

2016.11.14

NOTES ON LIMIT MHS AND REGULARIZED PERIODS

§1 Canonical extensions

Suppose that

$$\mathbb{V} \rightarrow \mathbb{D}^*$$

is a local system of finite dimensional complex vector spaces over a punctured disk. By rescaling the holomorphic coordinate, we may (and will) assume that $1 \in \mathbb{D}^*$.

Denote the fiber of \mathbb{V} over $t \in \mathbb{D}^*$ by V_t . Let

$$h_t : V_t \rightarrow V_t$$

be the monodromy operator. The local system is determined by any one

2.

of these. In particular, it is determined by

$$h := h_1 \in \text{Aut } V_1.$$

Let U be a contractible neighborhood of $1 \in \mathbb{D}^*$,
 $g_t : V_1 \rightarrow V_t, \quad t \in U$
be the isomorphism, satisfying $g_1 = id$,
given by parallel translation. (This can be continued to a multi-valued section of $\text{Aut } \mathbb{V}$.) Then

$$(1) \quad h_t = g_t \circ h \circ g_t^{-1}$$

(2) $h =$ result of "analytically continuing" h_t around the unit circle.

Let $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{C}} \mathbb{D}^*$ be the corresponding

3.

holomorphic vector bundle. Denote its canonical flat connection by ∇ .

Choose a logarithm of h and

$$\text{set } N = \frac{t}{2\pi} \cdot \log h.$$

Remark:

(1) Deligne chooses H such that its eigenvalues λ satisfy $0 \leq \operatorname{Re}(\lambda) < 1$.

Such a choice will give rise to Deligne's canonical extension of N to D .

(2) In our case, h is unipotent. The canonical choice of $\log h$ is

$$\sum_{k=1}^{\infty} (-1)^{k+1} (h^{-1})/k.$$

4.

This sum is finite as h^{-1} is nilpotent, as is N .

Set

$$N_t = g_t H g_t^{-1} \in \operatorname{End} V_t.$$

This satisfies

$$e^{2\pi i N_t} = h_t.$$

Deligne trivializations: Suppose $v \in V_1$.

Let $v(t)$, $t \in U$, be the flat section of ∇ over U satisfying $v(1) = v$. Note that $v(t) = g_t v$.

Set

$$g(t) = g_t t^{-N} v \in V_t.$$

$$\text{Since } N_t = g_t H g_t^{-1},$$

$$g(t) = t^{-N_t} v(t).$$

This can be analytically continued to a possibly multi-valued section of

5.

\mathcal{V} . Here

$$t^A := e^{A \log t}.$$

When analytically continued around the unit circle it becomes

$$e^{A(\log t + 2\pi i)} = e^{\text{const}_t A}.$$

So when $g(t)$ is analytically continued around the unit circle it becomes

$$g_{t+h} e^{-imh} t^{-h} v$$

$$= g_t t^{-h} v \\ = g(t)$$

This proves:

Prop: for each $v \in V_1$, the section

$$g(t) = g_t t^{-h} v \quad t \in D^*$$

of \mathcal{V} is single valued. \square

6.

Trivialize \mathcal{V} over D^* by

$$\begin{aligned} D_x^* V_1 &\rightarrow \mathcal{V} \\ (t, v) &\mapsto g_t t^{-h} v \end{aligned}$$

Prop: The pullback of the connection

on \mathcal{V} to $D_x^* V_1$ is

$$\nabla_v = -Nv \frac{dt}{t},$$

where $N \in V_1$ is identified with

the corresponding constant section of $D_x^* V_1 \rightarrow D^*$.

Proof. Since g_t is a flat section of $\text{Aut } \mathcal{V}$,

$$\nabla(g_t t^{-h} v) = -g_t t^{-h} Nv \frac{dt}{t}$$

This section of $V \otimes \mathcal{L}_{D^*}^1$

7.

corresponds to $-N \cup \frac{dt}{f}$ under
the trivialization above. \square

This trivialization gives an extension
of V to D . Namely, $D \times V_t \rightarrow D$.

The embedding $V \hookrightarrow D \times V_t$ is

$$\begin{array}{ccc} V & \hookrightarrow & D \times V_t \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & D \end{array}$$

given by the identification above.

NB: A priori, this extension depends
on the choice of the holomorphic
coordinate t in D .

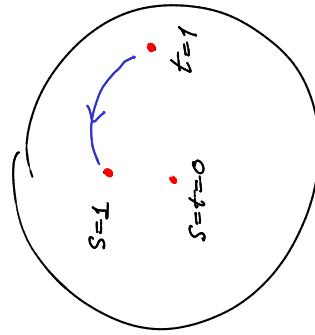
Prop: This extension does not
depend on the choice of holomorphic
coordinate t in D .

Proof: Suppose s is another holomorphic
coordinate in D , centered at o . Write

8.

coordinate in D , centered at o . write
 $s = t \cdot f(t)$

where $f(o) \neq 0$. (Log, $s=t$ is
in D^* . (If not, rescale s .)



Let g_0 be the isomorphism

$$g_0 : V_{t=1} \longrightarrow V_{s=1}$$

given by parallel transport. Set

$$\begin{aligned} h' &= g_0 h g_0^{-1} \in \text{Aut } V_{s=1} \\ N' &= g_0 N g_0^{-1} \in \text{End } V_{s=1}, \end{aligned}$$

9.

$$\text{and } g'_s = g_{t(0)} g_0^{-1}$$

Then one has the trivialization

$$\begin{array}{ccc} V_{t=1} \times \mathbb{D}^* & \xrightarrow{\quad} & V \\ \downarrow & & \downarrow \\ V_{s=1} \times \mathbb{D}^* & \xrightarrow{\quad} & V \end{array}$$

given by $(v, t) \mapsto g'_s s^{-\mu'} v$.

10.

have the diagram

$$\begin{array}{ccc} V_{t=1} \times \mathbb{D}_t^* & \xrightarrow{\quad} & V_{s=1} \times \mathbb{D}_s^* \\ \downarrow (t, v) & \searrow & \downarrow (s, v) \\ g_t t^{-\mu} v & \xrightarrow{\quad} & g'_s s^{-\mu'} v. \end{array}$$

This diagram commutes as

$$(v, t) \mapsto (g_0 \#(t)^N v, s(t))$$

$$\begin{array}{ccc} & & \downarrow \\ (g_t g_0^{-1}) s(t)^{-\mu'} g_0 \#(t)^N v & & \\ = g_t s(t)^N \#(t)^N v & & \\ = g_t t^{-\mu} \#(t)^N \#(t)^N v & & \\ = g_t t^{-\mu} v. & & \square \end{array}$$

So this extension, as a holomorphic vector bundle with connection, depends

Restricting to punctured disk, we

holomorphic vector bundles.

and $\#(0) \neq 0$, this is an isom of

11.

only on the choice of the logarithm N of h .

Remarks:

① Given a choice of N , the extension is determined by a unique isomorphism that is the identity on the subsheaf \mathbb{V} over D^* .

The central fiber V_0 is therefore well defined up to a unique isomorphism.

Because of this, we can write all trivializations of the extension as

$$\sigma' = h \sigma h^{-1} - dh h^{-1} = h N h^{-1} \frac{dt}{t} + dh h^{-1}$$

Since h is analytic and since

$$\begin{array}{ccc} \mathbb{U} & \hookrightarrow & V_0 \times D \\ \downarrow & & \downarrow \\ D^* & \hookrightarrow & D \end{array}$$

so N should be regarded as

12.

an element of $\text{End}(V_0)$.

