Richard Hain November 8, 2017

Math 625
Riemann Surfaces
Assignment 5

Due: November 16, 2017.

1. Show that if D is a divisor on a compact Riemann surface X and $p \in X$, then
$$\ell(D) \leq \ell(D+p) \leq \ell(D) + 1.$$

2. Show that if D is a divisor on a compact Riemann surface X of positive genus, then $\ell(D) \leq 1 + \deg D$. Hint: Write $D = D_0 + p_1 + \dots + p_d$, where D_0 has degree 0. Prove the result for D_0 and then use the previous result.

3. Suppose that X is hyperelliptic. (In particular, X is compact and has genus ≥ 2.) Suppose that $x : X \to \mathbb{P}^1$ is a double covering, which we assume to be branched over $\infty \in \mathbb{P}^1$. Let $p \in X$ be the unique point lying over ∞ and $\sigma : X \to X$ the involution of the double covering.

(a) How many critical values does $x : X \to \mathbb{P}^1$ have?
(b) Compute the divisor of the meromorphic 1-form dx on X.
(c) Show that $\sigma^*\omega = -\omega$ for all $\omega \in H^0(\Omega^1_X)$. Hint: use a trace argument.
(d) Show that there is a meromorphic function $y \in L\big((2g+1)p\big)$ such that $\sigma^* y = -y$ and dx/y is holomorphic. Show that it vanishes at the critical points $\neq p$.
(e) Suppose that $a_0, \dots, a_N$ are the critical values of x that lie in $\mathbb{C}$. Show that $y^2 = c\prod_{j=0}^N (x - a_j)$ in $\mathcal{M}(X)$, where $c \in \mathbb{C}^*$.
(f) Show that $(x, y) : X - \{p\} \to \mathbb{C}^2$ maps $X - \{p\}$ isomorphically onto the smooth plane curve $y^2 = \prod_{j=0}^N (x - a_j)$.