Richard Hain October 23, 2017

Math 625
Riemann Surfaces
Assignment 4

Due: October 31, 2017.

Throughout, X is a *compact* Riemann surface of genus $g \geq 0$, and for a divisor D on X

$$\ell(D) = h^0(\mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D))$$
$$i(D) = h^1(\mathcal{O}_X(D)) = \dim H^1(X, \mathcal{O}_X(D)) = h^0(K_X(-D)).$$

1. Show that $\ell(D)$ and $i(D)$ depend only on the divisor class of D.

2. Show that if D is a divisor on X of negative degree, then $\ell(D) = 0$.

3. Show that if $\deg D = 0$, then $\ell(D)$ is either 0 or 1, and that $\ell(D) = 1$ if and only if D is principal.

4. Show that the degree of a canonical divisor of X is $2g - 2$.

5. Suppose that $P \in X$. Give two proofs of the fact that $h^0(K_X(P)) = h^0(K_X)$.

6. Show that if P and Q are distinct point of X, then

$$h^0(K_X(P + Q)) = g + 1.$$

Deduce that there is meromorphic differential w, whose only poles are P and Q, both simple, with residue 1 at P and -1 at Q. Show that this differential is unique up to the addition of a holomorphic differential.

7. Show that if $\deg D > 2g - 2$, then $i(D) = 0$. Deduce that

$$\ell(D) = \deg D + 1 - g > g - 1.$$

8. Show that if $\deg D = 2g - 2$, then $\ell(D) = g$ or $g - 1$, and that $\ell(D) = g$ if and only if D is a canonical divisor.