2016.09.05

TIME ORDERED SIMPLICES

<u>The standard</u> n-<u>simplex</u>:

$$\Delta^n = \left\{ (s_0, \dots, s_n) : s_j \geq 0, \ \sum_{j=0}^{n} s_j = 1 \right\}$$

The s_j are called barycentric coordinates.

The standard orientation is

all in $\mathbb{R}^{n+1}$

$$\left(\frac{\partial}{\partial s_0} + \dots + \frac{\partial}{\partial s_n} \right) \lrcorner \ ds_0 \wedge \dots \wedge ds_n$$

$$= ds_1 \wedge \dots \wedge ds_n$$

as both take the value 1 on

$$\frac{\partial}{\partial s_1} \wedge \dots \wedge \frac{\partial}{\partial s_n}$$

as

$$\left(\frac{\partial}{\partial s_0} + \dots + \frac{\partial}{\partial s_n} \right) \wedge \frac{\partial}{\partial s_1} \wedge \dots \wedge \frac{\partial}{\partial s_n}$$

$$= \frac{\partial}{\partial s_0} \wedge \frac{\partial}{\partial s_1} \wedge \dots \wedge \frac{\partial}{\partial s_n}$$

The orientation also equals

$$(-1)^j \, ds_0 \wedge ds_1 \wedge \dots \wedge \widehat{ds_j} \wedge \dots \wedge ds_n \qquad j = 0, \dots, n.$$

The vertices of Δ^n are

$$e_0, \dots, e_n$$

jth place

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$

The jth face is

$$\langle e_0, \dots, \hat{e}_j, \dots, e_n \rangle$$

and has equation $s_j = 0$.

<u>The time ordered</u> n-<u>simplex</u>

$$\Delta^n = \{ (t_1, \dots, t_n) : 0 \leq t_1 \leq \dots \leq t_n \leq 1 \}$$

Set $t_0 = 0$ and $t_{n+1} = 1$. Define

$$s_j = t_{j+1} - t_j \qquad j = 0, \dots, n.$$

Then (1) $s_j \geq 0$

(2) $s_0 + \dots + s_n = 1$.

So

$$(t_1, \dots, t_n) \longmapsto (s_0, \dots, s_n)$$

sets up a bijection between the standard n-simplex and the time ordered n-simplex. The inverse is

$$t_j = s_0 + \dots + s_{j-1}$$

<u>Orientations:</u>

$$
\begin{aligned}
& dt_1 \wedge \dots \wedge dt_n \\
&= ds_0 \wedge (ds_0 + ds_1) \wedge \dots \wedge (ds_0 + \dots + ds_{n-1}) \\
&= ds_0 \wedge ds_1 \wedge \dots \wedge ds_{n-1} \\
&= (-1)^n \, ds_1 \wedge \dots \wedge ds_n .
\end{aligned}
$$

<u>Vertices</u> <u>and</u> <u>faces</u>:

jth face is defined by $s_j = 0$. So by

$$t_{j+1} = t_j \qquad j = 0, \dots, n$$

The jth vertex e_j has

$$s_j = 1, \quad \text{all other } s_k = 0.$$

So

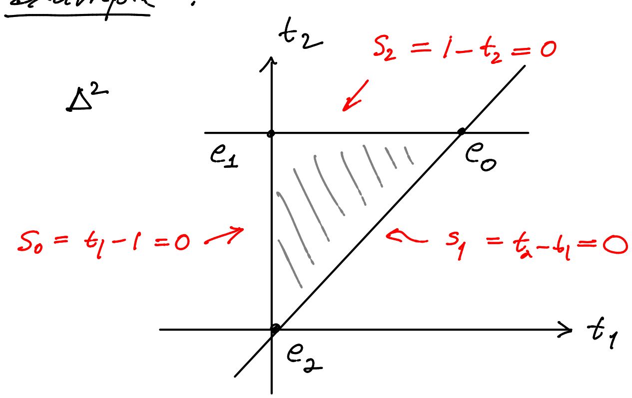
$$e_j = (0, \dots, \overset{j-1}{0}, \overset{j}{1}, \dots, 1) \in \mathbb{R}^n$$

$\nwarrow$ $(t_1, \dots, t_n)$ coords

$$
\begin{aligned}
e_0 &= (1, \dots, 1) \\
e_1 &= (0, 1, \dots, 1) \\
e_2 &= (0, 0, 1, \dots, 1) \\
&\vdots \\
e_n &= (0, \dots, 0)
\end{aligned}
$$

<u>Example</u>:

