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MATH 612
FINAL PROBLEM SET

Due: 5 pm., Thursday May 4, 2023

1. The goal of this problem is to define the Euler class and to construct the Gysin sequence of a sphere bundle. Suppose that $\pi : S \to X$ is a locally trivial sphere bundle whose fiber is S^n.

(i) Set
$$B = (S \times [0,1])/\sim$$
where $(u,s) \sim (v,t)$ if and only if either $u = v$ and $s = t$, or $s = t = 0$ and $\pi(u) = \pi(v)$. Show that $B \to X$ is a ball bundle whose associated sphere bundle is $S \to X$. Show that the "zero section" $\zeta : X \to B$ of and $\pi : B \to X$ are homotopy equivalences.

(ii) Suppose that $S \to X$ is R-oriented in the obvious sense. (Equivalently, assume that $B \to X$ is R-oriented.) Define the *Euler class* $e(S) \in H^{n+1}(X;R)$ of the sphere bundle by
$$e(S) := \zeta^* \tau_B.$$
Show that any two sections of B are homotopic. Deduce that for all sections s of B we have $e(S) = s^* \tau_B$.

(iii) Show that the Euler class of a trivial sphere bundle vanishes.

A (real or complex) vector bundle $V \to X$ is R-orientable if the associated ball bundle is R-orientable. An orientation of V is, by definition, an orientation of the associated ball bundle. The *Euler class* $e(V)$ of an R-oriented vector bundle V is defined to be the Euler class of the associated sphere bundle.

(iv) Construct and prove the exactness of the (period 3) long exact sequence

$$\cdots \to H^{n+j}(X;R) \xrightarrow{\pi^*} H^{n+j}(S;R) \xrightarrow{r} H^j(X;R)$$
$$\xrightarrow{\smile e(S)} H^{n+j+1}(X;R) \xrightarrow{\pi^*} H^{n+j+1}(S;R) \xrightarrow{r} H^{j+1}(X;R) \to \cdots$$

Hint: Apply the Thom isomorphism to the cohomology LES of the pair (B,S).

In the de Rham version, r is integration over the fiber — see Bott and Tu for details. In general, r is Kronecker dual to the "tube map", which is the map $H_j(X) \to H_{n+j}(X)$ that takes a cycle z to its inverse image

$\pi^{-1}(z)$ in S. One can use the Künneth theorem to show that it is well defined.

2. Prove that complex vector bundles (and their associated ball and sphere bundles) have a natural orientation. (Hint: show that the orientation $i^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$ of $\mathbb{C}^n$ is invariant under $\mathrm{GL}_n(\mathbb{C})$. Alternatively, show that $\mathrm{GL}_n(\mathbb{C})$ is connected so that $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{R})^+$.)

3. Suppose that $L \to X$ is a complex line bundle. Show that the Euler class of the dual line bundle $\check{L}$ is $-e(L)$.

4. (The tautological line bundle over projective space) Suppose that F is a field and that V is a non-zero finite dimensional vector space over F. Define

$$L = \{(\ell, v) \in \mathbb{P}(V) \times V : v \in \ell\}.$$

Denote the projection onto the first factor by $\pi : L \to \mathbb{P}(V)$.

(i) Show that the fiber of L over ℓ is the corresponding 1-dimensional subspace of V.

(ii) Each non-zero linear functional $\phi : V \to F$ defines a Zariski open subset $U_\phi := \mathbb{P}(V) - \mathbb{P}(\ker \phi)$. Show that there is a unique section s_ϕ of L over U_ϕ with the property that

$$s_\phi(\ell) = (\ell, v)$$

where $\phi(v) = 1$. Show that if $\phi, \psi \in V^*$ are non-zero linear functionals on V, then

$$s_\psi = \phi\psi^{-1} s_\phi.$$

Deduce that L is an algebraic line bundle over $\mathbb{P}(V)$ and that s_ϕ has a "simple pole" along the hyperplane $\mathbb{P}(\ker \phi)$.