Any two such trivializations will differ by a map

$$h : D \rightarrow \text{Aut}(V_0)$$

where $h(0) = id_{V_0}$. The corresponding connection on $V_0 \times D$ will be

$$\nabla_V = -h N h^{-1} \frac{dt}{t} + dh h^{-1}$$

So the connection form is

$$\text{Res}_0 \mathcal{R} = \text{Res}_0 \mathcal{R}' = N.$$

13.

- ② for all choices of local parameter t at 0,
- $$\lim_{p \rightarrow 0} t(p)^{-Np} v(p) \quad \left\{ \begin{array}{l} p \in \mathbb{D}^* \\ \text{flat section} \end{array} \right.$$
- exists in V_0 . This induces the isomorphism $V_p \rightarrow V_0$ corresponding to t . Perhaps a better way to describe this is to say that the map

$$H^0(\widehat{\mathbb{D}}^*, \pi^* V) \rightarrow V_0$$

$$v \mapsto \lim_{p \rightarrow 0} t(p)^{-Np} v(p)$$

is an isomorphism for each choice of local parameter t .

- ③ Suppose that

$$V = V_0 \times \mathbb{D}$$

14.

- is a Deligne trivialization of V .
 i.e.: for a given t , $\nabla = d - N \frac{dt}{t}$.
- In this trivialization,
- $$Np = N = \text{Res}_0 \nabla$$
- for all $p \in \mathbb{D}^*$. Every other trivialization of V differs from this one by a map $\varphi: \mathbb{D} \rightarrow \mathcal{L}(V_0)$ where $\varphi(0) = \text{id}$. In this trivialization
- $$Np = \varphi(p) N \varphi(p)^{-1}$$
- indep of trivialization.
- \star Now suppose that N is nilpotent.
- In this case, the extension is Deligne's canonical extension of $V \otimes \mathbb{D}^*$ to \mathbb{D} . It is characterized by the properties:
- $\nabla: V \rightarrow V \otimes \Omega^1_{\mathbb{D}}(\log 0)$

15.

$i\varphi: (\mathcal{V}, \nabla)$ has a regular singular point at o .

(2) $\text{Res}_o \nabla$ is nilpotent.

These conditions imply that the canonical extension of the tensor product of two local systems on \mathbb{D}^* with unipotent monodromy is the tensor product of their canonical extensions. (Similarly with duals.)

Several variables

This works the same way:

Suppose \mathcal{V} is a local system over $(\mathbb{D}^*)^m$. Then

coords (t_1, \dots, t_m)

$$\mathfrak{A}_{\gamma}(\mathbb{D}^*) = \sigma_1^{-\alpha_1} \times \cdots \times \sigma_m^{-\alpha_m}$$

where $\sigma_j = \text{positive loop about}$

$t_j = 1$.

16.

Denote the fiber of \mathcal{V} over $\underline{t} = (t_1, \dots, t_m)$ by $V_{\underline{t}}$. Choose

$N_1, N_2, \dots, N_m \in \text{End } V_{\underline{t}}$ such that

$$(1) \quad e^{2\pi i N_j} = \text{monodromy}$$

$$\sigma_j: V_{\underline{t}} \rightarrow V_{\underline{t}}$$

$$(2) \quad [N_j, N_k] = 0.$$

Rk: When the local monodromy operators $\sigma_j: V_{\underline{t}} \rightarrow V_{\underline{t}}$ are unipotent, the canonical (nilpotent) choices of N_j will work as they are polynomials in the σ_j^{-1} .

Then on trivializes $\mathcal{V} \otimes \mathcal{O}_{(\mathbb{D}^*)^m}$ by

$$V_{\underline{t}} \times (\mathbb{D}^*)^m \rightarrow \mathcal{V} \otimes \mathcal{O}_{(\mathbb{D}^*)^m}$$

$$(v, \underline{t}) \mapsto \sigma_{\underline{t}}^{-N_1} \cdots \sigma_{\underline{t}}^{-N_m} v$$

where $g_{\underline{t}} : V_{\underline{1}} \rightarrow V_{\underline{t}}$
 is obtained by analytically continuing
 the id : $V_{\underline{1}} \rightarrow V_{\underline{1}}$. This gives the
 extension

$$\mathcal{U} := V_{\underline{1}} \times \mathbb{D}^m$$

of $\mathbb{W} \otimes \mathcal{O}_{\mathbb{D}^m}$ to \mathbb{D}^m . The connection
 is given by

$$\nabla v = - \sum_{j=1}^m N_j v \frac{dt_j}{t_j}.$$

As above, one proves that (\mathcal{U}, ∇)
 depends only on the choices of the
 N_j , and not on t . Details are
 left as an exercise.

\mathbb{Q} -structures on V_0

Suppose that \mathbb{W} is a \mathbb{Q} - (or \mathbb{R} -)
 local system over \mathbb{D}^* . As above,
 the choice of a monodromy logarithm

$$N = \frac{i}{2\pi i} \log h,$$

where $h : V_{\underline{1}} \rightarrow V_{\underline{1}}$, determines an
 extension $\mathcal{U} \rightarrow \mathbb{D}$ of $\mathbb{W} \otimes \mathcal{O}_{\mathbb{D}^*}$.

Each choice of local coordinate t
 on \mathbb{D} determines a \mathbb{Q} -structure on
 V_0 as follows: let $v(t)$ be a
 (multivalued) section of \mathcal{V}_0 . Then

$$\lim_{t \rightarrow 0} t^{-N_j} v(t) \in V_0$$

is an element of $V_{0,\alpha}$. More
 precisely, the \mathbb{Q} -structure on V_0 is
 given by the map

19

$$H^0(\widetilde{\mathbb{D}^*}, \pi^* \mathbb{V}_{\mathbb{Q}}) \rightarrow V_0$$

$$\nu(t) \mapsto \lim_{t \rightarrow 0} t^{-N} \nu(t)$$

where $\pi: \widetilde{\mathbb{D}^*} \rightarrow \mathbb{D}^*$

is a universal covering. This is

a \mathbb{Q} -structure because

$$H^0(\widetilde{\mathbb{D}^*}, \pi^* \mathbb{V}_{\mathbb{Q}}) \otimes \mathbb{C} \rightarrow V_0$$

is an isomorphism.

Prop $\frac{N}{2}$: This \mathbb{Q} -structure depends

only on $\frac{d}{dt} t \in T_0 \mathbb{D}$. If

$\frac{d}{ds} s = \lambda \frac{d}{dt} t$, then the \mathbb{Q} -structure
 $V_{\frac{d}{ds} s}$ on V_0 associated to $\frac{d}{ds} s$

$$is \quad V_{\frac{d}{ds} s} = \lambda^{-N} V_{\frac{d}{dt} t}$$

where $\frac{d}{ds} s = \lambda \frac{d}{dt} t$ and

$$N = \text{Res}_0 \nabla.$$

20.

proof: Suppose s is another local parameter at 0. Then

$$s = t \#(t)$$

where $\#(0) \neq 0$. Then

$$ds = \#(s) dt$$

$$\text{and } \frac{\partial}{\partial s} = \#(s)^{-1} \frac{\partial}{\partial t}$$

$$\text{Let } \lambda = \#(s)^{-1}.$$

$$If \quad \nu \in H^0(\mathbb{D}^*, \mathbb{V}_{\mathbb{Q}}), \text{ then}$$

$$\lim_{t \rightarrow 0} s(t)^{-N} \nu(t)$$

$$= \lim_{t \rightarrow 0} \#(t(s))^{-N} t(s)^{-N} \nu(s)$$

$$= \#(0)^{-N} \lim_{t \rightarrow 0} t(t)^{-N} \nu(t)$$

$$= \lambda^N \lim_{t \rightarrow 0} t(t)^{-N} \nu(t).$$

□

21

§ 2 Polarized Variations of Hodge Structure.

