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Math 612
Comments on Assignment 1

Suppose that Y is a surface with boundary that is obtained by removing n disks from a compact oriented surface $\overline{Y}$ of genus g. Let X be Y with n Möbius bands attached along its boundary:

$$X = Y \cup_h \big(\coprod_{j=1}^{n} M_j\big)$$

where each M_j is a Möbius band and where $h = \sqcup h_j$ is a homeomorphism

$$\coprod \partial M_j \to \partial Y$$

that identifies the boundary of M_j with the jth boundary component of Y. You were asked to compute the cohomology of X with $\mathbb{Z}$ and $\mathbb{F}_2$ coefficients (and last semester, you were asked to compute the corresponding homology groups). I'll try to give an efficient account. I will not use the UCT, but you can also compute the cohomology from the homology and change from $\mathbb{Z}$ coefficients to $\mathbb{Z}/2$ coefficients using it.

The first step is to compute $H_\bullet(M, \partial M; R)$ and $H^\bullet(M, \partial M; R)$ for a single Möbius band M. Our model for M is

$$([0, 2\pi] \times [-1, 1])/\sim$$

where $(0, t) \sim (2\pi, -t)$. I'll call the circle $[0, 2\pi] \times \{0\}$ the *spine* S of M. The inclusion $S \hookrightarrow M$ is a homotopy equivalence as there is a retraction $r : M \to S$, which is defined by $(\theta, t) \mapsto (\theta, 0)$. The boundary ∂M is also a circle. So there are isomorphisms

$$H_1(S; R) = H_1(M; R) \cong R \text{ and } H_1(\partial M; R) \cong R.$$

The map $\partial M \to M \to S$ is a 2:1 covering. Covering space theory implies (as all fundamental groups here are abelian) that (with respect to these isomorphisms)

$$H_1(\partial M; R) \to H_1(M; R) \cong H_1(S; R)$$

is multiplication by ± 2. Adjusting the generator, we can assume it is 2.

The homology LES for the pair $(M, \partial M)$ produces the exact sequence

$$0 \to H_2(M, \partial M; R) \to H_1(\partial M; R) \xrightarrow{\times 2} H_1(M; R) \to H_1(M, \partial M; R) \to 0$$

from which we conclude that

$$H_j(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2 & j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_j(M, \partial M; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & j = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Using the fact that for all spheres (and hence for S^1) the Kronecker map

$$\kappa : H^j(Z; R) \to \operatorname{Hom}_{\mathbb{Z}}(H_j(Z; \mathbb{Z}), R)$$

is an isomorphism, we see that $H^1(M; R)$ and $H^1(\partial M; R)$ are both isomorphic to R and that (for appropriate integral generators)

$$H^1(M; R) \to H^1(\partial M; R)$$

is multiplication by 2. Plugging this into the LES

$$0 \to H^1(M, \partial M; R) \to H^1(M; R) \stackrel{\times 2}{\to} H^1(\partial M; R) \to H^2(M, \partial M; R) \to 0$$

of $(M, \partial M)$, we see that

$$H^j(M, \partial M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2 & j = 2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H^j(M, \partial M; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & j = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Note: One good error check is to use Euler characteristic: if X can be obtained from A by attaching a finite number of cells and if R is a PID, then

$$\chi(X, A) = \sum_{j \geq 0} \operatorname{rank}_R H_j(X, A; R) = \sum_{j \geq 0} \operatorname{rank}_R H^j(X, A; R).$$

This does not depend on R. Above $\chi(M, \partial M) = \chi(M) - \chi(\partial M) = 0$. You can also use the UCTs.

Back to the original problem. In my opinion, the best way to approach this is to consider the LES of the pair (X, Y). Set $Z = \sqcup M_j$. Observe that, by excision, the inclusion

$$(Z, \partial Z) \to (X, Y)$$

induces an isomorphism on homology and cohomology (all coefficients).

