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## HOPF ALGEBRAS

Graded Hopf algebras NB: This includes  
as  $A = A_0$

(1)  $\mathbb{k}$  = field of char 0

(2)  $A = \bigoplus_{n \in \mathbb{Z}} A_n$

where each  $A_n$  is a  $\mathbb{k}$ -vector space

(3) associative multiplication

$$\mu: A \otimes A \rightarrow A$$

This preserves grading:  $A_n \otimes A_m \rightarrow A_{n+m}$ .

(4) Augmentation

$$\varepsilon: A \rightarrow \mathbb{k}$$

a graded algebra homom. (NB:  $\mathbb{k}$  has to be in degree 0.)

(5) Unit  $\eta: \mathbb{k} \rightarrow A_0 \hookrightarrow A$ .

an algebra homom satisfying  $\varepsilon \circ \eta = \text{id}_{\mathbb{k}}$ .

$$(6) A = \mathbb{k} \oplus I$$

where

$$I = IA := \ker \varepsilon$$

$$\mathbb{k} = \text{im } \eta.$$

$$(7) \quad \Delta: A \rightarrow A \otimes A \text{ a graded,}$$

augmentation preserving algebra homom

which is coassociative:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow 1 \otimes \Delta \\ A \otimes A & \xrightarrow{\Delta \otimes 1} & A \otimes A \end{array}$$

commutes

and "counital":

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \text{nat isom} \swarrow & & \downarrow \varepsilon \otimes 1 \\ a \mapsto 1 \otimes a & & k \otimes A \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \approx \searrow & & \downarrow 1 \otimes \varepsilon \\ A \otimes \mathbb{k} & & \text{commute.} \end{array}$$

$$\text{i.e., } \Delta \alpha \equiv \alpha \otimes 1 + 1 \otimes \alpha \text{ mod } I \otimes I$$

$A$  is a **bialgebra** (up to this point).

(8) an antipode  $\tau: A \rightarrow A$ .

$$\begin{array}{ccc} & \Delta: A \otimes A & \xrightarrow{\text{id}, \text{id}} A \otimes A \\ & \downarrow & \downarrow \mu \\ A & \xrightarrow{\varepsilon} k & \xrightarrow{\gamma} A \\ & \Delta \xrightarrow{\text{id}, i} A \otimes A & \xrightarrow{\mu} A \end{array}$$

$\tau$  commutes.

Commutativity and cocommutativity

Suppose  $V = \bigoplus_{n \in \mathbb{Z}} V_n$  is a graded

$k$ -vector space. Define the isomorphism

$$\tau: V \otimes V \rightarrow V \otimes V$$

$$\text{by } \tau(v_1 \otimes v_2) = (-1)^{m_1 m_2} v_2 \otimes v_1$$

where  $v_1 \in V_m$  and  $v_2 \in V_n$ .

Def: A graded Hopf algebra  $A$

is (graded) commutative if

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \tau \downarrow & \nearrow & \downarrow \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

$\tau$  commutes.

$$\text{i.e. } a_1 \cdot a_2 = (-1)^{l(a_1) l(a_2)} a_2 \cdot a_1$$

$\mathcal{H}$  is cocommutative if

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ & \searrow & \swarrow \varepsilon \\ & A & \end{array}$$

commutes.

Examples  $\downarrow$  ungraded. I.e.  $A = A_0$ .

①  $\mathcal{R} = \text{discrete group}$   $\xleftarrow{\text{functorial in } \mathcal{R}}$

$A = k\mathcal{R} = \text{group algebra of } \mathcal{R}/k$ .

$\Delta: \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$

$\tau: \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$  all  $x \in \mathcal{R}$ .

Ex:  $k(\mathcal{R}/\mathcal{R}) \cong k\mathcal{R} \otimes k\mathcal{R}$

$$(\tau \circ \mu) \leftrightarrow \tau \otimes \mu$$

$\Delta$  is induced by diagonal map  $\mathcal{R} \rightarrow \mathcal{R} \times \mathcal{R}$ .