This is a summary of Hilbert Schmid's fundamental results about PVHS.

(1)

Variations of Hodge Structure (VHS)

A \mathbb{Q} (or \mathbb{Z} or \mathbb{R}) VHS of weight m over a smooth \mathbb{C} -variety T consists of

(i) a \mathbb{Q} -local system \mathbb{V} over T of finite rank

(2) holomorphic sub-bundles \mathcal{F}^p of

$$\mathbb{V} = \bigoplus_{\alpha} \mathbb{Q}_{\alpha} \quad \text{such that} \\ \cdots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \cdots$$

and

$$(\mathbb{V}_t, \alpha, (\mathbb{V}_t, c, F^\bullet))$$

is a hodge structure of weight m

for all $t \in T$.

(3) The canonical flat connection ∇ on \mathbb{V} satisfies Griffiths transversality

$$\nabla: \mathcal{F}^p \rightarrow \mathcal{F}^{p-1} \otimes \Omega_T^1$$

Example: If $f: X \rightarrow T$ is a

complex analytic family of compact Kähler manifolds, then

$$\mathbb{V} := R^m f_* \mathbb{Q}$$

is a VHS of weight m over T .

Polarized VHS:

A polarization of a Hodge structure $V = (V_\alpha, (V_\alpha, F^\bullet))$ of weight m is a $(-1)^m$ -symmetric bilinear form

$$\langle , \rangle : V_\alpha \otimes V_\alpha \rightarrow \mathbb{Q}$$

22

23.

Satisfying :

$$(1) \quad \langle V^{p,m-p}, V^{s,m-s} \rangle = 0$$

unless $s=m-p$,

$$(2) \quad V^{p,q} \times V^{p,q} \rightarrow \mathbb{C}$$

$$(v_1, v_2) \mapsto i^{p-q} \langle v_1, \bar{v}_2 \rangle$$

is a positive definite hermitian form for all (p,q) satisfying $p+q=m$.

Example: X = smooth projective variety of dim $\geq m$. Let

$$\omega \in H^2(X; \mathbb{Q})$$

be the class of a projective embedding.

Let

$$V = PH^m(X) \xrightarrow{d=d_{\text{max}}} \ker \{ \text{and } H^m(X) \rightarrow H^{m+2}(X) \}$$

24.

and

$$\langle \xi, \gamma \rangle = (-1)^{\frac{m(m-1)}{2}} \int_X \xi \gamma \omega^m$$

$$\xi, \gamma \in PH^m(X).$$

This is part of the "Hard Lefschetz theorem":

Def: A polarized variation of Hodge structure (PVHS) of weight m over a complex manifold T is a VHS

\mathbb{V} of weight m over T plus a flat innerproduct

$$\langle , \rangle : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{Q}_T$$

whose restriction to each fiber V_t is a polarized HS of weight m .

25.

Example: Suppose that $f: X \rightarrow T$ is a holomorphic family of smooth projective varieties over a complex manifold T .

Cie: locally have embedding

$$X_{t_0} \hookrightarrow P^M \times_U$$

$$\begin{array}{ccc} & / & \\ \downarrow & & \downarrow \\ U & & U \end{array}$$

s.t. the polarizations match on $U \cap V$)

Suppose $\dim X_t = d \geq m$. Let $V \rightarrow T$ be the local system whose fiber over $t \in T$ is $P\text{H}^m(X_t; \mathbb{Q})$. This is a PVHS of weight m over T .

Defn: A HS is polarizable if it has at least one polarization. A VHS

26.

is polarizable if it admits a polarization.
(We can regard a PHS as a PVHS over a point.)

Prop: Suppose that \mathcal{V} is a \mathbb{Q} -PVHS

over T . If A is a sub VHS of \mathcal{V} , then

- (1) the restriction of the polarization of \mathcal{V} to A polarizes A ,

- (2) A^\perp is a sub VHS of \mathcal{V}

- (3) $\mathcal{V} = A \otimes A^\perp$ as PVHS.

pf: Exercise: Use the Riemann-Hodge bilinear relations to show that the restriction of the polarization to A is non-degenerate, etc. \square

over T
COR: The category of polarizable \mathbb{Q} -PVHS is a semi-simple tannakian category.

27.

Remark : If $f: X \rightarrow T$ is a family of smooth projective varieties, then $R^m f_* \mathbb{Q}$ is polarizable. Just use the decomposition

$$H^m(X_t; \mathbb{Q}) \cong R^m f_*(X_t) \oplus \omega \cdot R^{m-2} f_*(X_t)$$

$$\oplus \omega^2 \cdot R^{m-4} f_*(X_t) \oplus \dots$$

when $m \leq \dim X_t$, and Poincaré duality when $m > \dim X_t$.

Since every VHS "coming from geometry" is a sub quotient of such a variation, every VHS of "geometric origin" is polarizable.

Schmid's Theorems :
Local monodromy theorem : If ∇ is a PVHS over D^* , then the base change $D \rightarrow D$, $t \mapsto t^\epsilon$, the local monodromy operator is unipotent.

28.

The monodromy operator $h: V_i \rightarrow V_j$ is quasi-unipotent. That is, there are positive integers e and n such that $(h^e - id)^{mtl} = 0$.

This is due (I think) to Grothendieck in the algebro-geometric case, where $\nabla = R^m f_* \mathbb{Q}$

- to Landman in the Kähler case?
- to Artin and Borel (and Schmid?) in the Pütte case.

Significance: it implies that, after the base change $D \rightarrow D$, $t \mapsto t^\epsilon$, the local monodromy operator is unipotent.

Nilpotent orbit Theorem (Schmid).

If $V \rightarrow D^*$ is a PVHS with unipotent monodromy then the Hodge sub-bundles \mathcal{F}^p of $V \otimes \mathbb{C}$

extend to holomorphic sub-bundles of the canonical extension $V \rightarrow D^{**}$ of V to D . In particular, they cut out the "limit Hodge filtration" on V_0 , the fiber of V over $t=0$.

Remark: The nilpotent orbit theorem also holds in the several variable case, provided that one has unipotent monodromy operators. (Cattani - Kaplan - Schmid)

We now have two of the three ingredients we need for the limit HHS on V_{tot} :

- (1) the \mathbb{Q} -structure

$$H^0(D^*, V_{\mathbb{Q}}) \rightarrow V_0$$

$$\mathcal{V} \mapsto \lim_{t \rightarrow 0} t^{-\text{rk}(D)}$$

which depends only on D .

- (2) The limit Hodge filtration on V_0 we still need a weight filtration.

Digression: monodromy weight filtration of a nilpotent operator:

Suppose V is a finite dimensional vector space over a field of char 0 and that $N: V \rightarrow V$ is nilpotent. Then there is a unique filtration

31.

$$0 = W_{r-1} \subseteq W_r \subseteq \dots \subseteq W_r = V$$

such that

$$(1) \quad N W_j \subseteq W_{j-2}$$

$$(2) \quad N^k : Gr_k^W V \rightarrow Gr_k^W V$$

is an isomorphism for all $k \geq 0$.

