Analyse mathématique/*Mathematical Analysis*

Multiresolution analysis, wavelets and fast algorithms on an interval

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Abstract — We adapt the standard construction of multiresolution analysis and orthonormal wavelet bases in $L^2(\mathbb{R})$ to the framework of functions defined on the interval [0,1]. The main properties of wavelet bases (regularity, space localization and vanishing moments) are preserved and a fast algorithm (with a special treatment at the borders of the interval) can be derived.

Analyses multirésolutions, ondelettes et algorithme rapide sur l'intervalle

Résumé — *Nous adaptons la construction classique des analyses multirésolutions et des bases orthonormées d'ondelettes au cadre des fonctions définies sur* [0,1]. *Les propriétés importantes des bases d'ondelettes (régularité, localisation spatiale et oscillations) sont maintenues ainsi que l'existence d'un algorithme rapide (avec un traitement particulier aux extrémités de l'intervalle).*

Version française abrégée — Les bases d'ondelettes et les analyses multirésolutions ont été définies pour l'analyse des fonctions de $L^2(\mathbb{R})$. Dans la pratique, il est clair que le signal est restreint à un intervalle borné (ou à un carré lorsqu'il s'agit d'une image). Comment s'adapter à une telle situation en conservant les propriétés intéressantes des ondelettes (régularité, cancellation, localisation), tant du point de vue des bases de fonctions que de l'algorithme FWT?

On part d'une base d'ondelettes $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ à support compact d'Ingrid Daubechies (*voir* [3]) possédant N moments nuls (on a alors $\mathrm{Supp}(\psi)=[0, 2N-1]$). Rappelons que dans ce cas, les polynômes de degré inférieur à $N-1$ peuvent être engendrés par des combinaisons linéaires des fonctions d'échelle $\varphi_{j,k}$ pour j fixé.

On construit tout d'abord une analyse multirésolution de [0,1], de la manière suivante : les espaces V_j sont engendrés par des fonctions $\varphi_{j,k}$ dont le support est strictement inclus dans [0,1] (on se restreint aux valeurs de j telles que la taille de ce support est plus petite que 1/2 de manière à pouvoir traiter les deux bords indépendamment) et par deux groupes de N fonctions localisées aux extrémités. Ces fonctions d'échelle particulières sont construites à partir des $\varphi_{j,k}$ dont les supports interceptent les bords de manière à ce que les polynômes de degré inférieur à $N-1$ soient tous contenus dans V_j. Les espaces ainsi construits sont emboîtés, grâce aux propriétés d'emboîtement pour l'analyse de $L^2(\mathbb{R})$ et aux relations d'échelle satisfaites par les polynômes. On vérifie aisément que la dimension de V_j est 2^j. Il ne reste plus qu'à orthonormaliser les 2N fonctions d'échelle des bords.

La construction des ondelettes est alors la suivante : on prend comme base de W_j les fonctions $\psi_{j,k}$ dont le support est strictement inclus dans [0,1] auxquelles on ajoute

$$\varphi^l_{i,j}(2x)-P_j(\varphi^l_{i,j}(2x)) \quad \text{et} \quad \varphi^r_{i,j}(2x-1)-P_j(\varphi^r_{i,j}(2x-1)) \quad \text{pour } 0\leq i\leq N-1$$

où $\{\varphi^l_{i,j}, \varphi^r_{i,j}\}_{i=0\ldots N-1}$ sont les fonctions d'échelle au bord construites précédemment et où P_j est le projecteur sur V_j. Il ne reste plus qu'à orthonormaliser ce nouveau système.

Ces ondelettes sont, par construction, régulières et bien localisées et elles vérifient les conditions de simplification à l'ordre $N-1$ puisque les espaces d'approximation contiennent les polynômes. L'algorithme de décomposition est identique à l'algorithme classique FWT lorsqu'on se trouve loin des bords : on effectue des convolutions par les mêmes filtres suivies

Note présentée par Yves MEYER.

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de décimations d'un échantillon sur deux. Le calcul aux extrémités utilise des coefficients particuliers qui sont liés aux fonctions spéciales qui ont été construites (on trouvera dans [2] les tables de ces coefficients).