Unit: induced by  $\mathbb{1} \hookrightarrow \mathcal{R}$   $\xleftarrow{\text{triv group}}$

$$k\mathbb{1} = \mathbb{1}.$$

Augmentation: induced by  $\mathcal{R} \rightarrow \mathbb{1}$

This is always cocommutative. It is commutative  $\Leftrightarrow \mathfrak{g}$  is abelian.

Antipode: This is induced by  $x \mapsto -x$  for all  $x \in \mathfrak{g}$ . It is an anti-homomorphism.

②  $\mathfrak{g} = \text{graded Lie algebra}$

$$= \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

$\mathcal{L}, \mathcal{J}: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  preserves grading

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\quad -[\cdot, \cdot] \quad} & \mathfrak{g} \\ \tau \downarrow & \nearrow \text{commutes} & \\ \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\quad [\cdot, \cdot] \quad} & \end{array}$$

$$[\tau y, x] = -(-1)^{[x][y]} [x, y].$$

$A = \mathfrak{g} = \text{universal enveloping algebra}$

$$= T\mathfrak{g} / \langle \langle xy - (-1)^{[x][y]} yx - [xy]: x, y \in \mathfrak{g} \rangle \rangle$$

tensor algebra of free associative alg on  $\mathfrak{g}$ .

This is a graded Hopf algebra: that's what Augmentation is induced by  $\mathfrak{g} \rightarrow \mathfrak{o}$  aug

as  $U_0 = \mathbb{k}$ . Unit is induced by  $\mathfrak{o} \rightarrow \mathfrak{g}$ .

Coproduct:

$$\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$$

$$x \mapsto 1 \otimes x + x \otimes 1 \quad x \in \mathfrak{g}.$$

This is well-defined as the relation

$$xy - (-1)^{[x][y]} x - (-1)^{[y][x]} y = 0.$$

$$\underline{\text{Rk: }} \mathfrak{U}(\mathfrak{g} \otimes \mathfrak{g}) \cong \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$$

$$(x, y) \mapsto x \otimes 1 + 1 \otimes y \quad x, y \in \mathfrak{g}.$$

So coproduct is induced by  $\mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  diag

This is cocommutative. It is commutative if and only if  $\mathfrak{g}$  is abelian.

Antipode: This is induced by  $x \mapsto -x$  for all  $x \in \mathfrak{g}$ . It is an anti-homomorphism.

(3)  $G = \text{affine group scheme } / \mathbb{k}$ .  
 ungraded  $\xrightarrow{\text{i.e.}} A = A_0$ .

$A = \mathcal{O}(G)$  = ring of functions on  $G$ .  
 As usual, the augmentation and unit are induced by the group homom

$\pi \hookrightarrow G$  and  $\iota \rightarrow \mathbb{H}$ .

The coproduct  $\mathcal{O}(G \times G)_{\text{Sh}}$

$$\Delta: \mathcal{O}(G) \rightarrow \mathcal{O}(G) \otimes \mathcal{O}(G)$$

is induced by  $\mu: G \times G \rightarrow G$ .

The antipode

$$\iota: \mathcal{O}(G) \rightarrow \mathcal{O}(G)$$

induced by  $g \mapsto g^{-1}$ .

$\nearrow$  eg: a topological group

(4) If  $X$  is a  $H$ -space, then

$$A = H_*(X; \mathbb{k})$$

graded

is a cocommutative Hopf algebra

with unit and counit induced by neutral element.

$$\xrightarrow{\text{f.e.}} f_{e_0} \hookrightarrow X, \quad X \rightarrow \{\ast\}$$

and antipode induced by  $x \mapsto x^{-1}$ .