Existence: Since N is nilpotent, it

can be put in Jordan canonical form over the field. This reduces the

existence proof to the case where N has a single Jordan block

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis

$$e_0, e_1, \dots, e_d$$

32.

$$e_0 \quad \left\{ \begin{array}{l} N e_j = e_{j-1} \\ N e_0 = 0 \end{array} \right. \quad j > 0$$

$$d = 2r : W_{r+1} = W_r = \text{span}\{e_0, \dots, e_{2r}\}$$

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet \\ e_0 & e_1 & e_r & e_{2r} \\ W_{-2r} & W_0 & W_{2r} \end{matrix}$$

W_r

$$d = 2r+1 : W_{r+2} = W_{r+1} = \text{span}\{e_0, \dots, e_{2r+1}\}$$

$$\begin{matrix} \bullet & \bullet & \dots & \bullet & \bullet & \dots & \bullet \\ e_0 & e_1 & e_r & e_{r+1} & e_{2r+1} \\ W_{-2r+1} & W_{-1} & W_1 & W_{2r+2} \end{matrix}$$

Uniqueness: exercise.

SLS-orbit theorem (Schmid)

If $\mathbb{V} \rightarrow D^*$ is a PVHS of weight m with unipotent monodromy, then

$$(V_{\partial/t}, M, (V_0, F))$$

is a HHS, where

$$(1) \quad V_{\partial/t} = \text{image of}$$

$$H^0(\tilde{D}^*, \mathbb{V}_{\mathbb{Q}}) \rightarrow V_0$$

$$\text{or } t \mapsto \lim_{t \rightarrow 0} t^{-M_0(t)}$$

$$(2) \quad M_0 = \text{monodromy weight}$$

filtration shuffled by m
(so that it is symmetric about m).

$$M_j = W(N)_{j-m}$$

(3) F^\bullet is the limit Hodge filtration

33. 34.

Moreover

$$N : V_{\partial/t} \rightarrow V_{\partial/t}(-x)$$

is a morphism of HHS. Equivalently,
 $N \in \text{Hom}_{\text{HHS}}(\mathcal{Q}(1), \text{End}(V_{\partial/t}))$.

Remark: There is a geometric construction
of the canonical extension of \mathbb{V} to
 D and of the extended Hodge bundles.
It is due to Steenbrink. More on
that later, in the section on periods.

Several variables: Schmid's theorems
extend to the several variable case.
See the papers of Cattani-Kaplan-Schmid.

35.

§3 Limits of mixed Hodge structures
 A variation of mixed Hodge structure (VHHS) over a complex manifold T consists of

- (1) a \mathbb{Q} -local system \mathbb{V} of finite rank over T and a filtration $0 \subseteq \mathbb{W}_0 \subseteq \dots \subseteq \mathbb{W}_{r-1} \subseteq \mathbb{W}_r \subseteq \dots \subseteq \mathbb{W}_p = \mathbb{V}$ of \mathbb{V} by \mathbb{Q} -local systems (2) holomorphic sub-bundles \mathcal{F}^p ($p \in \mathbb{Z}$) of \mathbb{V} := $\mathbb{V} \otimes_{\mathbb{Q}} \mathbb{C}_T$

satisfying Griffiths transversality

$$\nabla: \mathcal{F}^p \rightarrow \mathcal{F}^{p-1} \otimes \Omega_T^1.$$

for each $t \in T$, the fiber V_t of \mathbb{V} should be a MHS with weight filtration cut out by \mathbb{W}_t and Hodge

36.

Filtration by \mathcal{F} :

A VHHS is graded polarizable if each $\text{Gr}_m \mathbb{V}$ is a polarizable variation of HS.

Examples

- (1) (Steenbrink – Zucker, Guillen – Navarro – Puerto). If $X \rightarrow T$ is a topologically locally trivial family of algebraic varieties (possibly singular, possibly non-compact)

then each $R^m \pi_* \mathbb{Q}$

is a VHS over T . (More on this later.)

- (2) (Hain) If $X \downarrow_T$ or

37.

is a locally topologically trivial family of smooth varieties, then the local system whose fiber over t is

$$\mathcal{O}(\pi_{\text{can}}(X_t, \sigma \cdot \epsilon_t))$$

is a direct limit of VMHS. (More later.)

(3) (Hain) If T is smooth and A is a PVHS over T , then the local system over T whose fiber over $t \in T$ is

$$\mathcal{O}(\pi_{\text{rel}}^{\text{rel}}(T, t))$$

is a VMHS. Here $\pi_{\text{rel}}(T, t)$ is the completion of $\pi_1(T, t)$ with the monodromy rep

$$\pi_1(T, t) \rightarrow \text{Aut } V_t.$$

38.

Admissible VMHS :

- (1) local behaviour : Suppose that $\mathbb{V} \rightarrow D^*$ is a VMHS. Assume that each $\text{Gr}_m \mathbb{V}$ is a polarizable VHS.
After a base change, we may assume that the monodromy is unipotent. Denote the canonical extension of \mathbb{V} to D by \mathbb{W} . It is filtered by w .
- Def: \mathbb{W} is admissible at 0 if
- (1) the Hodge bundles \mathbb{W}^p extend to holomorphic sub-bundles of \mathbb{V}
 - (2) there is a filtration \mathbb{N}_+ of V_0 satisfying
- (a) $N \mathbb{W}_r \subseteq \mathbb{W}_{r+2}$ $N = \text{less. P}$
- (b) the filtration \mathbb{W}_n induced

39

on $\text{Gr}_m^W V_0$ is the monodromy weight filtration of

$$\eta: \text{Gr}_m^W V_0 \rightarrow \text{Gr}_m^W V_0$$

shifted by m . (It is called the relative weight filtration of the nilpotent endomorphism of the filtered vectorspace (V, W) .)

$$(3) \quad \left(V_{\text{reg}}, \eta, (V_0, F^\bullet) \right)$$

as
usual

$$\left\{ \begin{array}{l} \text{im } H^0(\tilde{\Omega}^*, V_\alpha) \rightarrow V_0 \\ \parallel \\ \text{im } H^0(\tilde{\Omega}^*, V_\alpha) \end{array} \right. \rightarrow V_0$$

$$v \mapsto \lim_{t \rightarrow 0} t^{-\mu_{\text{reg}}(t)}$$

Have SES $\text{Res} \rightarrow H_0(\tilde{\Omega}^*)$

$$0 \rightarrow H^1(E - \{p, q\}, \omega) \quad p \neq q$$

$$\text{each } (W_m V_{\text{reg}}, \eta, (W_m V_0, F^\bullet))$$

We say that the MHS on V_{reg} is filtered by W .

40.

Remarks: Unlike the case of PVHS

- (1) the existence of the limit MHS is not automatic,
- (2) for the generic nilpotent endomorphism $\eta: (V, W) \supset$ of a filtered vector space, there is no relative weight filtration.

Example of nilpotent MHS (V, W) 2

with no relative weight filtration.

$$V = H^1(E - \{p, q\}, \omega) \quad E = \text{elliptic curve}$$

$\text{Res} \rightarrow H_0(\tilde{\Omega}^*)$

$$0 \rightarrow H^2(E) \rightarrow V \rightarrow Q \rightarrow$$

Define W as usual :

$$\begin{array}{c} 0 \subseteq H^1(E) \subseteq V \\ \parallel \quad \parallel \\ W_0 \quad W_1 \quad W_2 \end{array}$$

41.