Δ^2



Products of simplices

Def: A shuffle of type (r,s) is a permutation σ of $\{1,2,\dots,r+s\}$ such that

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \dots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \dots < \sigma^{-1}(r+s).$$

$Sh(r,s) := \{\text{shuffles of type } (r,s)\}$

Remark

$\sigma^{-1}(j)$ is the position of j in the list

$$\begin{array}{ccccccc} 1 & 2 & & \sigma^{-1}(j) & & & r+s \\ \sigma(1) & \sigma(2) & \dots & j & \dots & & \sigma(r+s). \end{array}$$

So if σ is a shuffle,

position of 1

$<$ position of 2

$< \dots <$ position of r

and

position of $r+1$

$<$ position of $r+2$

$< \dots <$ position of $r+s$. //

Let

$$\Delta^r = \{(t_1,\dots,t_r) : 0 \le t_1 \le \dots \le t_r \le 1\}$$

$$\Delta^s = \{(t_{r+1},\dots,t_{r+s}) : 0 \le t_{r+1} \le \dots \le t_{r+s}\}$$

For each point

$$(t_1,\dots,t_{r+s}) \in \Delta^r \times \Delta^s$$

there is a shuffle $\sigma \in Sh(r,s)$ such that

$$0 \le t_{\sigma(1)} \le t_{\sigma(2)} \le \dots \le t_{\sigma(r+s)} \le 1.$$

This shuffle is unique if the t_j are distinct.

Prop:

$$\Delta^r \times \Delta^s = \bigcup_{\sigma \in Sh(r,s)} \{ (t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(r+s)}) : 0 \le t_1 \le \ldots \le t_{r+s} \le 1 \}$$

$$= \bigcup_{\sigma \in Sh(r,s)} \Delta^{r+s}_\sigma \quad \Delta^{r+s}_\sigma := \{ (t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(r+s)}) : 0 \le t_1 \le \ldots \le t_{r+s} \le 1 \}$$

Note:

$$\Delta^r \times \Delta^s \supseteq \coprod_{\sigma \in Sh(r,s)} \mathring{\Delta}^{r+s}_\sigma$$

← interiors are disjoint.

So intersections of the Δ_σ have measure 0.

Decompositions of Simplices

For each $(t_1, \ldots, t_n) \in \Delta^n$ there is (r,s) with $r+s=n$ such that

$$(*) \qquad 0 \le t_1 \le \ldots \le t_r \le 1/2 \le t_{r+1} \le \ldots \le t_n \le 1$$

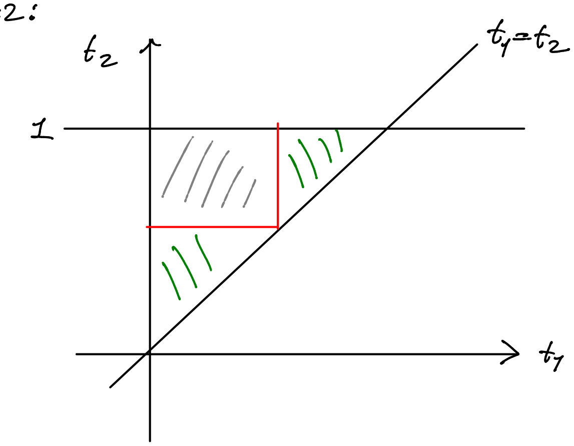
This (r,s) is unique if no $t_j = 1/2$.

The $(t_1, \ldots, t_n)$ satisfying $(*)$ can be identified with $\Delta^r \times \Delta^s$ via

$$(t_1, \ldots, t_n) \mapsto ((2t_1, \ldots, 2t_r), (2t_{r+1}-1, \ldots, 2t_n - 1))$$

Prop: $\Delta^n \cong \bigcup_{r+s=n} \Delta^r \times \Delta^s$.

$n=2$:



Boundary orientations

The standard orientation of Δ^n

induces $(-1)^j \times$ the standard orientation on its j^{th} face.

Call $dt_1 \wedge \dots \wedge dt_n$ the "natural orientation" of the time ordered Δ^n. Since the natural orientation of Δ^n is $(-1)^n$ times the standard orientation, we see that the natural orientation of Δ^n induces $(-1)^{j+1}$ times the natural orientation on its j^{th} face.

This can be proved directly: $\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_{j+1}}$ is an outward normal to the j^{th} face $t_j = t_{j+1}$. So the induced orientation on the j^{th} face is

$$- \frac{\partial}{\partial t_{j+1}} \lrcorner \, dt_1 \wedge \dots \wedge dt_n$$

$$= (-1)^{j+1} dt_1 \wedge \dots \wedge \widehat{dt_{j+1}} \wedge \dots \wedge dt_n$$