(iii) Deduce that when $F = \mathbb{R}$ or $\mathbb{C}$ that L is a C^∞ line bundle. (It is a holomorphic line bundle when $F = \mathbb{C}$.)

When $F = \mathbb{R}$ or $\mathbb{C}$ a norm on V induces a natural metric on L — i.e., a way of measuring length of vectors in each fiber. Namely

$$\|(\ell, v)\| = \|v\|.$$

(iv) Show that, when $F = \mathbb{R}$ or $\mathbb{C}$, the total space of the sphere bundle of $L \to \mathbb{P}(V)$ is the unit sphere in V.

(v) Show that when $F = \mathbb{R}$ and $V = \mathbb{R}^{n+1}$ with its euclidean standard metric, the sphere bundle is the universal covering $S^n \to \mathbb{P}(V) = \mathbb{RP}^n$.

(vi) Show that when $F = \mathbb{C}$ and $V = \mathbb{C}^{n+1}$ with its hermitian standard metric, the sphere bundle is a circle bundle

$$S^{2n+1} \to \mathbb{P}(V) = \mathbb{CP}^n.$$

This is called the *Hopf fibration*. The complex structure of L gives it a natural orientation.

(vii) Show that the Euler class $e \in H^1(\mathbb{RP}^n; \mathbb{F}_2)$ of the double covering $S^n \to \mathbb{RP}^n$ is the generator of $H^1(\mathbb{RP}^n; \mathbb{F}_2)$. Hint: use the Gysin sequence.

(viii) Show that the Euler class e of the Hopf fibration $S^{2n+1} \to \mathbb{CP}^n$ is a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$. Hint: use the Gysin sequence.

Remark: algebraic geometers denote the line bundle $L \to \mathbb{P}(V)$ by $\mathcal{O}_{\mathbb{P}(V)}(-1)$ and its dual by $\mathcal{O}_{\mathbb{P}(V)}(1)$. The line bundle $\mathcal{O}_{\mathbb{P}(V)}(d)$ is defined to be the dth tensor power of $\mathcal{O}_{\mathbb{P}(V)}(1)$ when $d > 0$ and the $-d$th power of $\mathcal{O}_{\mathbb{P}(V)}(-1)$ when $d < 0$.

5. The goal of the problem is to understand the normal bundle of a hyperplane in $\mathbb{P}(V)$. Suppose that F is a field and that $V = W \oplus Fe$, where $e \in V$, where V is finite dimensional and W is non-zero. Set

$$N := \{[w, \lambda e] \in \mathbb{P}(V) : \lambda \in F,\ w \in W,\ w \neq 0\}$$

and define $\pi : N \to \mathbb{P}(W)$ by $\pi([w, \lambda e]) = [w]$.

(i) Suppose that $\phi \in W^*$ is non-zero. Define $s_\phi : \mathbb{P}(W) \to N$ by

$$s_\phi([w]) = [w, \phi(w)e].$$

Show that s_ϕ is a well-defined section of $N \to \mathbb{P}(W)$ which vanishes on the hyperplane $\mathbb{P}(\ker \phi)$ in $\mathbb{P}(W)$.

(ii) Prove that $N \to \mathbb{P}(W)$ is an algebraic line bundle over $\mathbb{P}(W)$. It is the normal bundle of $\mathbb{P}(W)$ in $\mathbb{P}(V)$.

(iii) Prove that it is dual to $\mathcal{O}_{\mathbb{P}(W)}(-1)$ and is therefore isomorphic to $\mathcal{O}_{\mathbb{P}(W)}(1)$.

(iv) Suppose that $F = \mathbb{C}$. Show that when $\dim \mathbb{P}(W) > 0$, the Euler class $e(N) \in H^2(\mathbb{P}(W); \mathbb{Z})$ of N is the positive generator [H] of $H^2(\mathbb{P}(W); \mathbb{Z})$.

