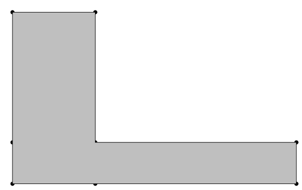
Richard Hain September 16, 2017

Math 625
Riemann Surfaces
Problem Set 2

Due: Tuesday, September 26, 2017.

1. The polygon P below lies in the complex plane. We have seen that the surface X obtained by identifying the opposite edges of P has a natural complex structure and that it has genus 2.



(i) Show that there is a holomorphic differential ω on X whose pullback to P is dz.
(ii) Find the zeros of ω and their orders.

2. Let Λ be a lattice in $\mathbb{C}$. Define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

(i) Show that $\wp_\Lambda$ is a doubly periodic meromorphic function with period lattice Λ. Show that its poles are at the points of Λ. Hint: first prove that its 'formal derivative'

$$\wp'_\Lambda(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3},$$

converges almost uniformly on $\mathbb{C} - \Lambda$, then integrate.
(ii) Deduce that $\wp_\Lambda : \mathbb{C}/\Lambda \to \mathbb{P}^1$ is a $2:1$ holomorphic map which is branched at the four points of order two of $\mathbb{C}/\Lambda$. Deduce also that $\wp'_\Lambda$ has only 3 zeros, counting multiplicity.
(iii) Show that if $f : \mathbb{C} \to \mathbb{P}^1$ is a doubly periodic meromorphic function (with period lattice Λ) with a Laurent expansion of the form

$$f(z) = \sum_{k=-2}^{\infty} c_k z^k$$

with $c_{-2} = 1$ and $c_{-1} = c_0 = 0$ about zero and poles only on Λ, then $f = \wp_\Lambda$.