Define N by :

- (a) $N/H^1(\mathcal{E}) = 0$
- (b) $Nx \in H^1(\mathcal{E})$, $\forall x \neq 0$
where $x \notin H^1(\mathcal{E})$.

Then the shifted monodromy filtration
on the $Gr_m V$ are

$$\begin{aligned} 0 &= M_1 \subseteq M_2 = Gr_2^w V \\ \text{et} \quad 0 &= M_0 \subseteq M_1 = Gr_1^w V \end{aligned}$$

as $Gr_w N = 0$. If there were a
relative weight filtration $H.$ on V , it
would satisfy

$$V = M_2 V \quad \text{and} \quad M_0 V = 0.$$

This would imply that $N \equiv 0$ as

$$NV = NM_0 \subseteq M_2.$$

Since $N \neq 0$, there is no relative
weight filtration.

42.

Definition: Suppose that T is a

smooth projective curve and that Σ is
a finite subtret. A variation of HHS
 ∇ over $T' = T - \Sigma$ is admissible if

- (1) it is graded polarizable;
- (2) it is admissible at each $t \in \Sigma$.

A VHHS ∇ over a smooth variety
 \mathcal{U} is admissible if its restriction to
each curve $T' \rightarrow \mathcal{U}$ is admissible.

NB: If the local monodromy is not
locally unipotent in the curve case,
kill the eigenvalues of the local monod
operators by passing to a finite branched
covering of T .

Remark: Steenbrink + Zucker defined
admissible VHHS over a curve; Kashiwara
extended this to higher dimensional bases.

43

Examples :

(1) Steenbrink-Zucker :

$$\mathbb{V} = R^mf_* \mathbb{Q}$$

where $f: X \rightarrow T'$ is a topologically locally trivial family of smooth varieties over an algebraic curve T'

(2) Guillén-Navarro-Pérez:

$$\mathbb{V} = R^mf_* \mathbb{Q}$$

where $X \rightarrow T'$ is a topologically loc trivial family of varieties - not necessarily smooth or complete.

(3) Main :

$$\begin{matrix} X & V_t = W_m \mathcal{O}(X_{t_1}, \dots, t_n) \\ \downarrow & \downarrow \\ T' & \end{matrix}$$

where $X \rightarrow T'$ is a topologically locally trivial family of smooth varieties over a curve.

44.

§ 4 Asymptotics of periods

Suppose we have dual local systems \mathbb{S}

$$\begin{matrix} \mathbb{V} & \mathbb{V} \\ \downarrow & \downarrow \\ \mathbb{D}^* & \mathbb{D}^* \end{matrix}$$

with unipotent monodromy. Suppose

$$\begin{matrix} \mathbb{U} & \mathbb{U} \\ \downarrow & \downarrow \\ \mathbb{D} & \mathbb{D} \end{matrix}$$

be their canonical extensions to \mathbb{D} . These have nilpotent residue at 0:

$$\begin{matrix} \text{Res}_0 \mathbb{D} = -N & \in \text{End } \mathbb{V} \\ \text{Res}_0 \mathbb{U} = N & \in \text{End } \mathbb{V} \end{matrix}$$

fix a parameter t in \mathbb{D} . Suppose that $\gamma(t) \in H^0(\mathbb{D}^*, \pi^* \mathbb{V})$ is a flat multivalued section of \mathbb{V} and that

45.

$\omega(t)$ is a holomorphic section of \mathcal{V} .

Prop N: The period

$$\int_{\mathcal{D}} \omega(t) := \langle \gamma(t), \omega(t) \rangle$$

is a polynomial $\sum_{j=0}^d \alpha_j(t) (\log t)^j$

in $\log t$ with coefficients in $\mathcal{O}(\mathbb{D})$.

Moreover "canonical regularization"

$$\lim_{t \rightarrow 0} \langle t^{-N} \gamma(t), \omega(t) \rangle = \varphi(0).$$

Proof. Suppose that $\{\gamma_1(t), \dots, \gamma_m(t)\}$ is a basis of $H^0(\mathbb{D}, \mathcal{V})$ and that

$$\{\omega_1(t), \dots, \omega_m(t)\}$$

is a framing of \mathcal{V} over \mathbb{D} . Let

$$\varphi_j(t) = t^{-N} \gamma_j(t) \quad j=1, \dots, m$$

be a Deligne framing of \mathcal{V} over \mathbb{D} and $\{\varphi_1, \dots, \varphi_m\}$ be the dual

46.

framing of \mathcal{V} . That is

$$\langle \varphi_j, \varphi_k \rangle = \delta_{jk}.$$

Identity V_0 with $\bigoplus_{j=1}^m \mathbb{C} \varphi_j$. Recall that $N = \text{Res}_0 \nabla$ acts on V_0 . The flat sections of \mathcal{V} , with respect to this framing, are \mathbb{C} -linear combinations of

$$t^N \varphi_j, \quad j=1, \dots, m.$$

Let $A \in M_m(\mathbb{C})$ be the matrix of N w.r.t. the basis $\{\varphi_1, \dots, \varphi_m\}$:

$$(N \varphi_1, \dots, N \varphi_m) = (g_1, \dots, g_m) A$$

A is nilpotent. We can write

$$(\omega_1(t), \dots, \omega_m(t)) = (\varphi_1, \dots, \varphi_m) B(t)$$

where $B \in \mathcal{L}_m(\mathcal{O}(\mathbb{D}))$. So

$$\left\langle \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_m(t) \end{pmatrix}, (\omega_1(t), \dots, \omega_m(t)) \right\rangle$$

47.

$$\begin{aligned}
 &= \left\langle \begin{pmatrix} t^N g_1 \\ \vdots \\ t^N g_m \end{pmatrix}, (g_1, \dots, g_m) \right\rangle \\
 &\quad \text{identity matrix} \\
 &= t^{+A^T} \left\langle \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}, (g_1, \dots, g_m) \right\rangle \mathcal{F}(t)
 \end{aligned}$$

$$\begin{aligned}
 &= t^{+A^T} \mathcal{F}(t). \\
 &= e^{A^T \log t} \mathcal{F}(t) \\
 &\in M_m(\mathcal{O}(\mathbb{D})[\log t]).
 \end{aligned}$$

Finally, write $\gamma = c_1 \gamma_1(t) + \dots + c_m \gamma_m(t)$
and $w(t) = f_1(t) w_1(t) + \dots + f_m(t) w_m(t)$
to deduce the first statement.

For the last statement, note that
 $t^{-N} \gamma(t) = \sum_{j=1}^m c_j \cdot t^{-N} \gamma_j(t) = \sum_{j=1}^m g_j \varphi_j$

The limit is then $\mathcal{F}(0) \begin{pmatrix} f_1(0) \\ \vdots \\ f_m(0) \end{pmatrix}$. \square

This is the const term \rightarrow

48.

In practice, one is given a trivialization $\mathcal{U} = V_0 \times D$ where it is easy to compute $N = \text{Res}_0 \cap \text{End } V_0$, but not so easy to compute the N_t . The following provided an alternative method for regularizing periods.