Now consider the LES of (X, Y) with $\mathbb{Z}$ coefficients. I will stick to cohomology, but you can do the homology computation as well if you like. This leads us to the diagram

$$
\begin{array}{ccccccccccc}
 & & & & K_X & \xrightarrow{\simeq} & K_Y & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H^1(X,Y) & \longrightarrow & H^1(X) & \hookrightarrow & H^1(Y) & \longrightarrow & H^2(X,Y) & \longrightarrow & H^2(X) \longrightarrow 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & \downarrow{\cong} & & \downarrow \\
0 & \longrightarrow & H^1(Z,\partial Z) & \longrightarrow & H^1(Z) & \longrightarrow & H^1(\partial Z) & \longrightarrow & H^2(Z,\partial Z) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \| & & \\
 & & 0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\times 2} & \mathbb{Z}^n & \longrightarrow & (\mathbb{Z}/2)^n & \longrightarrow & 0
\end{array}
$$

I have filled in the bottom row using the computations above and using the fact that

$$H^\bullet(Z, \partial Z; R) = \bigoplus_j H^\bullet(M_j, \partial M_j; R)$$

Here K_X and K_Y are the kernels of the restriction mappings induced by $Z \to X$ and $\partial Z \to Y$, respectively. It is clear that the inclusion $H^1(X) \hookrightarrow H^1(Y)$ induces an inclusion $K_X \to K_Y$. A simple diagram chase implies that it is onto and therefore an isomorphism as indicated.

To complete the computation, we need to understand the column containing $H^1(Y)$. Since ∂Z is identified with ∂Y, we can replace ∂Z by ∂Y. Then we have the cohomology LES

$$0 \to H^1(Y, \partial Y) \to H^1(Y) \to H^1(\partial Y) \to H^2(Y, \partial Y) \to 0$$

Write $Y = \overline{Y} - D$, where D is the union of the n open disks removed from $\overline{Y}$ to obtain Y. Excision then implies that

$$H^2(Y, \partial Y) \cong H^2(\overline{Y}, D) \cong H^2(\overline{Y}) \cong \mathbb{Z}.$$

This implies that the image of $H^1(Y) \to H^1(\partial Z)$ is the kernel of the "trace map"

$$H^1(\partial Z) \to \mathbb{Z}, \quad (k_1, \ldots, k_n) \mapsto \sum_j k_j$$

and that the image of $H^1(X) \to H^2(\partial Z)$ is $2\times$ the kernel of the trace map. It also implies (via the diagram) that the image of $H^1(Y) \to H^2(Z, \partial Z)$ is the kernel of the mod 2 reduction

$$H^2(Z, \partial Z) \to \mathbb{Z}/2.$$

It follows that $H^2(X;\mathbb{Z}) \cong \mathbb{Z}/2$.

Assembling this, we see that

$$H^j(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j=0 \\ \mathbb{Z}^{2g+n-1} & j=1 \\ \mathbb{Z}/2 & j=2 \\ 0 & \text{otherwise} \end{cases}$$

A similar, but simpler, argument can be used to show that

$$H^j(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & j=0 \\ \mathbb{F}_2^{2g+n} & j=1 \\ \mathbb{F}_2 & j=2 \\ 0 & \text{otherwise} \end{cases}$$

You should try to give an efficient (and convincing) proofs that

$$H_j(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j=0 \\ \mathbb{Z}^{2g+n-1} \oplus \mathbb{Z}/2 & j=1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_j(X;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2 & j=0 \\ \mathbb{F}_2^{2g+n} & j=1 \\ \mathbb{F}_2 & j=2 \\ 0 & \text{otherwise} \end{cases}$$

Note that these are compatible with both UCTs. You can also compute the Euler characteristic using any of these. They should all give the same answer, namely $\chi(X) = 2 - (2g+n)$. You can see directly that this is the Euler characteristic using the formula

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cup B)$$

and the fact that $\chi(M) = \chi(M \cap Y) = 0$, so that

$$\chi(X) = \chi(Y \cup M) = \chi(Y) = 2 - (2g+n).$$