I. Multiresolution analysis and orthonormal wavelets in $L^2(\mathbb{R})$. — Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis with compactly supported scaling function φ, *i.e.* a sequence of approximation spaces

$$\{0\}\to\ldots V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\ldots\to L^2(\mathbb{R}) \tag{1}$$

each V_j having $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}=\{2^{j/2}\varphi(2^jx-k)\}_{k\in\mathbb{Z}}$ as an orthonormal basis. In this framework (*see* [4], [6]), wavelets characterize the missing details between two successive levels of approximation. More precisely, $\{\psi_{j,k}\}_{k\in\mathbb{Z}}=\{2^{j/2}\psi(2^jx-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of W_j, the orthogonal complement of V_j in V_{j+1}.

Such functions have been constructed in [3] with arbitrarily high regularity. Recall that one uses a conjugate quadrature filter pair (CQF, *see* [7]), that can be reduced to the data of a trigonometric polynomial $m_0(\omega)=\sum_{n=0}^{2N-1}h_n e^{-in\omega}$ such that $m_0(0)=1$ and

$$|m_0(\omega)|^2+|m_0(\omega+\pi)|^2=1 \tag{2}$$

The functions φ and ψ are then defined in the Fourier domain by

$$\hat{\varphi}(\omega)=\prod_{k=1}^{+\infty}m_0(2^{-k}\omega)\quad\text{and}\quad\hat{\psi}(\omega)=m_1(\omega/2)\,\hat{\varphi}(\omega/2) \tag{3}$$

As a consequence, it satisfies the following scaling equation

$$\varphi(x)=2\sum_{n=0}^{2N-1}h_n\varphi(2x-n) \tag{4}$$

and the wavelet is defined with the conjugate filter $g_n=(-1)^n h_{2N-1-n}$

$$\psi(x)=2\sum_{n=0}^{2N-1}g_n\varphi(2x-n) \tag{5}$$

In the discrete setting, these filters are used to decompose a sampled signal (that can be viewed as the coordinates of an element of V_0) into two subsampled channels representing the approximation of the signal and the missing details, at the coarser scale. The original signal can be reconstructed from these channels by interpolation with the same filters. The algorithm iterates the decomposition on the coarse approximation, but one can also do it on the details: this corresponds to the wavelet packets bases introduced in [1].

II. Adaptation to "life in the interval". — In many practical situations, the signal (continuous or discrete) to be analyzed is defined (or restricted) on an interval, say [0,1], or a rectangle in the case of an image. Since $L^2[0,1]$ is not invariant by translations and dilations, multiresolution analysis needs to be redefined. In particular, we only consider the scales $j\geqq j_0\geqq 0$ for which the supports of the scaling functions and wavelets are smaller than the interval, *i. e.* a half sequence of approximation subspaces $V_{j_0}\subset V_{j_0+1}\subset V_{j_0+2}\ldots\to L^2[0,1]$.

Here, V_j will be generated by the functions $\varphi_{j,k}$ in the interior of [0, 1] and by some special functions adapted to (and localized at) the borders. The same will hold for W_j

and the functions $\psi_{j,k}$. This means that inside the interval, the decomposition (and thus the algorithm) will be the same as if one worked in $L^2(\mathbb{R})$.

This approach was first used in a construction by Y. Meyer [5] in which the scaling functions at the edges are obtained by orthonormalizing the restrictions in [0,1] of the $\varphi_{j,k}$ that are supported in both the inside and the outside of the interval. Since $\mathrm{Supp}(\varphi)=[0, 2N-1]$, N being the number of vanishing moments for the wavelet ψ (*see* [3]), we have $\dim(V_j)=2^j+2N-2$ and $\dim(W_j)=2^j$. This arithmetic rule is a problem for wavelet-packets decompositions in which one needs to have the same format for the approximation and the detail channels that are processed in a similar fashion. Another problem is that the orthonormalization process of the restricted scaling functions is ill-conditionned for $N>3$, due to the very small ties of these functions that have to be normalized. This results in some difficulties in the computation of the adapted filters to be used at the borders in the algorithm.

The goal of our construction is to circumvent these disadvantages while keeping the main qualities of Meyer's bases on the interval: regularity, vanishing moments, spatial localization and existence of a fast algorithm.

III. Multiresolution analysis on [0,1]. — We shall build V_j for $j \geq j_0 > 0$ such that $\mathrm{Supp}(\varphi_{j,0})=[0,2^{-j}(2N-1)]$ is included in [0,1/2]. By this assumption the borders will not "interact" and the special functions at the left and right edges will be defined independently. Regularity and spatial localization will be ensured by choosing these functions as linear combinations of the restricted $\varphi_{j,k}$ previously mentioned.