Prop 2: $I \# v : D^* \rightarrow V_0$ is a flat section of $\mathcal{U} = V_0 \times D$, then "canonical regularization" "monstar board regularization".

$$\lim_{t \rightarrow 0} t^{-N_t} v(t) = \lim_{t \rightarrow 0} t^{-N} v(t).$$

For t in any angular sector.

Remark. The local sections $t^{-N_v(t)}$ of $\mathcal{U} = V_0 \times D$ are single-valued on D^* if and only if this is a "designe trivialization",

proof. If $\mathcal{U} = V_0 \times D$ is a designe trivialization of \mathcal{U} , then every flat

49.

sector is of the form $t^N v$, where

$v \in V_0$. In this case, $N_t = N$ for all $t \in D^*$ so that $t^{-N} v(t) = v$ and

$$\lim_{t \rightarrow 0} t^{-N} v(t) = \lim_{t \rightarrow 0} t^{-N} v(t) = v.$$

In general, the trivialization differs from a Deligne trivialization by a holomorphic

map $g: D \rightarrow GL(V_0)$ with $g(0) = id$.

In this case

$$N_t = g(t) N g(t)^{-1}.$$

so, in any angular sector $\theta < \arg t < \theta_T$,

$$\|t^{N_t} - t^N\| = \|g(t) e^{N \log t} g(t)^{-1} - e^{N \log t}\|$$

holo on D , van at 0

$$\leq \sum_{k=0}^K \|g(t) N^k g(t)^{-1} - N^k\| \frac{|\log t|^k}{k!}$$

$$\leq C |t| |\log t|^K$$

$$\rightarrow 0 \quad \text{as } |t| \rightarrow 0.$$

Remark: As we shall see, this gives Brown's "mortar board" regularization.

50.

§ 5 Regularizing periods

Suppose that K is a number field.

Fix an embedding $K \hookrightarrow \mathbb{C}$. Suppose

that

$$\begin{array}{c} \overline{X} \\ \downarrow \\ T \end{array}$$

\downarrow a curve

is a proper family of smooth projective varieties defined over K . Suppose that

$D \subseteq \overline{X}$ is a DNC, each of whose

components is transverse to the fibers

of f . Suppose that $\Sigma \subseteq T(K)$ is finite.

Set

$$E = f^{-1}(\Sigma).$$

assume for simplicity that $E = E^{\text{reg}}$

Suppose that $D \subseteq E$ is a DNC and that the restriction $f': X' \rightarrow T'$, where

$$X' := \overline{X} - (D \cup E) \quad \text{and} \quad T' = T - \Sigma,$$

is locally topologically trivial.

51.

Results of Steenbrink and Steenbrink
Zucker imply that the connection

$$\nabla_K^{\text{DR}} = \left(R_{\frac{d}{dt}} \mathcal{L}_{\frac{d}{dt}} \log(D_{0,\mathbb{C}}), \nabla \right)$$

$$|$$

$$T$$

satisfies

- (1) it is a vector bundle defined over \mathbb{K}
- (2) ∇ has regular singular points at each $t \in \Sigma$ with nilpotent residue
- (3) after tensoring with \mathbb{C} , it is isomorphic to the canonical extension of

$$H^m(X'_t) \subseteq R^m \mathcal{F}_t^* \mathbb{C}$$

$$| \quad |$$

$$t \in T' = T - \Sigma$$

to T .

52.

So we have

$$(V_K^{\text{DR}}, \nabla)$$

$$| \quad \text{and} \quad V_{\mathbb{Q}}^B$$

$$T/K$$

$$|$$

$$\text{DeRham Betti}$$

and a comparison isomorphism

$$V_{\mathbb{Q}}^B \otimes_{T'} \rightarrow V_K^{\text{DR}} \otimes \mathbb{C} |_{T'}$$

The formulas in the previous section tell us how to regularize this comparison map over $\tilde{v} \in \overline{\rho}(\Sigma)$, where $\rho \in \Sigma$ and $\tilde{v} \neq \bar{o}$.

53.

§ 6 Example .

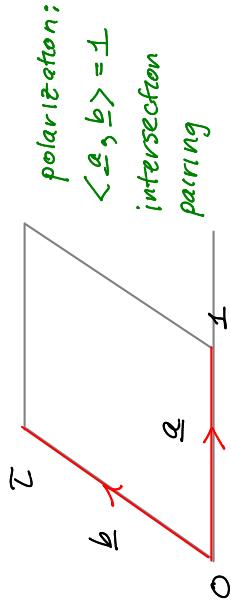
Let \mathcal{E} be the universal elliptic curve and A/\mathcal{H} be the local system

$f|_{\mathcal{M}_{1,1}}$

and the \mathcal{D} disk D^* :

$$\begin{array}{ccccc} \mathcal{H}_{\mathcal{D}} & \longrightarrow & \mathcal{H}_{D^*} & \longrightarrow & \mathcal{H} \\ | & & | & & | \\ \mathcal{D} & \longrightarrow & D^* & \longrightarrow & \mathcal{M}_{1,1} \\ \tau \mapsto g = e^{2\pi i \tau} & & & & \end{array}$$

Let $\underline{\alpha}$ and \underline{b} be the standard basis of $H_1(E_\tau)$, where $E_\tau = \mathbb{C}/\mathbb{Z}\underline{\alpha} + \mathbb{Z}\underline{b}$.



54.

Let $\underline{\alpha}, \underline{b}$ be the dual basis of $H^1(E_\tau)$. The Poincaré duality isomorphism

$$H_1(E_\tau) \rightarrow H^1(E_\tau)$$

takes $\underline{\alpha}$ to \underline{b}^\vee and \underline{b} to $-\underline{\alpha}^\vee$. The pull back of \mathcal{H} to \mathcal{D} is

$$\mathcal{H}_{\mathcal{D}} = \mathcal{D} \times (\mathbb{Z}\underline{\alpha} \oplus \mathbb{Z}\underline{b})$$

The left $SL_2(\mathbb{Z})$ -action is

$$\gamma: (\underline{\alpha}, -\underline{b}) \mapsto (\underline{\alpha}, -\underline{b}) \circ$$

$$\text{Set } \mathcal{H} = \mathcal{H} \otimes \mathcal{O}_{\mathcal{M}_{1,1}} \text{ etc.}$$

Then

$$\begin{aligned} \underline{\omega}(\tau) &= 2\pi i (\underline{\alpha} + \tau \underline{b}^\vee) \\ &= -2\pi i \underline{b} + \log g \underline{\alpha}, \end{aligned}$$

the class of $2\pi i \underline{d}\underline{z}$ in $H^1(E_\tau; \mathbb{C})$, is a section of \mathcal{H} over \mathcal{D} and \mathcal{D}^* .

Note that

$$\mathcal{H}^{\otimes k} = \mathcal{O}_{\mathcal{D}^k} \otimes \mathcal{O}_{\mathcal{M}_{1,1}}^{\otimes k}$$

55.

Extend it to D as
 $H_D = \mathcal{O}_D \underline{\alpha} \oplus \mathcal{O}_D \underline{\omega}$

I claim that this is the canonical extension of H_1 to D^* . To see this, note that the connection satisfies

$$\nabla \underline{\alpha} = \nabla \underline{\omega} = 0.$$

so that $\nabla \underline{\alpha} = 0$ and $\nabla \underline{\omega} = \underline{\alpha} \frac{dg}{g}$.