It is an anti-homomorphism.

$$\underline{\text{eg:}} \quad X = \rho_{\alpha, H}, \quad \rho_0 = \gamma_\alpha = \text{const}/\text{loop}.$$

$$\underline{\text{NB:}} \quad H_0(\rho_{\alpha, H}; \mathbb{k}) = \mathbb{k} \pi_\ell(H, \alpha).$$

(4)\* If  $X$  is an  $H$ -space with a finite # of path components, then

$$A = H^*(X; \mathbb{k})$$

is a graded commutative Hopf algebra.

$$\underline{\text{eg:}} \quad X = \rho_{\alpha, H} \quad \text{where } \pi_\ell(H, \alpha) \text{ is finite.}$$

In (4) and (4)\* there is an

antipode induced by  $x \mapsto x^!$ . It

Satisfies

$$i(uv) = (-1)^{|u||v|} i(v) i(u).$$

Non-neg  $A = \bigoplus_{n \geq 0} A_n$

(5)  $\mathbb{I}_A$  is a graded Hopf algebra  
of finite type ( $\underline{\text{rk}} : \dim A_n < \infty \text{ all } n$ ),

then  $A^*$  is a Hopf algebra, where

$$A^* := \bigoplus_{n \geq 0} A_n^* \quad \text{Hom}(A_n, k)$$

$A \rightsquigarrow A^*$  swaps multiplication and  
comult, augmentation and unit, etc.

### Primitives and indecomposables

$A$  = graded Hopf algebra

$$PA := \{ a \in \mathbb{I}_A : \Delta a = a \otimes 1 + 1 \otimes a \}$$

$$QA := \mathbb{I}_A / \mathbb{I}_A^2.$$

$PA$  = "primitive elements"

$QA$  = "indecomposables of  $A$ ".

Rk: While  $A = k \otimes \mathbb{I}_A$ . Then

$$\Delta : A \rightarrow A \otimes A \text{ induces}$$

$$\bar{\Delta} : \mathbb{I}_A \rightarrow \mathbb{I}_A \otimes \mathbb{I}_A$$

$$\rho_A = \ker \{ \bar{\Delta} : \mathbb{I}_A \rightarrow \mathbb{I}_A^{\otimes 2} \}$$

$$QA = \text{coker} \{ \mu : \mathbb{I}_A^{\otimes 2} \rightarrow \mathbb{I}_A \}.$$

So these notions are dual.

Prop: If  $A$  is cocommutative, then  
 $PA$  is a graded Lie algebra with  
bracket  $[u, v] = uv - (-1)^{|u||v|} vu$ .

Proof: Suppose  $a, v \in PA$ . Then

$$\begin{aligned} \Delta [a, v] &= \Delta (av - (-1)^{|u||v|} vu) \\ &= \Delta u \Delta v - (-1)^{|u||v|} \Delta v \Delta u \\ &= (a \otimes 1 + 1 \otimes a) (v \otimes 1 + 1 \otimes v) \\ &\quad - (-1)^{|u||v|} (v \otimes 1 + 1 \otimes v) (a \otimes 1 + 1 \otimes a) \\ &= (av \otimes 1 + a \otimes v + (-1)^{|u||v|} vu \otimes 1 + 1 \otimes uv) \\ &\quad - (-1)^{|u||v|} (vu \otimes 1 + v \otimes u + (-1)^{|u||v|} uv \otimes 1 + 1 \otimes vu) \end{aligned}$$

$$= [u, v] \otimes 1 + 1 \otimes [u, v]. \quad \square$$

Remark: If  $A$  is commutative, then  
 $\mathbb{Q}A$  is a "Lie coalgebra" with cobracket

$$\mathbb{Q}A \rightarrow \mathbb{Q}A \otimes \mathbb{Q}A$$

defined by  $\bar{\alpha} \mapsto \bar{\Delta}\alpha - \varepsilon \circ \bar{\Delta}\alpha$ .

Here  $\bar{\alpha}$  denotes the image of  $\alpha \in A$  in  $\mathbb{Q}A$ .

$\mathbb{Q}A$ .

Example:  $G = \text{affine group scheme}/k$ .

$$A = \mathcal{O}(G)$$

$\underline{m}_e = \text{augmentation ideal}$   
 $= \text{functions that vanish at the identity}$

$$\begin{aligned} \mathbb{Q}\mathcal{O}(G) &= \underline{m}_e / \underline{m}_e^2 \\ &= T_e^* G = \text{cotangent space} \\ &\quad \text{of } G \text{ at } e. \end{aligned}$$

$$= \mathfrak{g}_e^*, \text{ where } \mathfrak{g} = \text{Lie } G \cong T_e G.$$

The bracket of  $\mathfrak{g}$  is  $\rightarrow$  and  
 cobracket

$$\underline{m}_e / \underline{m}_e^2 \rightarrow \underline{m}_e / \underline{m}_e^2 \otimes \underline{m}_e / \underline{m}_e^2$$

are dual.

