Richard Hain September 30, 2017

Math 625
Riemann Surfaces
Problem Set 3

Due: Thursday, October 12, 2017

1. Suppose that a group Γ acts on a space (or set, Riemann surface, etc) X on the left. Suppose that V is another space (or set, Riemann surface, etc). Suppose that for each $\gamma \in \Gamma$ and $x \in X$, one has $M_\gamma(x) \in \operatorname{Aut} V$. Show that the function $\Gamma \to \operatorname{Aut}(X \times V)$ defined by

$$\gamma : (x, v) \mapsto (\gamma x, M_\gamma(x)v)$$

defines an action $\Gamma \to \operatorname{Aut}(X \times V)$ if and only if

$$M_{\gamma_1\gamma_2}(x) = M_{\gamma_1}(\gamma_2 x)M_{\gamma_2}(x).$$

A function $M : \Gamma \times X \to \operatorname{Aut} V$ satisfying this condition is called a *factor of automorphy.*

2. Suppose that X is a Riemann surface and that Γ acts fixed point freely and properly discontinuously on X. Show that if $M_\gamma(x)$ is a factor of automorphy with values in $\mathbb{C}^*$, then the quotient

$$\Gamma\backslash(X \times \mathbb{C}) \to \Gamma\backslash X$$

is a complex line bundle over the Riemann surface $\Gamma\backslash X$. A function $f : X \to \mathbb{C}$ corresponds to the section $x \mapsto (x, f(x))$ of $X \times \mathbb{C} \to \mathbb{C}$. Show that f descends to a section of the line bundle $\Gamma\backslash(X\times\mathbb{C}) \to \Gamma\backslash X$ if and only if

$$f(\gamma x) = M_\gamma(x)f(x).$$

3. Now suppose that $X = \mathbb{C}$ and $\Gamma = \mathbb{Z}^2$. Fix τ in the upper half plane $\mathfrak{h}$ and let Γ act on $\mathbb{C}$ by $(m, n) : z \mapsto z + m\tau + n$. Show that

$$M_{(m,n)}(z) = \exp(-2\pi imz - \pi im^2\tau) \in \mathbb{C}^*$$

is a factor of automorphy. Let $E_\tau = \Gamma\backslash\mathbb{C}$ and $L \to E_\tau$ be the line bundle $\Gamma\backslash(\mathbb{C} \times \mathbb{C}) \to E_\tau$. Show that an entire function $f : \mathbb{C} \to \mathbb{C}$ defines a section of $L \to E$ if and only if

$$f(z + 1) = f(z) \text{ and } f(z + \tau) = \exp(-2\pi iz - \pi i\tau)f(z) \tag{1}$$

for all $z \in \mathbb{C}$. Hint: it might be helpful to set $q = e^{\pi i\tau}$ and $w = e^{2\pi iz}$.

4. Fix $\tau \in \mathfrak{h}$. Show that the series

$$\vartheta_\tau(z) = \sum_{n=-\infty}^{\infty} \exp(\pi in^2\tau + 2\pi inz)$$

converges to a holomorphic function $\mathbb{C} \to \mathbb{C}$ that satisfies the conditions in (1) and thus defines a section of the line bundle $L \to E_\tau$.

5. Compute the number of zeros of ϑ_τ in the fundamental domain

$$F = \{t + s\tau : 0 \leq s \leq 1,\ 0 \leq t \leq 1\}.$$

Hint: Compute

$$\int_{\partial F} \frac{d\vartheta_\tau}{\vartheta_\tau}.$$

Compute the degree of the line bundle L. Locate the zeros by computing

$$\int_{\partial F} z \frac{d\vartheta_\tau}{\vartheta_\tau}.$$