6. The goal of this problem is to understand the normal bundle of a linear subspace of $\mathbb{P}(V)$. Suppose that F is a field and that V is a non-zero finite dimensional vector space over F. Suppose that W is a non-zero proper subspace of V. Choose a complement T of W in V:

$$V = W \oplus T.$$

(i) Set $N = \mathbb{P}(V) - \mathbb{P}(T)$. Show that

$$N = \{[w,t] : w \in W,\ t \in T; w \neq 0\}.$$

(ii) Define a projection $\pi : N \to \mathbb{P}(W)$ by $\pi([w,t]) = [w]$.

(iii) Choose a basis $e_1, \dots, e_q$ of T. For $j = 1, \dots, q$, set

$$L_j = \{[w, \lambda e_j] : w \in W,\ w \neq 0,\ \lambda \in F\}.$$

Show that $L_j \to \mathbb{P}(W)$ is a line bundle isomorphic to $\mathcal{O}_{\mathbb{P}(W)}(1)$.

(iv) Show that N is isomorphic to the vector bundle

$$\bigoplus_{j=1}^{q} L_j$$

and is therefore a vector bundle over $\mathbb{P}(W)$ isomorphic to $\mathcal{O}_{P(W)}(1)^{\oplus q}$.

7. (Construction and basic properties of the first Chern class of a complex line bundle.) Every complex line bundle $L \to X$ has a natural orientation, and therefore an Euler class $e(L) \in H^2(X;\mathbb{Z})$. Define the first Chern class of the complex line bundle $L \to X$ by $c_1(L) = e(L)$.

(i) Show that if $f : Y \to X$, then $c_1(f^*L) = f^*c_1(L)$.

(ii) Show that $c_1(\check{L}) = -c_1(L)$.

(iii) Show that the first Chern class of the tautological line bundle $L \to \mathbb{CP}^n$ is $-[H]$, where H is a hyperplane in $\mathbb{CP}^n$. Hint: (1) what is the normal bundle of H in $\mathbb{CP}^n$? (2) What is the restriction of L to H?

Remark: it is true that if L_1 and L_2 are complex line bundles over X, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$. I could not think of a simple way for you to prove this using the construction of c_1 above.

8. (Construction and basic properties of the first Stiefel-Whitney class of a real line bundle.) Every real line bundle $L \to X$ has a natural $\mathbb{F}_2$ orientation, and therefore an Euler class $e(L) \in H^1(X;\mathbb{F}_2)$. Define the first Stiefel-Whitney class of the real line bundle $L \to X$ by $w_1(L) = e(L)$.

(i) Show that if $f : Y \to X$, then $w_1(f^*L) = f^*w_1(L)$.

(ii) Show that $L \to X$ is trivial if and only if $w_1(L) = 0$. Hint: Reduced to the case when X is path connected. Then use the Gysin sequence — start from the degree 0 terms.

(iii) Show that the first Stiefel-Whitney class of the tautological line bundle $L \to \mathbb{RP}^n$ is $[H]$, where H is a hyperplane. (Hint: What is the restriction of L to H?

(iv) The S^0-bundle B associated to $L \to X$ is a double covering. It therefore determines a homomorphism $\phi_L : \pi_1(X, x) \to \mathbb{F}_2$. Show that ϕ_L can be viewed as an element of $H^1(X; \mathbb{F}_2)$. Show that L is trivial if and only $\phi_L = 0$ provided that X is path connected.

(v) Denote the S^0 bundle associated to the real line bundle $L \to X$ by $\pi : S \to L$. Show that the map $H^j(S; \mathbb{F}_2) \to H^j(X; \mathbb{F}_2)$ in the Gysin sequence is the pushforward π_*.

(vi) Show that $w_1(L) = \phi_L$ when X is path connected. Hint: Use the Gysin sequence to show that both span the kernel of $\pi^* : H^1(X; \mathbb{F}_2) \to H^1(S; \mathbb{F}_2)$.

(vii) Show that if L_1 and L_2 are real line bundles over X, then $w_1(L \otimes L') = w_1(L) + w_1(L')$. Hint: reduce to the case X path connected and then show that $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$.