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Math 612
Worksheet 1

The 3-dimensional (real) Heisenberg group is the group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Fix an integer $d \geq 1$ and set

$$\Gamma_d = \left\{ \begin{pmatrix} 1 & m & k \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, dk \in \mathbb{Z} \right\}.$$

This acts on G by left translation. Set $M = \Gamma_d \backslash G$.

(1) Show that M is a smooth 3-manifold. What is its fundamental group?

(2) Show that the map $G \to \mathbb{R}^2$ defined by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y)$$

is a group homomorphism and that it induces a smooth map $p : M \to T$, where T is the torus $\mathbb{R}^2/\mathbb{Z}^2$. Show that each fiber of p is a circle.

(3) Compute $H_1(M; \mathbb{Z})$.

(4) Show that the 3-form $dx \wedge dy \wedge dz$ on G is left invariant. Deduce that M is orientable.

(5) Compute the integral cohomology groups of M.

(6) Compute the cup product on $H^\bullet(M; \mathbb{Z})$. This is harder!

Some help in how to proceed:

(7) Denote the Lie algebra of G by $\mathfrak{g}$. Identify the left-invariant forms on G with $\Lambda^\bullet \mathfrak{g}^\vee$.
(8) Construct a DG algebra homomorphism $\Lambda^\bullet \mathfrak{g}^\vee \to E^\bullet(M)$.
(9) Set
$$\gamma = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$
Compute
$$\gamma^* x,\ \gamma^* y,\ \gamma^* z,\ \gamma^* dx,\ \gamma^* dy,\ \gamma^* dz.$$
(10) Find a basis of the left-invariant 1-forms on G.
(11) Find a basis of the *closed* left-invariant 2-forms on M. Which of these represent non-zero cohomology classes? Why?
(12) Compute the cup product $H^1(M;\mathbb{R})^{\otimes 2} \to H^2(M;\mathbb{R})$.
(13) Fix the isomorphism of $\pi_1(M, x_o)$ with Γ_d by taking id to be the basepoint of G and the identity coset x_o to be the base point of M. Set
$$\alpha := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \beta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \gamma := \begin{pmatrix} 1 & 0 & 1/d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
What loops represent $\alpha\beta\alpha^{-1}\beta^{-1}$ and γ?
(14) What loop represents a generator of the torsion subgroup of $H_1(M;\mathbb{Z})$?
(15) Find transversally intersecting 2-cycles in M that represent a basis of $H_1(M;\mathbb{Z})$. What does this tell you about the cup product $H^1(M;\mathbb{Z})^{\otimes 2} \to H^2(M;\mathbb{Z})$?