Richard Hain February 28, 2023

Math 612
Problem Set 4

Due: Tuesday, March 21, 2023

1. For a real number $a > 0$, set

$$S^n(a) = \{x \in \mathbb{R}^{n+1} : \|x\| = a\} \text{ and } B^{n+1}(a) = \{x \in \mathbb{R}^{n+1} : \|x\| \leq a\}.$$

(When the a does not appear, it will be assumed to be 1.)

(i) Show that

$$\int_{S^n(a)} \sum_{j=0}^n (-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n = (n+1)C_{n+1}a^{n+1}$$

where C_m is the volume of B^m. Hint: use Stokes' Theorem.

(ii) Show that the form

$$\omega := \sum_{j=0}^n (-1)^j \frac{x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n}{(x_0^2 + x_1^2 + \cdots + x_n^2)^{(n+1)/2}}$$

is closed. Deduce that it generates $H^n_{\mathrm{dR}}(\mathbb{R}^{n+1} - \{0\})$.

(iii) Show that

$$\int_{S^n(a)} \omega = \mathrm{vol}(S^n).$$

Hint: use the fact that

$$\mathrm{vol}(S^n(a)) = \frac{d}{dr}\bigg|_{r=a} \mathrm{vol}(B^{n+1}(r)).$$

Deduce that the class of $((n+1)C_{n+1})^{-1}\omega$ is an *integral generator* of $H^n(S^n;\mathbb{R})$. That is, it is the image of a generator of $H^n(S^n;\mathbb{Z})$.

2. ("Math 212 in a Box") Suppose that U is an open subset of $\mathbb{R}^3$. Denote the space of vector fields defined on U by $\mathcal{X}(U)$. Note that $E^0(U)$ is the space of "scalar fields" $\mathcal{F}(U)$ on U.

(i) Show that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\mathrm{grad}"] \arrow[d, equal] & \mathcal{X}(U) \arrow[r, "\mathrm{curl}"] \arrow[d, "\varphi_1"] & \mathcal{X}(U) \arrow[r, "\mathrm{div}"] \arrow[d, "\varphi_2"] & \mathcal{F}(U) \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
0 \arrow[r] & E^0(U) \arrow[r, "d"] & E^1(U) \arrow[r, "d"] & E^2(U) \arrow[r, "d"] & E^3(U) \arrow[r] & 0
\end{tikzcd}$$

commutes, where

$$\begin{aligned}\varphi_1(f,g,h) &= f\,dx + g\,dy + h\,dz\\ \varphi_2(f,g,h) &= f\,dy\wedge dz - g\,dx\wedge dz + h\,dx\wedge dy\\ \varphi_3(f) &= f\,dx\wedge dy\wedge dz\end{aligned}$$

(ii) Show that a necessary condition for a vector field $F \in \mathcal{X}(U)$ to be the curl of a vector field $G \in \mathcal{X}(U)$ is that $\operatorname{div} F = 0$. Show that a divergence free vector field $F \in \mathcal{X}(U)$ is the curl of a vector field $G \in \mathcal{X}(U)$ if and only if

$$\int_{Z_j} \varphi_2(F) = 0$$

where $\{Z_j\}$ are 2-cycles in U whose homology classes generate $H_2(U)$.

(iii) Suppose that T is a smooth oriented surface in U. Show that

$$\int_T F\cdot \mathbf{n}\,dA = \int_T \varphi_2(F)$$

for all $F \in \mathcal{X}(U)$. Here $\mathbf{n}$ is the outward unit normal vector to Y that defines its orientation and dA is the standard area form on Y.

(iv) Give an explicit example of a divergence free vector field defined on $U = \mathbb{R}^3 - \{0\}$ that is not the curl of any $G \in \mathcal{X}(U)$.