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ITERATED INTEGRALS

Recall that we have the diagram

$$\begin{array}{ccc} \Delta^n \times PM & \xrightarrow{\text{ev}} & M \times M^n \times M \\ \pi \downarrow & \swarrow \text{piecewise smooth paths} & \\ & \text{regarded as a differentiable} & \\ & \text{space} & \\ & PM & \end{array}$$

- ② if ω_j has degree $n_j > 0$, then
 $\int(\omega_1 \dots / \omega_n) \in E^N(PM)$

where

$$N = \sum_{j=1}^n (n_j - 1).$$

- ③ This definition agrees with
 PM
the standard one:

$$\int \omega_1 \dots \omega_r = \int \dots \int_{\alpha \leq t_1 \leq \dots \leq t_r \leq 1} f_1(t_1) \dots f_r(t_r)$$

$$\text{where } \text{ev}((t_1, \dots, t_n), x) = (x_0, x(t_1), \dots, x(t_n), x(1))$$

for $\omega_1, \dots, \omega_r \in E^*(M)$, define

$$\int(\omega_1 \dots / \omega_n) = \pi_{\#} \text{ev}^* \tau \times \omega_1 \times \dots \times \omega_n \times 1$$

where $\tau \omega_j(t) = f_j(t) dt$, when
all ω_j have degree 1.

- ④ If instead, we parameterize paths
on $[a, b]$, where $a < b$, and let

$$\Delta^r = \{t : a \leq t_1 \leq \dots \leq t_r \leq b\}$$

$$\int(\omega_1 \dots / \omega_n) = 0$$

- so we might have well assumed
that each $\omega_j \in E^+(M)$. ← forms of positive degree

$$\begin{aligned} & e: \Delta^r \times PM \rightarrow M^{r+2} \\ & (\underline{t}, x) \mapsto (x(a), x(t_1), \dots, x(t_r), x(b)) \\ & \text{then } \int(\omega_1 \dots / \omega_n) = \pi_{\#} e^*(\tau \times \omega_1 \times \dots \times \omega_n \times 1) \end{aligned}$$

In general, an iterated path integral is an element of $E^*(PM)$ that is a linear combination of

$$\phi_0^* \omega' \wedge f(\omega_1 / \dots / \omega_r) \wedge \phi_1^* \omega''.$$

EXTERIOR DERIVATIVE FORMULA
NOTATION:

$$(1) \quad d(\omega_1 / \dots / \omega_r) :=$$

$$\sum_{j=1}^r (-1)^{(\omega_1 + \dots + \omega_{j-1})} (\omega_1 / \dots / \omega_{j-1} / d\omega_j / \omega_r)$$

This is the \otimes differential. It obeys the Koszul convention, where ω_j has weight $| \omega_j |$.

$$(2) \quad s(\omega_1 / \dots / \omega_r) := \sum_{j=1}^{r-1} (-1)^j (\omega_1 / \dots / \omega_j \wedge \omega_{j+1} / \dots / \omega_r).$$

This also obeys the Koszul convention as \wedge has degree 0.

The formula

$$(-1)^r d f(\omega_1 / \dots / \omega_r)$$

$$\begin{aligned} &= \int d(\omega_1 / \dots / \omega_r) + \int s(\omega_1 / \dots / \omega_r) \\ &\quad + (-1)^r \int (\omega_1 / \dots / \omega_{r-1}) \wedge \phi_1^* \omega_r \\ &\quad + (-1)^{(r-1)} \omega_1 \wedge \int (\omega_2 / \dots / \omega_r). \end{aligned}$$

$$\in E^*(PM).$$

The formula is simpler when one restricts to

$$(a) \quad \rho_{\alpha, M} = \{ r \in PM : r(c) = \alpha \} \\ (b) \quad \rho_{\alpha, b} M = \{ r \in PM : r(0) = \alpha, r'(0) = b \}$$

Eg: In $E^*(\rho_{\alpha, M})$

$$\begin{aligned} (-1)^r d f(\omega_1 / \dots / \omega_r) &= \int d(\omega_1 / \dots / \omega_r) + \int s(\omega_1 / \dots / \omega_r) \\ &\quad + (-1)^r \int (\omega_1 / \dots / \omega_{r-1}) \wedge \phi_1^* \omega_r. \end{aligned}$$

Special case: iterated line integrals.

