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BRIEF REMARKS ON THE THOM ISOMORPHISM

§1 Suppose that X is a "nice" topological space. (More about that below). Suppose that R is a ring. (Two main cases are where $\mathbb{Z} \subseteq R$ and $\mathbb{F}_2 \subseteq R$.)

<u>Basic Setup</u>: Suppose that

$$\pi : (B, S) \to X \qquad (\pi = \pi_B, \pi_S)$$

is a bundle pair with fiber (B^n, S^{n-1}). That is, each $x \in X$ has a neighbourhood U s.t the "restriction of π to U"

$$(B_U, S_U) := (\pi_B^{-1} U, \pi_S^{-1} S) \to U$$

is a trivial (B^n, S^{n-1}) bundle. Ie. we have a homeo $\varphi : (B_U, S_U) \to (B^n, S^{n-1}) \times U$ such that

$$\begin{tikzcd} (B_U, S_U) \arrow[rr, "\varphi", "\approx"'] \arrow[dr, "\pi"'] & & (B^n, S^{n-1}) \times U \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

commutes.

Suppose that R is a commutative ring and that $(B, S) \to X$ is "R-oriented". That is one can consistently choose a generator

an "orientation of B_x" $\to$ $\tau_x \in H^n(B_x, S_x; R)$.

(See Hatcher). Here

$$(B_x, S_x) = \pi^{-1}(x) \qquad x \in X.$$

This is homeo to (B^n, S^{n-1}). We insist that each τ_x is the image of a generator of $H^n(B_x, S_x; \mathbb{Z})$ under

$$H^n(B_x, S_x; \mathbb{Z}) \to H^n(B_x, S_x; R).$$

This means that for each x, there are 2 possible choices for τ_x when $2 \neq 0$ in R and only one choice when $2=0$.

This means that every (B^n, S^{n-1})-bundle is $\mathbb{F}_2$-orientable.

<u>Example</u>: The Möbius band M is a (B^1, S^0) bundle over S^1. It is $\mathbb{F}_2$ orientable, but not $\mathbb{Z}$-orientable.

§ 2 <u>Statement of Results</u>

To simplify the proofs, we will assume that every open covering of X has a "good" refinement <u>OR</u> that X is a CW-complex.

eg C^∞ manifolds, simplicial complexes.

<u>PROP</u>: ① If $(B,S) \xrightarrow{\pi} X$ is a (B^n, S^{n-1}) bundle (not necessarily orientable), Then for all R

$$H^j(B,S;R) = 0 \quad \text{all } j < n.$$

② If $(B,S) \to X$ is R-orientable, and X is path connected, then

$$H^n(B,S;R) = R\tau$$

In general $H^n(B,S) = \prod_{\text{cpts } \alpha} H^n(B_\alpha, S_\alpha)$

where $\tau \in H^n(B,S;R)$ has the property that

$$\tau|_{(B_x, S_x)} = \tau_x \in H^n(B_x, S_x; R)$$

uniquely characterizes τ.

this is the orientation on the fiber over $x \in X$.

<u>Definition</u>: $\tau \in H^n(B,S;R)$ is called the <u>Thom</u> <u>class</u> of the oriented (B^n, S^{n-1}) bundle $(B,S) \to X$.

Now that we have the Thom class, we can define a map

$$\text{Thom}: H^j(X;R) \to H^{j+n}(B,S;R)$$

as the composite

$$\begin{tikzcd} H^j(X;R) \arrow[r, "\pi_B^* \otimes \tau"] \arrow[dr, "\mathrm{Thom}"'] & H^j(B;R) \otimes H^n(B,S;R) \arrow[d, "\mathrm{cup}"] \\ & H^{n+j}(B,S;R) \end{tikzcd}$$

$$\begin{tikzcd} u \arrow[r, mapsto] & \pi_B^* u \otimes \tau \arrow[d, mapsto] \\ & (\pi_B^* u) \cup \tau \end{tikzcd}$$

<u>Theorem</u>: If $\pi : (B,S) \to X$ is R-orientable, then

$$\mathrm{Thom}: H^j(X;R) \to H^{n+j}(B,S;R)$$

is an isomorphism for all $j \in \mathbb{Z}$.

