Richard Hain February 13, 2023

MATH 612
THE COHOMOLOGY OF $S^n \times X$

The goal of this note is to show that, for all $n \geq 0$, the cross product

$$\times : H^\bullet(S^n) \otimes H^\bullet(X) \to H^\bullet(S^n \times X)$$

is a graded ring isomorphism. (Coefficients will be in a fixed commutative ring R.) The definition of the cross product implies that it is a ring homomorphism. We need to show that it is an isomorphism. We will prove the result by induction on n. The result is easily verified when $n = 0$. (Exercise) So suppose that $n > 0$ and that we know the result for $S^{n-1} \times X$.

The first observation is that $H^\bullet(X)$ is a direct summand of $H^\bullet(S^n \times X)$. To see this, fix a point $e \in S^n$. One has the inclusion j defined by

$$X \xrightarrow{\simeq} \{e\} \times X \longrightarrow S^n \times X$$

and the projection $q : S^n \times X \to X$. Since $q \circ j = \mathrm{id}_X$, it follows that j^* is surjective and that

$$H^\bullet(S^n \times X) \cong q^* H^\bullet(X) \oplus \ker j^*.$$

The definition of the cross product implies that $q^* H^\bullet(X)$ is just the image of

$$1 \times \_ : H^\bullet(X) \to H^\bullet(S^n \times X).$$

We'll complete the proof by showing that the image of

$$H^n(S^n) \otimes H^\bullet(X) \to H^\bullet(S^n \times X)$$

is $\ker j^*$. Observe that (by naturality of the cross product), the diagram

$$\begin{tikzcd}
H^n(S^n) \otimes H^j(X) \arrow[r, "\times"] \arrow[d, "0"'] & H^{n+j}(S^n \times X) \arrow[d, "j^*"] \\
H^n(\{e\}) \otimes H^j(X) \arrow[r, "\times"] & H^{n+j}(S^n \times X)
\end{tikzcd}$$

commutes. Since $n > 0$, the left hand vertical map is zero. From this we conclude that the image of the top row is contained in $\ker j^*$.

To understand $\ker j^*$, consider the cohomology LES

$$\cdots \longrightarrow H^{j-1}(S^n \times X) \xrightarrow{j^*} H^{j-1}(e \times X) \longrightarrow$$
$$H^j(S^n \times X, e \times X) \longrightarrow H^j(S^n \times X) \xrightarrow{j^*} H^j(e \times X) \longrightarrow$$

of the pair $(S^n \times X, e \times X)$. Since j^* is surjective, this splits into short exact sequences

$$0 \longrightarrow H^j(S^n \times X, e \times X) \longrightarrow H^j(S^n \times X) \xrightarrow{j^*} H^j(e \times X) \longrightarrow 0$$

from which we conclude that the natural map

$$H^j(S^n \times X, e \times X) \longrightarrow H^j(S^n \times X)$$

is injective with image $\ker j^*$ so that

$$H^\bullet(S^n \times X) \cong (1 \times H^\bullet(X)) \oplus H^\bullet(S^n \times X, e \times X).$$

To complete the proof, we will show that

$$\times : H^n(S^n) \times H^\bullet(X) \to H^\bullet(S^n \times X, e \times X)$$

is an isomorphism. To this end, consider the diagram

$$\begin{tikzcd}
\tilde{H}^{n-1}(S^{n-1}) \otimes H^j(X) \arrow[r, "\times"] \arrow[d, "\delta"'] & H^{n+j-1}(S^{n-1} \times X, e \times X) \arrow[d, "\delta"] \\
H^n(B^n, S^{n-1}) \otimes H^j(X) \arrow[r, "\times"] \arrow[d] & H^{n+j}(B^n \times X, S^{n-1} \times X) \arrow[d] \\
H^n(S^n) \otimes H^j(X) \arrow[r, "\times"] & H^{n+j}(S^n \times X, e \times X)
\end{tikzcd}$$

where $e \in S^{n-1} \subset S^n$. This commutes and the top horizontal map is an isomorphism by our inductive hypothesis. (You need to check that cross product is compatible with the connecting homomorphism to see that the top square commutes. You also need to check that the top map is an isomorphism when $n = 1$.) The bottom two vertical maps are isomorphism by excision. The two connecting homomorphisms δ are isomorphisms for all $n \geq 1$. So the bottom horizontal map is an isomorphism. This completes the proof.

As a consequence, we see that the cross product induces a graded ring isomorphism

$$H^\bullet(S^{n_1}) \otimes \cdots \otimes H^\bullet(S^{n_r}) \xrightarrow{\simeq} H^\bullet(S^{n_1} \times \cdots \times S^{n_r}).$$

This gives the presentation

$$H^\bullet(S^{n_1} \times \cdots \times S^{n_r}) \cong R\langle e_1, \ldots, e_r\rangle/(e_j^2 = 0, e_j e_k = (-1)^{n_j n_k} e_k e_j),$$

where $e_j \in H^{n_j}(S^{n_j})$ is a generator.