Here each ω_j has degree 1. In this case

$$d \int (\omega_1 / \dots / \omega_r) = (-1)^r d(\omega_1 / \dots / \omega_r)$$

$$\begin{aligned} & + (-1)^r \int s(\omega_1 / \dots / \omega_r) \\ & + \int (\omega_1 / \dots / \omega_{r-1}) \wedge \phi^* \omega_r \\ & - \phi^* \omega_1 \wedge \int (\omega_2 / \dots / \omega_r). \\ & = \sum_{j=1}^r (-1)^{r+j-1} \int (\omega_1 / \dots / d\omega_j / \dots / \omega_r) \\ & + (-1)^r \sum (-1)^j \int (\omega_1 / \dots / \omega_j \wedge \omega_{j+1} / \dots / \omega_r) \\ & + \int (\omega_1 / \dots / \omega_{r-1}) \wedge \phi^* \omega_r \\ & - \phi^* \omega_1 \wedge \int (\omega_2 / \dots / \omega_r). \end{aligned}$$

Example: Suppose $\omega_1, \dots, \omega_m, \xi \in \mathcal{E}'(n)$

and $\alpha_{jk} \in \mathbb{C}$. If $d\omega_j = 0$, then

$$d \left(\int \sum \alpha_{jk} (\omega_j / \omega_k) + \int (\xi) \right)$$

$$= - \sum_{j,k} \alpha_{jk} \int (\omega_j \wedge \omega_k) - \int (\omega)$$

so if

$$d\xi + \sum_{j,k} \alpha_{jk} \omega_j \wedge \omega_k = 0$$

then $\sum \alpha_{jk} \int (\omega_j / \omega_k) + \int (\xi)$ is closed. (Later we will see that this is "if and only if".)

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Before giving the proof of the exterior derivative, we need to dispense with some preliminary issues ...

Preliminaries :

$$\begin{array}{ccc} \textcircled{1} & \text{The diagram} & \xrightarrow{\quad P^r M \quad} \\ \Delta^n \times PM & \xrightarrow{e} & M \times M^n \times M \\ \pi \downarrow & & \downarrow \text{proj}_{M^n} \\ PM & \xrightarrow{p_0 \times p_1} & M \times M \end{array}$$

commutes:

$$\begin{array}{ccc} ((t_1, \dots, t_n), x) & \mapsto & (x(0), x(t_1), \dots, x(t_n)) \\ \downarrow & & \downarrow \\ x & \longmapsto & (x(0), x(1)) \end{array}$$

This implies that

$$\begin{aligned} e^*(\omega_0 \times \omega_1 \times \dots \times \omega_{n+1}) \\ = \pi^* \rho_0^* \omega_0 \wedge e^*(1 \times \omega_1 \times \dots \times \omega_{n+1}) \wedge \pi^* \rho_1^* \omega_{n+1} \end{aligned}$$

- ② Because of the difference between the "standard" and "time ordered" orientations of Δ^n , we will use the notation $\pi^{(r)} : \Delta^r \times PM \rightarrow PM$ for the projection. Because the "time ordered" orientation of Δ^n is $(-1)^n \times$ standard orientation,
- $$(\partial \pi^{(r)})_* = - \sum_{j=0}^r (-1)^j (\pi_j^{(r-1)})_*$$
- where
- $$\pi_j^{(r-1)} : \Delta^{r-1} \times PM \rightarrow PM$$

is the restriction of $\Delta^r \times PM$ to the j th face $t_j = t_{j+1} \dots t_r$ of Δ^r .

(3) Combining thus with the formula

$$\begin{aligned} (-1)^r d \pi_*^{(r)} &= \pi_*^{(r)} d - (\partial \pi^{(r)})_* \\ &= \pi_*^{(r)} d + \sum_{j=0}^r (-1)^j \pi_j^{(r-1)} \end{aligned}$$