Thus, in this extension,
 $\nabla = d + \underline{\alpha} \frac{\partial}{\partial \underline{\omega}} \frac{dg}{g}$.

So the extended connection has a regular singular point at $\underline{\omega}=0$. The residue at $\underline{\omega}=0$ is the nilpotent endomorphism

$$H = -\underline{\alpha} \frac{\partial}{\partial \underline{\omega}}$$

It follows that

- (1) $\mathcal{O}_D \underline{\alpha} \oplus \mathcal{O}_D \underline{\omega}$ is the canonical extension of H_1 to D . $\left| \begin{array}{l} SO \\ N_{\underline{\alpha}} = N \end{array} \right.$
- (2) $\{\underline{\alpha}, \underline{\omega}\}$ is a Deligne frame. all $g \in \mathbb{D}$

56.

The Hodge bundle \mathcal{F}^1 is $\mathcal{O}_D \underline{\omega}$. It clearly extends to

$V_0 = \mathbb{C} \underline{\alpha} \oplus \mathbb{C} \underline{\omega}$
flat
The $\mathbb{Z}/2$ (even \mathbb{Z}) sections are spanned by

$\underline{\alpha}$ and

$$\underline{b}(g) = -\frac{1}{2\pi i} \cdot \underline{\omega}(g) + \frac{\log g}{2\pi i} \underline{\alpha}$$

Since $N^2=0$,

$$\underline{g}^{-N} = e^{-N \log g} = 1 + \log g \frac{\alpha}{\partial \underline{\omega}}$$

and $\underline{g}^{-N} \underline{\alpha} = \underline{\alpha}$

$$2\pi i \underline{g}^{-N} \underline{b}(g) = -\underline{\omega}(g) + \log g \underline{\alpha}$$

$$-\log g \underline{\alpha} \\ = -\underline{\omega}(g).$$

$$\text{So } V_{g/g} = \mathbb{Z} \underline{\alpha} \oplus \frac{1}{2\pi i} \mathbb{Z} \underline{\omega}$$

It follows that

57.

$$\begin{aligned} V_{\partial/\partial}^B &= \mathbb{Z}\underline{\alpha} \oplus \mathbb{Z}\underline{b} \\ V_{\partial/\partial}^{DR} &= \mathbb{C}\underline{\alpha} \oplus \mathbb{C}\underline{\omega}, \quad F^*V_{\partial/\partial}^{DR} = \mathbb{C}\underline{\omega} \end{aligned}$$

and $-x\underline{\alpha} \underline{b} \mapsto \underline{\omega}$

$$\therefore V_{\partial/\partial}^B = \mathbb{Z}(0) \oplus \mathbb{Z}(-x).$$

This extends (by linear algebra) to

 $S^m H$: over \mathbb{D} :

$$S^m H = \mathcal{O}_{\mathbb{D}} \underline{\alpha}^m \oplus \mathcal{O}_{\mathbb{D}} \underline{\alpha}^m \underline{\omega}$$

$$\oplus \dots \oplus \mathcal{O}_{\mathbb{D}} \underline{\omega}^m.$$

$$\text{exp } F^p H = \bigoplus \mathcal{O}_{\mathbb{D}} \underline{\alpha}^m \underline{\omega}^s.$$

$$\nabla = d + \underline{\alpha} \frac{\partial}{\partial \underline{w}} \frac{dq}{q}$$

This satisfies Griffiths transversality

$$\nabla q^p \in \mathcal{F}^p \otimes \mathcal{O}_{\mathbb{D}}(\log 0)$$

It is polarized by \langle , \rangle , extended

58.

to $S^m H$ by linear algebra. The fiber over ∂/∂ is

$$V_{\partial/\partial}^B = \mathbb{Z}\underline{\alpha}^m \oplus \mathbb{Z}\underline{\alpha}^{m-1} \underline{b} \oplus \dots \oplus \mathbb{Z}\underline{b}^m$$

$$V^{DR} = \mathbb{C}\underline{\alpha}^m \oplus \mathbb{C}\underline{\alpha}^{m-1} \underline{\omega} \oplus \dots \oplus \mathbb{C}\underline{\omega}^m.$$

The limit HHS is

$$V_{\partial/\partial} = \mathbb{Z}(0) \oplus \mathbb{Z}(-1) \oplus \dots \oplus \mathbb{Z}(-m).$$

Fiber of H^1 over ∂/∂ : use the coordinate $t = \underline{\alpha}/\underline{\beta}$. Then $\partial_t t = 1/\underline{\beta}$.

$$(\underline{\alpha}/\underline{\lambda})^{-N} = 1 + \log \underline{\beta} \not\subseteq \mathbb{Z}\underline{w} - \log \underline{\lambda} \not\subseteq \mathbb{Z}\underline{w}.$$

$$\text{so } (\underline{\alpha}/\underline{\lambda})^{-N} \underline{\alpha} = \underline{\alpha} \text{ and}$$

$$2\pi i (\underline{\alpha}/\underline{\lambda})^{-N} \underline{b} = -\underline{w} + \log \underline{\lambda} \underline{a}$$

So $V_{\partial/\partial}$ is the element 1 of
 $\text{Ext}_{HHS}^1(\mathbb{Z}(1), \mathbb{Z}) \cong \mathbb{C}^*$.

59.

§ 7 Regularizing Iterated Integrals

I'll illustrate this with a few examples.

Suppose that X is a smooth projective curve and that D is an effective reduced divisor in X . Suppose that

$$(1) \quad \omega_1, \dots, \omega_r \in H^0(X, \Omega_X^1(\log D))$$

$$(2) \quad P \in D, \quad \mathbf{c} \in X - D.$$

$$(3) \quad \vec{v} \in T_P X, \quad \vec{v} \neq 0.$$

We want to compute the regularized integral

$$\int_{\mathcal{I}} (\omega_1 \cdot \dots \cdot \omega_r)$$

$$\mathcal{I} = (x_1, \dots, x_n)$$

To this end, let

$$A = \mathbb{C}\langle x_1, \dots, x_n \rangle / \overline{I}^{(r+1)}$$

Embed A in $\text{End}(A)$ by left multiplication.

$$\text{Set } \underline{\Omega}_0 = \sum_{j=1}^n \text{Res}_{\omega_j} \frac{dt}{t} x_j.$$

60.

This defines a flat connection on

$A \times X$

/

X

with regular singular points along D .

Its residue at P is left multiplicative by

$$\text{Res}_{\Omega} = \sum_{j=1}^n \text{Res}_{\omega_j} x_j.$$

This is nilpotent in A . Set

$$T(x) = (1 + f(x) + f(x)^2 + \dots + f(x)^Q)$$

Then

$dT = -\Omega T$ } by formula of
d (not int.)