STRUCTURE THEOREMS: Here we need graded  
 the concept of a connected Hopf algebra.

So we will assume that

$$A = \bigoplus_n A_n = \bigoplus_{m,n} A_{m,n}$$

is a graded Hopf algebra, where

$$A_n = \bigoplus_{m>0} A_{m,n}$$

The product, coproduct, antipode  
 augmentation and unit are required to  
 respect the bigrading. (View  $\mathfrak{k} = \mathbb{k}_{0,0}$ .)

Def:  $A$  is connected if.

$$\bigoplus_n A_{0,n} = A_{0,0} = \mathbb{k}.$$

that is,

$$A_0 = \overset{= A_{0,0}}{\underset{k \oplus \bigoplus}{\oplus}} A_{m,0}$$

$$A_n = \underset{n \neq 0}{\bigoplus} A_{m,n}$$

Example:

① If  $\mathfrak{g} = \bigoplus_{n>0} \mathfrak{g}_n$  is a positively graded Lie algebra

$$A = \bigcup \mathfrak{g} = \bigoplus_{n>0} A_n = k \otimes \bigoplus_{n>0} A_n$$

Here  $A_n = A_{n,n}$

② If  $X$  is a connected  $H$ -space

$$A = H_*(X; k)$$

is a connected Hopf algebra with

$$A_{m,n} = \begin{cases} A_n & m=n \\ 0 & m \neq n \end{cases}$$

③ If  $n$  is a discrete group  
 $= A_{0,0}$   
 $k \oplus \bigoplus_{m>0} A_{m,0}$   
then  $k\mathfrak{n}$  is not graded. but

$$A = k\mathfrak{I} \otimes k\mathfrak{n} := \bigoplus_{m>0} I^m / I^{m+1}$$

$\cong$

$\mathfrak{A}_{m,0}$   
is a connected graded Hopf algebra.

Theorem (Milnor - Moore) If

$$A = \bigoplus_n (\bigoplus_{m>0} A_{m,n})$$

is a connected graded cocommutative  
Hopf algebra, then the natural Hopf  
algebra homomorphism

$$UPA \rightarrow A$$

is an isomorphism.

Remark: This is not true without  
the connectedness hypothesis. For  
example, take  $\mathfrak{r} = \langle t\rangle = t^2$ , an

infinite cyclic group. Let

$$A = \mathbb{Q}[t, t^{-1}].$$

The coproduct is  $\Delta t^n = t^n \otimes t^n$ . Then  
(exercise)  $\rho_A = 0$ . So  $U^P A = \mathbb{K}$ .

Then (Poincaré-Birkhoff-Witt) If  
 $\mathfrak{g}$  is a graded Lie algebra, then

$$\mathfrak{g} \rightarrow P U \mathfrak{g}$$

is an isomorphism. In particular,  
the natural map  $\varphi: \mathfrak{g} \rightarrow U \mathfrak{g}$  is

injective.

Stronger version: as a coalgebra,  
 $U \mathfrak{g}$  is isomorphic to the symmetric  
coalgebra  $S^c \mathfrak{g}$  of on  $\mathfrak{g}$ . The isomorphism  
takes

$$x_1 \dots x_n \in S^c \mathfrak{g}$$

to

$$\sum_{n \in \mathbb{Z}_n} n! g(x_m) \dots g(x_0) \in U \mathfrak{g}.$$

Free graded commutative algebras

Suppose that  $V.$  is a graded vector  
space over  $\mathbb{K}$ . The free associative  
algebra on  $V.$  is the tensor algebra

$$T(V.) = \bigoplus_{n \geq 0} V.^{\otimes n}.$$

It is graded.