The vanishing moments property deserves more attention. In $L^2(\mathbb{R})$, it can be expressed directly on the scaling function by the Fix-Strang rules, *i. e.* identities of the type

(6)
$$\left|\begin{array}{l} \sum_k \varphi(x-k)=1=P_0(x) \\ \sum_k k\,\varphi(x-k)=P_1(x) \\ \quad\dots \\ \sum_k k^{N-1}\varphi(x-k)=P_{N-1}(x) \end{array}\right.$$

where $P_i(x)$ is a polynomial of degree i. The function $\psi(x)$ being orthogonal to all the $\varphi(x-k)$, all its moments of order $i \leq N-1$ are then equal to zero. In order to mimic this property on the interval, we define, for $0 \leq i \leq N-1$, N functions compactly supported on $x \geq 0$ by

(7)
$$\tilde{\varphi}^{l,i}(x)=\Big(P_i(x)-\sum_{k>0} k^i\varphi(x-k)\Big)\chi_{[0,+\infty]}=\sum_{k\leq 0} k^i\varphi(x-k)\chi_{[0,+\infty]}$$

and similarly N functions compactly supported on $x \leq 0$ by

(8)
$$\tilde{\varphi}^{r,i}(x)=\Big(P_i(x)-\sum_{k<1-2N} k^i\varphi(x-k)\Big)\chi_{[-\infty,0]}=\sum_{k\geq 1-2N} k^i\varphi(x-k)\chi_{[-\infty,0]}$$

The subspace $V_j (j \leq j_0)$ will be generated by the interior functions $\{\varphi_{j,k}\}_{1\leq k\leq 2^j-2N}$ completed by $\{\tilde{\varphi}^{l,i}(2^j x)\}_{0\leq i\leq N-1}$ at the left edge and by $\{\tilde{\varphi}_{r,i}(2^j(x-1))\}_{0\leq i\leq N-1}$ at the right edge.

With this definition, V_j contains all the polynomials of degree $N-1$ restricted to [0,1] and clearly, the projection of any square-integrable function f onto V_j converges to f in

$L^2(\mathbb{R})$ as j goes to $+\infty$. Furthermore, we have the following result:

THEOREM 1. — *For all* $j \geq j_0$, $V_j \subset V_{j+1}$ *and the set of functions*

$$\{\tilde{\varphi}^{l,i}(2^j x), \tilde{\varphi}^{r,i}(2^j(x-1))\}_{0 \leq i \leq N-1} \cup \{\varphi_{j,k}\}_{1 \leq k \leq 2^j-2N}$$

form a basis of V_j.

Sketch of proof. — The inclusion can be proved by showing that the generators of V_j are linear combinations of those of V_{j+1}. By the scaling equation (4), this is clear for the interior functions $\{\varphi_{j,k}\}_{1 \leq k \leq 2^j-2N}$. For the functions at the edges, one remarks that the polynomials also satisfy trivial scaling relations of the type $x^n = 2^{-n}(x/2)^n$ and the result follows immediately from the definitions (7) and (8).

It is clear that the three groups of generators are orthogonal and that the scaling functions in the interior form an orthonormal family. To proove that the $\{\tilde{\varphi}^{l,i}(x)\}_{0 \leq i \leq N-1}$ and linearly independent, one remarks that they are obtained by applying a transformation of rank N on the restricted functions $\{\varphi(x+k)\}_{0 \leq k \leq 2N-2}$ which are linearly independent because of their embedded supports. The same holds at the right edge.

It follows from this results that $\dim(V_j) = 2^j$. Remark that we can apply an invertible triangular transformation on the functions at the edges to preserve the property that their support are embedded. This property will also be preserved by a Gram-Schmidt orthonormalization process if it follows the order of the supports. This leads to N orthonormal functions $\{\varphi^{l,i}(x)\}_{0 \leq i \leq N-1}$ with $\mathrm{Supp}(\varphi^{l,i}) = [0, N+i]$ and similarly at the right edge.