§3 <u>Proofs</u>:

(1) getting started: each $x \in X$

has a neighbourhood U where $(B_U, S_U) \to U$ is trivial. By assumption, we can find a good covering $\mathcal{U}$ of X such that

$$(B_U, S_U) \to U$$

is trivial for all $U \in \mathcal{U}$.

Since $\mathcal{U}$ is a good covering, each $U \in \mathcal{U}$ and each

$$U_0 \cap \dots \cap U_n \qquad U_j \in \mathcal{U}$$

is contractible.

Observation: Since U is contractible

$$\begin{aligned} H^\bullet(B^n, S^{n-1}) \times U; R) &\cong H^\bullet(B^n, S^{n-1}; R) \\ &\cong \begin{cases} 0 & \bullet \neq n \\ R & \bullet = n. \end{cases} \end{aligned}$$

So the Proposition and the Thom isomorphism hold in this case

(2) The basic step in the proof of

(a) both assertions of the PROP

(b) the Thom isom.

is:

if it is true for U, V and $U \cap V$, it is true for $U \cup V$. Point:

$$(B_{U\cup V}, S_{U\cup V}) = (B_U, S_U) \cup (B_V, S_V)$$
$$(B_{U\cap V}, S_{U\cap V}) = (B_U, S_U) \cap (B_V, S_V).$$

Sketch:

(a) vanishing of $H^j(B, S; \mathbb{R})$, $j<n$.

Suppose $j<n$. Have

$$\begin{matrix} H^{j-1}(B_U,S_U) \\ \oplus \\ H^{j-1}(B_V,S_V)\end{matrix} \to H^{j-1}(B_{U\cap V},S_{U\cap V}) \xrightarrow{\delta} H^j(B_{U\cup V},S_{U\cup V}) \to \begin{matrix} H^j(B_U,S_U) \\ \oplus \\ H^j(B_V,S_V)\end{matrix} \to H^j(B_{U\cap V},S_{U\cap V})$$

so vanishes

vanish by assumpt

(b) Existence of Thom class:

assume OK for U, V, $U\cap V$. Then Mayer–Vietoris becomes exact sequence

$$0 \to H^n(B_{U\cup V}, S_{U\cup V}) \to H^n(B_U,S_U) \oplus H^n(B_V,S_V) \to H^n(B_{U\cap V}, S_{U\cap V})$$

Let $\tau_U \in H^n(B_U, S_U)$

$\tau_V \in H^n(B_V, S_V)$

$\tau_{U\cap V} \in H^n(B_{U\cap V}, S_{U\cap V})$

be the Thom classes over U, V, $U\cap V$.

Uniqueness of Thom class implies that

$$\tau_U\big|_{U\cap V} = \tau_V\big|_{U\cap V} = \tau_{U\cap V}$$

So MV sequence above implies that there is unique

$$\tau_{U\cup V} \in H^n(B_{U\cup V}, S_{U\cup V})$$

which restricts to

9. On each path component

$\tau_U \in H^n(B_U, S_U)$ & $\tau_V \in H^n(B_V, S_V)$. The MV sequence above becomes

$$0 \to R\tau_{U\cup V} \to R\tau_U \oplus R\tau_V \to R\tau_{U\cap V} \to \cdots$$

$$\tau_{U\cup V} \mapsto (\tau_U, \tau_V)$$

$$(\tau_U, 0) \longmapsto \tau_{U\cap V}$$

$$(0, \tau_V) \longmapsto -\tau_{U\cap V}.$$

<u>Proving the Thom isomorphism</u>: The basic step is to show that the Thom isom holds for $U, V, U\cap V$, then it holds for $U\cup V$. To do this consider diagram on next page.

10.

$$\begin{tikzcd}
H^{n+j-1}((B,S)_U) \oplus H^{n+j-1}((B,S)_V) \arrow[r] & H^{n+j-1}((B,S)_{U\cap V}) \arrow[r, "\delta"] & H^{n+j}((B,S)_{U\cup V}) \arrow[r] & H^{n+j}((B,S)_U) \oplus H^{n+j}((B,S)_V) \arrow[r] & H^{n+j}((B,S)_{U\cap V}) \\
H^{j-1}(U) \oplus H^{j-1}(V) \arrow[r] \arrow[u, "Th_U \oplus Th_V"] & H^{j-1}(U\cap V) \arrow[r, "\delta"] \arrow[u, "Th_{U\cap V}"] & H^j(U\cup V) \arrow[r] \arrow[u, "Th_{U\cup V}"] & H^j(U) \oplus H^j(V) \arrow[r] \arrow[u, "Th_U \oplus Th_V"] & H^j(U\cap V) \arrow[u, "Th_{U\cap V}."]
\end{tikzcd}$$

