Richard Hain September 1, 2017

Math 625
Riemann Surfaces
Problem Set 1

Due: Thursday, September 14, 2017.

1. Show that a holomorphic bijection between Riemann surfaces is a biholomorphism.

2. Show that if $f : X \to Y$ is a non-constant holomorphic map between Riemann surfaces, then the inverse image of each $y \in Y$ is a closed, discrete subset of X.

3. Suppose that $p : X \to Y$ is a covering map. Show that if Y is a Riemann surface, then X has a unique complex structure such that p is holomorphic. Note that, in particular, the universal covering of every Riemann surface is naturally a Riemann surface. Show that if Z is a Riemann surface, then a function $f : Z \to X$ is holomorphic if and only if $p \circ f : Z \to Y$ is holomorphic.

4. Suppose that $p : X \to Y$ is a Galois (i.e., normal or regular) covering map with Galois group (i.e., group of deck transformations) G. Suppose that X is a Riemann surface and that G acts on X as a group of biholomorphisms. Show that Y has a unique complex structure such that p is holomorphic.

5. Suppose that U is an open neighbourhood of 0 in $\mathbb{C}$ and that $\phi : U \to U$ is a biholomorphism of order n that fixes the origin. Show that there is a local holomorphic coordinate w in $\mathbb{C}$ centered at 0 and an $\epsilon > 0$ such that

(i) $\mathbb{D}(0,\epsilon) \subset U$,
(ii) $\phi(\mathbb{D}) \subseteq \mathbb{D}$, where $\mathbb{D} = \mathbb{D}(0,\epsilon)$,
(iii) $\phi(w) = \zeta_n w$, where $\zeta_n = e^{2\pi i/n}$.

Hint: show that the function $U \to \mathbb{C}$ defined by $z \mapsto \prod_{j=0}^{n-1} \phi^j(z)$ has a holomorphic nth root in a neighbourhood of 0. Take w to be an nth root.

6. Suppose that X is a Riemann surface and that G is a finite subgroup of $\operatorname{Aut} X$. Denote the stabilizer of $P \in X$ by G_P.

(i) Show that each $P \in X$ has a G_P-invariant coordinate neighbourhood.
(ii) Show that the stabilizer of each $P \in X$ is cyclic.

(iii) Show that the quotient $G\backslash X$ of X by G has a natural complex structure such that the projection $X \to G\backslash X$ is holomorphic.