IV. WAVELETS ON [0,1]. — We have now an orthonormal basis for each V_j. We define W_j as the subspace generated by the interior wavelets $\{\psi_{j,k}\}_{1 \leq k \leq 2^j-2N}$ completed by N functions at the left edge $\{\tilde{\psi}^{l,i}(2^j x)\}_{0 \leq i \leq N-1}$ and N functions at the right edge $\{\tilde{\psi}^{r,i}(2^j(x-1))\}_{0 \leq i \leq N-1}$ that are defined by

$$\tilde{\psi}^{l,i}(2^j x) = \varphi^{l,i}(2^{j+1} x) - \mathrm{Proj}_{V_j}(\varphi^{l,i}(2^{j+1} x)) \tag{9}$$

and

$$\tilde{\psi}^{r,i}(2^j(x-1)) = \varphi^{r,i}(2^{j+1}(x-1)) - \mathrm{Proj}_{V_j}(\varphi^{r,i}(2^{j+1}(x-1))) \tag{10}$$

With this definition, it is clear that W_j is orthogonal to V_j and contained in V_{j+1}. To proove that W_j is exactly the orthogonal complement, it suffices to show that $\dim(W_j) = 2^j$. This is ensured by the following result:

THEOREM 2. — $\{\tilde{\psi}^{l,i}(2^j x), \tilde{\psi}^{r,i}(2^j(x-1))\}_{0 \leq i \leq N-1} \cup \{\psi_{j,k}\}_{1 \leq k \leq 2^j-2N}$ *is a basis of* W_j.

Skectch of proof. — Here again, it is sufficient to check that the wavelets at the edges are linearly independent. Suppose that there exists a non-trivial family $a_0, \ldots, a_{N-1}$ such that $\sum_{l,i} a_i \psi^{l,i}(2^j x) = 0$ for all x. This would imply that the function $g = \sum a_i \varphi^{l,i}(2^{j+1} x)$ is a non-trivial element of V_j. But $\mathrm{Supp}(g) = [0, 2^{-j-1}(2N-1)]$ which is strictly included into the support of the smallest function $\varphi^{l,0}(2^j x)$. By the embedded structure of the supports for the básis of V_j, it follows that $g(x) = 0$ everywhere on [0,1] which contradicts the assumption. The same holds for the right edge wavelets.

By a Gram-Schmidt process, we obtain N orthonormal wavelets at each edge $\{\psi^{l,i}(2^j x), \psi^{r,i}(2^j(x-1))\}_{0 \leq i \leq N-1}$ and we thus have an orthonormal basis for W_j.

These wavelets have the same regularity as the standard wavelet in the interior, but they also have the same vanishing moments since they are orthogonal to the polynomials of degree $N-1$ that are contained by the approximation subspaces V_j. As a consequence they constitute an unconditional basis for the Hölder and Sobolev spaces on the interval (of exponent inferior to the regularity index of the wavelet).

V. Fast algorithms. – The fast decomposition and reconstruction algorithms corresponding to these wavelets have the same pyramidal structure as the standard algorithms. At each scale, a sequence of 2^j samples in divided into 2^{j-1} approximation coefficients and 2^{j-1} detail coefficients and it can be recovered from these subsampled channels.

The coefficients that are situated at a distance of more than N points from the edges are still obtained by standard convolution. For the N first and last coefficients, one uses special filters that correspond to the scaling relations satisfied by the scaling functions and the wavelets that we have constructed at the edges.

Remark that contrarily to the standard algorithm, the constant sequence $\{1, 1, \ldots, 1, 1\}$ does not have only zeros in the detail coefficients at each scale because it does not correspond to a constant function. Indeed we have

$$1=\sum_{i=0}^{N-1} b_i 2^{-j/2}\varphi^{l,i}(2^j x)+\sum_{i=0}^{N-1} c_i 2^{-j/2}\varphi^{r,i}(2^j(x-1))+\sum_{k=1}^{2^j-2N} 2^{-j/2}\varphi_{j,k} \qquad (11)$$

and the coefficients b_i and c_i are not all equal to 1. The same problem holds for higher order moments.

To circumvent this disadvantage, one can use a preconditioning linear transformation that acts on the N first and last samples in the discrete signal, before applying the algorithm. This preconditioning transforms the sequence $\{1, 1, \ldots, 1, 1\}$ into $\{b_0, \ldots, b_{N-1}, 1, 1, \ldots, 1, 1, c_{N-1}, \ldots, c_0\}$ (that has all its wavelets coefficients equal to zero) and similarly for higher order moments up to $N-1$. The inverse transformation is applied on the reconstructed signal.

Tables for the filter coefficients at the edges and the entries of the preconditioning matrices can be found in [2].

Note remise le 11 mai 1992, acceptée le 16 décembre 1992.

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