By assumption, $Th_U, Th_V, Th_{U\cap V}$ are isoms. To prove assertion, you need to prove that the diagram above commutes. Squares ① ③ & ④ commute. (Easy). You need to check that <u>square ②</u> commutes up to a sign.

↳ see next page.

proof that

$$\begin{tikzcd} H^{n+j-1}(B_{U\cap V}, S_{U\cap V}) \arrow[r, "\delta"] & H^{n+j}(B_{U\cup V}, S_{U\cup V}) \\ H^{j-1}(U\cap V) \arrow[u, "Th_{U\cap V}"] \arrow[r, "\delta"] & H^{j}(B_{U\cup V}, S_{U\cup V}) \arrow[u, "Th_{U\cup V}"'] \end{tikzcd}$$

commutes up to a sign:

Given $\xi \in H^{j-1}(U\cap V)$, choose representing cocycle $z \in C^{j-1}(\mathcal{U})$. Since

$$C^{j-1}(U) \oplus C^{j-1}(V) \to C^{j-1}(U\cap V)$$

$$(c_1, c_2) \longmapsto c_1/_{U\cap V} - c_2/_{U\cap V},$$

have (c_U, c_V) such that

$$z = c_U/_{U\cap V} - c_V/_{U\cap V}$$

<u>Rk:</u> You can take $c_V = 0$ as

$$C^{\bullet}(U) \to C^{\bullet}(U\cap V)$$

is surjective

Since z is a cocycle

$$\delta c_U/_{U\cap V} - \delta c_V/_{U\cap V} = \delta z = 0$$

so have $e \in C^{j}_{\mathcal{U}}(U\cup V)$ such that

$$e/_U = \delta c_U, \quad e/_V = \delta c_V$$

where $\mathcal{U} = \{U, V\}$. This is because

$$0 \to C^{\bullet}_{\mathcal{U}}(U\cup V) \to C^{\bullet}(U) \oplus C^{\bullet}(V)$$

$$\to C^{\bullet}(U\cap V) \to 0$$

is exact. By proof of Mayer–Vietoris

$$H^{j-1}(U) \xrightarrow{\delta} H^{j}(U\cup V)$$

$$\xi \longmapsto [e]$$

To prove the diagram commutes, choose cocycle t that represents $\tau_{U\cup V}$. The characterization of Thom classes implies that

$t|_U$ represents τ_U

$t|_V$ " " τ_V

$t|_{U\cap V}$ " " $\tau_{U\cap V}$.

Now $Th(\xi)$ is represented by $t \cup z \in C^{n+j-1}(B_{U\cap V}, S_{U\cap V})$. This lifts to

$\pi: B_U \to U$ etc

$(\pi^* c_U \cup t, \pi^* c_V \cup t) \in C^\cdot(B_U, S_U) \oplus C^\cdot(B_V, S_V)$.

and the $\hat{\mathcal{U}}$-small cocycle

$$e \cup t \in C^{n+j}(B_{U\cup V}, S_{U\cup V})$$

whose restriction to U is

$$\delta(\pi^* c_U \cup t) = \pi^*(\delta c_U) \cup t \qquad \text{as } \delta t = 0$$

$\therefore$

$$\delta: H^{n+j-1}(B_U \cap B_V) \to H^{n+j}(B_{U\cup V}, S_{U\cup V})$$

$$(\pi^* \xi) \cup \tau_{U\cap V} \mapsto (\pi^* e) \cup \tau_{U\cup V}$$

$$\begin{tikzcd}
\tau_{U\cap V} \cup \pi^* \xi \arrow[r, mapsto, "\delta"] & \pm \pi^* e \cup \tau_{U\cup V}. \\
\xi \arrow[u, mapsto, "Thom"] \arrow[r, mapsto, "\delta"] & {[e]} \arrow[u, "Thom"']
\end{tikzcd}$$