Proof of the exterior derivative formula

$$\begin{aligned} &= \int d(\omega_1 / \dots / \omega_r) + \int \delta(\omega_1 / \dots / \omega_r) \\ &\quad + (-1)^r \pi_*^{(r-1)} \left(e^*(1 \times \omega_1 \times \dots \times \omega_{r-1}) \wedge \pi^* \beta_*^* \omega_r \right) \\ &\quad + \text{prelim 1} \left\{ \begin{array}{l} + (-1)^r \pi_*^{(r-1)} \left(e^*(1 \times \omega_1 \times \dots \times \omega_{r-1}) \wedge \pi^* \beta_*^* \omega_r \right) \\ + \pi_*^{(r-1)} \left(\pi^* \beta_*^* \omega_1 \wedge e^*(1 \times \omega_2 \times \dots \times \omega_r \times 1) \right) \end{array} \right. \\ &= \int d(\omega_1 / \dots / \omega_r) + \int \delta(\omega_1 / \dots / \omega_r) \\ &\quad + (-1)^r \int (\omega_1 / \dots / \omega_r) \wedge \beta_*^* \omega_r \\ &\quad + (-1)^{(r-1)} \omega_1 \wedge \beta_*^* \omega_1 \wedge \int (\omega_2 / \dots / \omega_r). \\ &\quad \text{projection formulas} \\ &= (-1)^r d \pi_*^{(r)} e^*(1 \times \omega_1 \times \dots \times \omega_r \times 1) \\ &= \pi_*^{(r)} e^* d (1 \times \omega_1 \times \dots \times \omega_r \times 1) \\ &\quad + \text{prelim 3} \left\{ \begin{array}{l} + \sum_{j=0}^r (-1)^j \pi_j^{(r-1)} e^*(1 \times \omega_1 \times \dots \times \omega_r \times 1) \\ + (-1)^j \pi_*^{(r-1)} e^*(1 \times \omega_1 \times \dots \times \omega_r \times 1) \end{array} \right. \\ &= \int d(\omega_1 / \dots / \omega_r) \\ &\quad + \sum_{j=1}^{r-1} (-1)^j \pi_j^{(r-1)} e^*(1 \times \omega_1 \times \dots \times \omega_j \wedge \omega_{j+1} \times \dots \times \omega_r \times 1) \\ &\quad + (-1)^r \pi_*^{(r-1)} e^*(1 \times \omega_1 \times \dots \times \omega_r) \\ &\quad + \pi_*^{(r-1)} e^*(\omega_1 \times \omega_2 \times \dots \times \omega_r \times 1) \end{aligned}$$

Remark. This obeys the Koszul convention if one views it in the "right way". Start with the double complex

$$A^{-riS} := [E^*(M)^{\otimes(r+2)}] \subseteq E^S(\mathrm{pr} X)$$

This has the two differentials

$$d: A^{-r,s} \rightarrow A^{-r,s+1}$$

and

$$\delta: A^{-r,s} \rightarrow A^{-r+1,s}$$

Define $D: A^{-r,s} \rightarrow A^{-r,s+1} \oplus A^{-r+1,s}$
by $D = d + (-1)^r \delta$. Then $D^2 = 0$.

So $d + (A'', D)$ is a complex.

Now denote

$$a_0 \otimes a_1 \otimes \dots \otimes a_{r+1} \in A^{r,s}$$

degrees

by $a_0 \otimes [a_1 / \dots / a_r] \otimes a_{r+1}$

This has total degree $s-r$. So we
should view

$$a_1 \otimes \dots \otimes a_r \mapsto [a_1 / \dots / a_r]$$

as an operator of degree $-r$. And

we should regard

$$\omega_0 \times \dots \times \omega_{r+1}$$

$$\mapsto \phi^* \omega_0 \wedge f(a_1 / \dots / a_r) \wedge \phi^* \omega_1$$

as an operator of degree $-r$. Specifically,
view f as an operator of degree r .
The operator f has degree 0

$$\text{“ } \quad \text{“ } \quad d \quad \text{“ } \quad \text{“ } \quad 1 \text{ “ }$$

with this understanding

$$\begin{aligned} & D f(a_1 / \dots / a_r) \xrightarrow{\text{degree } -r} \\ &= (d + (-1)^r \delta) f(a_1 / \dots / a_r) \\ &= (-1)^r \int d(a_1 / \dots / a_r) \xrightarrow{\text{degree } -r+1} \\ &+ (-1)^r \int \delta(a_1 / \dots / a_r) \\ &+ (-1)^r \cdot (-1)^{(a_1 / \dots / a_r)} a_1 \otimes f(a_2 / \dots / a_r) \\ & \quad \text{[} (-1)^r \text{]}^2 + f(a_1 / \dots / a_r) \otimes a_r \end{aligned}$$

Later, we will formalize this as the
"bar construction".