The free (graded) commutative algebra  
on  $V.$ , denoted  $\Lambda^*(V.)$ , is the graded  
algebra

$$\Lambda^*(V.) = T(V.) / \langle v w - (-1)^{mn} w v : v \in V_n, w \in V_m \rangle$$

Remark:

$$\Lambda^*(V.) = \text{Exterior}(V_{\text{odd}}) \otimes \text{Sym}(V_{\text{ev}})$$

$$\text{Exterior}(W) = T(W) / \langle w_1 w_2 + w_2 w_1 : w_i \in W \rangle$$

$$\text{Sym}(W) = T(W) / \langle w_1 w_2 - w_2 w_1 : w_i \in W \rangle$$

$\cong \mathbb{K}[x_1, x_2, \dots]$   
where  $x_1, x_2, \dots$  is a basis of  $W$ .

— • —

As above, suppose that  $A = A_\bullet$  is a  
graded Hopf algebra, where

$$A_n = \bigoplus_{m \geq 0} A_{m,n}.$$

Recall that  $A$  is connected if

- (1)  $A_{0,n} = 0$  when  $n > 0$
- (2)  $A_{0,0} = \mathbb{K}$ .

PROPOSITION : If  $A$  is a connected,  
graded commutative Hopf algebra,  
then  $A$  is free as a (graded)  
commutative algebra.

proof. choose a bigraded section of

$$\mathcal{I}_A \longrightarrow \mathbb{Q}A.$$

Denote its image by  $V = V_\bullet$ . The  
inclusion  $V \hookrightarrow A$  induces a bigraded  
algebra homomorphism  $T(V) \rightarrow A$ .

Since  $A$  is connected, this is surjective.  
So we can write

$$A = N(V) / R$$

where  $R$  is a bigraded ideal. Let

$$\mathcal{I}_A = (V) = \text{augmentation ideal}.$$

Since the coproduct  $\Delta : A \rightarrow A \otimes A$  is  
augmentation preserving, it induces

$$\overline{\Delta} : \overline{\mathcal{I}_A^n} / \mathcal{I}_A^{n+1} \longrightarrow \bigoplus_{\substack{i+j=n \\ i,j \geq 0}} \overline{\mathcal{I}_A^i} / \mathcal{I}_A^{i+1} \otimes \overline{\mathcal{I}_A^j} / \mathcal{I}_A^{j+1}$$

Iterate this to obtain a map

$$\overline{\Delta}^{n-1} : \overline{\mathcal{I}_A^n} / \mathcal{I}_A^{n+1} \longrightarrow (\overline{\mathcal{I}_A^2})^{\otimes n}$$

By induction, this is injective for all  $n \geq 2$ .

We know  $R \subseteq I_A^n$ . Suppose  $R \subseteq I_A^m$

where  $m \geq 2$ . Then

$$I_A^j / I_A^{j-1} \approx I_A^j / I_A^{j+1}$$

all  $j < n$ . Since

$$\begin{array}{ccc} I_A^n / I_A^{n+1} & \xrightarrow{\text{onto}} & (I_A / I_A^2)^{\otimes n} \\ \downarrow & \nearrow \text{is inj} & \downarrow \text{is } \otimes \\ I_A^n / I_A^{n+1} & \xrightarrow{\quad} & R / R^n I_A^{n+1} \end{array}$$

$\therefore R \subseteq I_A^{n+1}$ . Since  $\cap I_A^{n+1} = 0$ ,  
 $R = 0$ . So  $A \cong \Lambda(V)$ .  $\square$

$\hookleftarrow$  over a field of char 0  
Cor: Every finite dimensional, connected Hopf algebra is an exterior algebra.

Cor: The rational cohomology ring of a topological group is an exterior algebra.

Ex: (1)  $G = \mathbb{R}^n / \mathbb{Z}^n$ .

$$\begin{array}{c} H^*(G) = \Lambda^* H^*(G) \\ (2) \quad H^*(GL_n(\mathbb{C})) \cong H^*(U(n)) = \Lambda^*(y_1, y_3, \dots, y_{2n}) \end{array}$$

where  $|y_{2i-1}| = 2i - 1$ .

