\xi|^2} d\xi \leq \int \left( 1 + |\xi|^2 \right)^2 |\hat{G}(\xi)|^2 d\xi. \]

We have for the first integral $I_1$ in (3.3)

\[ I_1 = 4 \int \frac{\sin^2 \pi x \cdot \xi}{(1 + |\xi|^2)^2} d\xi = 4 \int_0^{\frac{\pi}{4}} \frac{1}{1 + |\xi|^2} d\xi. \]

The first integral $I_2$ in (3.4) satisfies

\[ I_2 \leq \pi^2|x|^2 \int \frac{|\xi|^2}{(1 + |\xi|^2)^2} d\xi \leq \pi^2|x|^2 \log(1 + \frac{1}{|x|^2}). \]
The second integral $I_3$ in (3.4) satisfies
\[ I_3 \leq \int_{|x| \geq \frac{1}{2}} \frac{d\xi}{(1 + |\xi|^2)^\frac{3}{2}} = \frac{\pi |x|^2}{1 + |x|^2}. \] (3.6)

Hence, by (3.1), (3.3), (3.5), and (3.6).
\[ |G(x)|^2 = 0 \left( |x|^2 \log \frac{1}{|x|^2} \right), \quad |x| \leq \frac{1}{2}. \] (3.7)

Since
\[ \int_{|x| \leq \frac{1}{2}} \frac{dx}{|x|^2 \log \frac{1}{|x|^2}} = \infty, \] (3.8)

the proposition follows.

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