Choose a holomorphic coordinate t at P such that $\vec{v} = \partial/\partial t$. Set

$$\Omega_0 = \sum_{j=1}^n \text{Res}_{\omega_j} \frac{dt}{t} x_j.$$

61. You can think of this as a log form
 on $T_p X$ with $\text{Res}_0 \Omega_0 = \text{Res}_p \Omega$.
 Francis Brown denotes

$\text{Res}_p \omega_j \cdot \frac{dt}{t}$ by $\bar{\omega}_j$.

$$\text{Since } \int_1^t \left(\frac{at}{t} \right) \dots \left(\frac{dt}{t} \right) = \frac{1}{m!} (\log t)^m$$

$$\left\langle 1 + \int(\alpha_0) + \int(\alpha_0 \log_0) + \dots \right\rangle_1^t = t \text{ Res}_0 \Omega.$$

So

$$t^{-N} T(t) = t \text{ Res}_0 \Omega T(t)$$

The regularized iterated integrals are the coefficients of

$$\lim_{t \rightarrow 0} t \text{ Res}_0 \Omega T(t)$$

and

$$\int_{\partial \Omega}^0 (\omega_1 \dots \omega_r) = \text{coefficient of } x_1 \dots x_r.$$

62. This gives

$$\int_{\partial \Omega}^0 (\omega_1 \dots \omega_r) = \sum_{j=0}^r \int_{\partial \Omega}^t (\bar{\omega}_1 \dots \bar{\omega}_j) \int_{\partial \Omega}^{\bar{\omega}} (\omega_{j+1} \dots \omega_r)$$

Example: (1) $X = P'$, $D = f_0, \alpha_0$, $\Phi \in C^\infty$

$$\vec{V} = \frac{\partial}{\partial z}.$$

$$\int_{\partial \Omega}^0 \frac{dz}{z} = \lim_{t \rightarrow 0} \left(\int_1^t \frac{dz}{z} + \int_t^0 \frac{dt}{z} \right)$$

$$\begin{aligned} &= \lim_{t \rightarrow 0} \int_1^t \frac{dz}{z} = \log 0 \\ &\quad (2) \text{ If } \vec{V} = \lambda \frac{\partial}{\partial z}, \text{ take } t = \pi/\lambda, \\ &\text{Then } \int_{\vec{V}}^0 \frac{dz}{z} = \lim_{t \rightarrow 0} \left(\int_{z=\lambda}^t \frac{dz}{z} + \int_t^0 \frac{dt}{z} \right) \\ &\quad = \log 0 - \log \lambda. \end{aligned}$$

This is the "mortar board" regularization

63.

The Drinfeld Associator. This is the regularization of

$$\langle T, \underline{dch} \rangle$$

where dch is the path in $P^1 - \{\infty, 1\}$

from $\vec{v}_0 := \partial/\partial x \in T_0 P^1$ to

$$\vec{v}_1 := -\partial/\partial x \in T_1 P^1 \xrightarrow[0]{} 1$$

and $T = 1 + \int(\omega) + \int(-\omega) + \dots$

$$\omega = \frac{dx}{x} X_0 + \frac{dx}{1-x} X_1 \quad \left\{ \begin{array}{l} KZ \\ \text{connection} \end{array} \right.$$

$$\mathcal{F} = \lim_{\substack{t \rightarrow 0 \\ s \rightarrow 0}} t^{X_0} \langle T, [t, 1-s] \rangle s^{X_1}.$$

Here the limit is taken with $s, t > 0$.

64.

APPENDIX
Here we show that if

$$V \xrightarrow[1]{D}$$

is a holomorphic vector bundle with flat connection

$$\nabla: V \rightarrow V \otimes \Omega_D^1(\log 0)$$

with residue $N \in \text{End } V_0$ satisfying

#1 no two eigenvalues of N differ by a non-zero integer eg: N is nilpotent.

then

- (1) each h_t is conjugate to $h = e^{2\pi i N}$

- (2) V is the Deligne extension

65.

of ∇ to D associated
 $|$
 D^*
to this choice of $\log t$.

Lemma: If $B \in M_m(\mathbb{C})$, then
the eigenvalues of
 $\text{ad}_B : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$
are $\{\lambda_j - \lambda_k : \lambda_j, \lambda_k \text{ eigenvalues of } B\}$.

Proof: Jordan canonical form implies
that $B = D + E$
where E is nilpotent, D is diagonal
and $[D, E] = 0$. The eigenvalues of
 ad_D are $\lambda_j - \lambda_k$. Since
 $\text{ad}_B = \text{ad}_D + \text{ad}_E$ semi-simple & commute
the eigenvalues of ad_B and ad_D are
the same. \square

66.

Cor: If $c \neq 0$ and
 $c \notin \{\lambda_j - \lambda_k : \lambda_j, \lambda_k \text{ eigenvalues of } B\}$
then the equation
 $cX = -[B, X] + c$
has a unique solution for all X in
 $M_m(\mathbb{C})$.

Proof: The lemma implies that
 $\text{ad}_B - cI_d : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$
is invertible. \square

Proof of the main assertion. Trivialize
 ∇ over a field of 0. (Since D is
Stein, ∇ is, in fact, trivial over D .) The
connection becomes
 $\nabla v = dv - A v \frac{dt}{t}$.
where $v : D^* \rightarrow V_0$. Note that

67.

$A(0) = N$. We need to show that there is a change of gauge $\rho: \mathbb{D} \rightarrow \text{Aut}(V_0)$ such that $\rho(0) = \text{id}_{V_0}$ and the transformed connection

$$(**) \quad \rho A \rho^{-1} \frac{dt}{t} - d\rho \rho^{-1} = N \frac{dt}{t}.$$

That is, we have to solve the equation

$$\rho A - N\rho = t \rho' \quad , \quad \rho(0) = \mathbb{I}$$

for ρ . Expand ρ and A :

$$\rho = \mathbb{I} + \sum_{k=1}^{\infty} \rho_k t^n \quad \text{and end } V_0 \\ A = N + \sum_{k=1}^{\infty} A_k t^n. \quad \text{and end } V_0$$

The equation $(**)$ becomes

$$N \rho_K = \sum_{k=0}^{K-1} \rho_k A_{K-k} + [\rho_K, N]$$

for all $K \geq 1$. The corollary above

68.

and the assumption that no two eigenvalues of N differ by a non-zero integer, imply that we can inductively solve these equations to find a formal solution $\rho(\mathbb{S})$ of

$$\rho A \rho^{-1} \frac{dt}{t} - d\rho \rho^{-1} = N \frac{dt}{t}.$$

Convergence: recall

$$\|A\|^2 = \text{largest eigenvalue of } A^T A.$$

$$\text{So}$$

$$\|A + K id\| = K \|ia + \frac{1}{K} A\| \\ \geq K/2 \quad \left. \begin{array}{l} \\ \\ K \gg 0 \end{array} \right\}$$

We have Id

$$(ad_N + K id) \rho_K = \rho_0 A_K + \dots + \rho_{K-1} A_1$$

$$\therefore \|ad_N + K id\| \| \rho_K \| \leq \| \rho_0 \| \| A_K \| + \dots + \| \rho_{K-1} \| \| A_1 \|$$

69.

So, for K sufficiently large,

$$\rho_K \leq \frac{2}{K} (\alpha_K + \alpha_{K-1}\rho_1 + \dots + \rho_{K-1}\alpha_1)$$

where $\rho_j = \|\beta_j\|$ and $\alpha_j = \|A_j\|$.

By rescaling t , we may assume that

$$\sum_{n=0}^{\infty} A_n t^n$$

converges absolutely on $|t| \leq 2$. This implies that $\alpha_n = \|A_n\| \leq \frac{1}{2}$ when $n \gg 0$.

So

$$\rho_K \leq \frac{1}{K} (\varepsilon + \rho_1 + \dots + \rho_{K-1}) \quad K \gg 0.$$

$$\leq \max \{ \rho_k : 0 \leq k < K \}$$

$\therefore \{ \rho_k : K \geq 0 \}$ is bounded above

and

$$\sum_{n=0}^{\infty} \rho_n t^n$$

converges when $|t| < 1$.